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Ehrhart's polynomial for equilateral triangles in $\mathbb{Z}^3$

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Abstract

In this paper we calculate the Ehrhart's polynomial associated with a 2-dimensional regular polytope (i.e. equilateral triangles) in $\mathbb{Z}^3$. The polynomial takes a relatively simple form in terms of the coordinates of the vertices of the triangle. We give some equivalent formula in terms of a parametrization of these objects which allows one to construct equilateral triangles with given properties. In particular, we show that given a prime number p which is equal to 1 or $-5 \pmod 8$, there exists an equilateral triangle with integer coordinates whose Ehrhart polynomial is $L(t) = (pt+2)(t+1)/2$, $t \in \mathbb{N}$.

1 Introduction

A description of all equilateral triangles with vertices in $\mathbb{Z}^3$ appeared first in [5] (with the proof of the full general case in [2]). An updated version of the same results but with a shorter analysis was included in [7].

In the 1960's, Eugène Ehrhart ([4]) proved that given a $\mathbb{Z}^k$ lattice κ-dimensional polytope in $\mathbb{R}^k$ $(1 \le \kappa \le k)$, denoted here generically by $\mathcal{P}$, there exists a polynomial $L(\mathcal{P}, t) \in \mathbb{Z}[t]$ of degree κ, associated with $\mathcal{P}$, satisfying

$$L(\mathcal{P}, t) = \text{the cardinality of } \{t\mathcal{P}\} \cap \mathbb{Z}^k,\ t \in \mathbb{N}. \quad (1)$$

One can learn about the Ehrhart polynomial from a multitude of sources nowadays (see for instance [1]). The simplest lattice polytopes that one can think of, besides the zero and the one-dimensional ones, are triangles together with their interior. In particular, it is natural to find out what the situation is with the equilateral triangles from this point of view. In general, when talking about the Ehrhart polynomial

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of a lattice polytope, we may assume without loss of generality that the polytope is *irreducible*, in the sense that one cannot obtain the polytope from an integer dilation of another lattice polytope. So, in what follows, we are going to consider only irreducible triangles. The problem of calculating their Ehrhart polynomial is a relatively simple matter and this follows more or less from the general theory. However, we are giving here an independent argument and we derive some simple form for this polynomial in terms of the parametrization in [2]. This allows us to construct such triangles having polynomials with certain desired properties. For example, given two positive integers λ, μ, one may ask whether there exists an (irreducible) equilateral triangle $\mathcal{P}$ such that $L(\mathcal{P},t)=\frac{\lambda t^2+\mu t}{2}+1$, $t\in\mathbb{N}$. We will see that a necessary but not sufficient condition for this to happen is

(a) λ is an arbitrary odd positive integer;

(b) μ is a positive odd number such that $\mu=\mu_1+\mu_2+\mu_3$ with μ_1, μ_2 and μ_3 mutually coprime divisors of λ (see Table 1).

It is interesting that the case $\mu=\lambda+2$ happens quite often and $L(\mathcal{P},t)=\frac{(\lambda t+2)(t+1)}{2}$, $t\in\mathbb{N}$. For these situations, we notice that the roots of $L(\mathcal{P},t)$ are -1 and $-\frac{2}{\lambda}$. The geometric interpretation of the fact that -1 is a root of $L(\mathcal{P},t)$ is that there are no lattice points in the interior of the polytope. It makes sense to consider the following sequence:

$$E(\lambda) \;:=\; \text{cardinality of } \{L(\mathcal{P},t)=\frac{\lambda t^2+\mu t}{2}+1 \,|\, \text{for some equilateral triangle } \mathcal{P}\},\quad \lambda\in\mathbb{N}.$$

We are going to start by recalling the parametrization from [7]. Clearly, every equilateral triangle in $\mathbb{Z}^3$ after a translation by a vector with integer coordinates can be assumed to have the origin as one of its vertices. Obviously, such a translation leaves the Ehrhart polynomial invariant. Let us denote such a triangle by OPQ: $O=(0,0,0)$, $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$. Then one can show that the triangle's plane (Figure 1), i.e. the plane passing through the origin and containing P and Q (see [5]), can be described as

$$\mathcal{P}_{a,b,c}:=\{(x,y,z)\in\mathbb{Z}^3 |\ \ ax+by+cz=0,\ \ a^2+b^2+c^2=3d^2,\ \gcd(a,b,c)=1,\ a,b,c,d\in\mathbb{Z}\}, \tag{2}$$

where a, b and c are integers such that a divides $p_2q_3-p_3q_2$, b divides $p_1q_3-p_3q_1$, and c is a divisor of $p_1q_2-p_2q_1$.

This is a lattice of points in $\mathbb{Z}^3$ which is, in general, much richer than the sublattice, ${\mathcal{P}^{eq}}_{a,b,c}$, of all points which are vertices of equilateral triangles with one of the vertices the origin (see Figure 1). It is natural to start with the integer d and a solution of the Diophantine equation $a^2+b^2+c^2=3d^2$. We observe that a trivial solution of this equation is $a=b=c=d$. Clearly we want $\gcd(a,b,c)=1$. It is not trivial to show that there is always such a solution if and only if d is an odd. Actually there is a pretty good description of all such solutions, with the additional restriction that $0 < a \leq b \leq c$, in terms of the prime factorization of d (see [6]). The vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ can be written in terms of two minimal ones [7] which can be constructed directly from the solution (a, b, c) by looking into the greatest common divisor of certain elements in the ring $\mathbb{Z}[i\sqrt{3}]$.

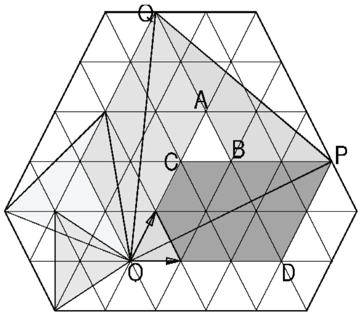

Figure 1: The lattice $\mathcal{P}^{eq}{}_{a,b,c}$

Theorem 1.1. *The sub-lattice* $\mathcal{P}^{eq}{}_{a,b,c}$ *is generated by two vectors* $\overrightarrow{\zeta}$ *and* $\overrightarrow{\eta}$ *with integers coordinates, in the following sense:* $\mathcal{T}^{m,n}_{a,b,c} := \triangle OPQ$ *with* P, Q *in* $\mathcal{P}_{a,b,c}$, *is equilateral if and only if for some integers* m, n

$$\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta}, \qquad \overrightarrow{OQ} = n\overrightarrow{\zeta} + (m-n)\overrightarrow{\eta}, \tag{3}$$

$$\text{with } \overrightarrow{\zeta} = (\zeta_1, \zeta_2, \zeta_3), \overrightarrow{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3), \overrightarrow{\eta} = \frac{\overrightarrow{\zeta} + \overrightarrow{\varsigma}}{2},$$

$$\begin{cases} \zeta_1 = -\dfrac{rac + dbs}{a^2+b^2} \\ \zeta_2 = \dfrac{das - bcr}{a^2+b^2} \\ \zeta_3 = r \end{cases}, \qquad \begin{cases} \varsigma_1 = \dfrac{3dbr - acs}{a^2+b^2} \\ \varsigma_2 = -\dfrac{3dar + bcs}{a^2+b^2} \\ \varsigma_3 = s \end{cases}, \tag{4}$$

where $s + i\sqrt{3}r = \gcd(ac - i\sqrt{3}bd, 2(a^2+b^2))$ *in the ring* $\mathbb{Z}[i\sqrt{3}]$.

Moreover, the following are also true:

(i) $2(a^2+b^2) = s^2 + 3r^2$, $2(b^2+c^2) = \varsigma_1^2 + 3\zeta_1^2$ *and* $2(a^2+c^2) = \varsigma_2^2 + 3\zeta_2^2$.

(ii) $|\overrightarrow{\zeta}| = d\sqrt{2}$, $|\overrightarrow{\varsigma}| = d\sqrt{6}$, *and* $\overrightarrow{\zeta} \cdot \overrightarrow{\varsigma} = 0$.

(iii) The sides-lengths of $\triangle OPQ$ *are equal to* $d\sqrt{2(m^2 - mn + n^2)}$.

(iv) $\triangle OPQ$ *is irreducible if and only if* $\gcd(m,n) = 1$.

(v) If $\omega = \gcd(a, b)$ then $r = \omega\widetilde{r}$ and $s = \omega\widetilde{s}$ with $\widetilde{r}, \widetilde{s} \in \mathbb{Z}$.

Remark: From a computational point of view, and without loss of generality, it is good to assume that $0 < a \leq b \leq c$ when applying the Theorem 1.1, since this reduces the number of possibilities of r and s.

An example here may be illuminating. If $d = 15$, we observe that we can take $a = 1$, $b = 7$ and $c = 25$ ($3d^2 = a^2 + b^2 + c^2$). Then, $r = -5$ and $s = 5$ give $\overrightarrow{\zeta} = (13, 16, -5)$ and $\overrightarrow{\eta} = 3(7, -1, 0)$. Also, properties (i), (ii), and (iii) in Theorem 1.1 are satisfied.

From the general theory (see [1] for a good account) we know that the Ehrhart polynomial of such a triangle Δ is given by

$$L(\Delta, t) = c_0 t^2 + \frac{c_1}{2} t + c_2, \ t \in \mathbb{N},$$

where $c_2 = 1$ since we are dealing with a (convex) polytope. It is easy to show that the coefficient c_0 is equal to the area of the triangle Δ normalized by the area of a fundamental domain of the sub-lattice $\mathcal{P}_{a,b,c}$. We will show that c_0 depends only on m, n and d. Let us show that in this case the coefficient c_1 is the number of points of the sub-lattice $\mathcal{P}_{a,b,c}$ on the sides of the triangle.

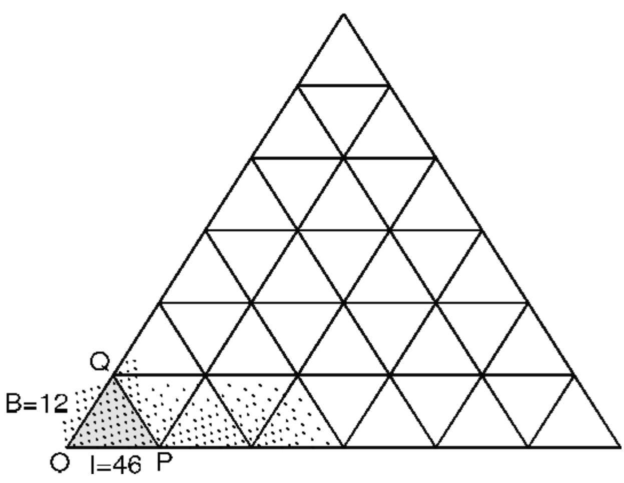

Figure 2: Dilation of Δ by $t = 6$, with $B = 12$, $I = 46$

We denote by I the number of lattice points in the interior of the triangle Δ and let B be the number of lattice points interior to the sides. It is not difficult to see that all the triangles of the same size as Δ, which appear as in the tessellation shown in Figure 2, filling out the interior of the dilation $t\Delta$, have the same property in terms of the numbers B and I defined earlier. There are $1+3+\cdots+(2t-1) = t^2$ such triangles in $t\Delta$. Each interior side of Δ gets translated into $1+2+\cdots+t = \frac{t(t+1)}{2}$ copies in $t\Delta$.

The number of vertices of all these triangles is $1+2+3+\cdots+(t+1)=\frac{(t+1)(t+2)}{2}$. Then simple counting shows that for $t \in \mathbb{N}$, we have

$$L(\Delta,t)=It^2+B\frac{t(t+1)}{2}+\frac{(t+1)(t+2)}{2}=(I+\frac{B}{2}+\frac{1}{2})t^2+\frac{B+3}{2}t+1.$$

This proves that $L(\Delta,t)$ is indeed a polynomial of degree two and the claim above about c_1 and c_2 follows. In Figure 2 we have a picture of a triangle $\Delta_0 = OPQ$ in which $B=12$ and $I=46$. This gives a polynomial $L(\Delta_0,t)=\frac{105t^2+15t}{2}+1$. Is there a realization Δ_0 in $\mathbb{Z}^3$ with such a polynomial? This polynomial does satisfy the necessary condition (part (b)) described above, since $15=3+5+7$ and $105=3(5)(7)$. We will explain the answer to this question in Section 4.

We point out that this polynomial is not a complete invariant, in the sense that we may have the same Ehrhart polynomial for two "different" triangles. By different triangles, we understand that one triangle cannot be obtained from the other by the usual transformations which leave the lattice $\mathbb{Z}^3$ invariant. For instance, if we take $\Delta_1 := \{(0,0,0),(13,-8,3),(0,-11,11)\}$ and $\Delta_2 := \{(0,0,0),(4,15,-1),(15,4,-1)\}$, then

$$L(\Delta_1,t)=L(\Delta_2,t)=\frac{11}{2}t^2+\frac{13}{2}t+1=\frac{1}{2}(t+1)(11t+2),\ t\in\mathbb{N}.$$

The triangles are essentially different since they live in totally different sub-lattices: $\mathcal{P}_{5,13,13}$ and $\mathcal{P}_{1,1,19}$ respectively. However, (as pointed out by the referee) one can check that $\Delta_1 B + (15,4,-1) = \Delta_2$, where $B \in SL(3,\mathbb{Z})$ (integer entries and determinant equal to 1) is given by

$$B := \begin{pmatrix} -9 & -24 & 2 \\ -21 & -61 & 5 \\ -22 & -60 & 5 \end{pmatrix}$$

which explains the equality of the two Ehrhart polynomials. This is not quite a coincidence and we will add some more information about this at the end of the paper. Of course, it is natural to wonder whether the Ehrhart polynomial for two equilateral triangles is the same if and only if the two triangles are equivalent via a transformation B, as above, $B \in SL(3,\mathbb{Z})$. We will see that the coefficient c_1 makes the difference if the two triangles correspond to the same d.

The sequence $E(\lambda)$ defined earlier begins in the way recorded in Table 1.

λ	All primitive solutions of $a^2+b^2+c^2=3\lambda^2$	$E(\lambda)$	c_1
1	$\{[1,1,1]\}$	1	$\{3\}$
3	$\{[1,1,5]\}$	2	$\{3,5\}$
5	$\{[1,5,7]\}$	1	$\{3\}$
7	$\{[1,5,11]\}$	1	$\{3\}$
9	$\{[1,11,11],[5,7,13]\}$	3	$\{3,5,11\}$
11	$\{[1,1,19],[5,13,13],[5,7,17]\}$	2	$\{3,13\}$
13	$\{[5,11,19],[7,13,17]\}$	1	$\{3\}$
15	$\{[5,11,23],[1,7,25],[5,17,19]\}$	2	$\{5,7\}$
17	$\{[7,17,23],[1,5,29],[13,13,23],[11,11,25]\}$	2	$\{3,19\}$
19	$\{[11,11,29],[1,11,31],[5,23,23],[13,17,25]\}$	2	$\{3,21\}$
21	$\{[11,19,29],[1,19,31],[13,23,25]\}$	3	$\{3,5,9\}$
23	$\{[11,25,29],[1,25,31],[1,19,35],[7,13,37]\}$	1	$\{3\}$
25	$\{[1,5,43],[17,25,31],[11,23,35],[5,13,41],[17,19,35]\}$	1	$\{3\}$
27	$\{[1,31,35],[11,29,35],[17,23,37],[7,17,43],[13,13,43]\}$	4	$\{3,5,11,29\}$
29	$\{[23,25,37],[1,11,49],[1,29,41],[7,25,43],[5,17,47]\}$	1	$\{3\}$
31	$\{[7,25,47],[11,19,49],[5,7,53],[17,35,37],[19,29,41]\}$	1	$\{3\}$
33	$\{[5,29,49],[7,37,43],[23,37,37],[19,35,41],[25,31,41],[23,23,47],$ $[13,17,53]\}$	5	$\{3,5,13,15,35\}$
35	$\{[5,29,53],[17,19,55],[11,23,55],[25,37,41],[25,29,47],[5,13,59]\}$	2	$\{3,7\}$
37	$\{[7,43,47],[5,19,61],[5,41,49],[1,25,59],[11,31,55],[23,37,47]\}$	1	$\{3\}$
39	$\{[13,37,55],[1,29,61],[5,7,67],[7,17,65],[11,31,59],[23,35,53]\}$	2	$\{3,5\}$
41	$\{[5,47,53],[1,1,71],[19,31,61],[13,43,55],[25,47,47],[5,23,67],$ $[31,41,49],[17,23,65]\}$	2	$\{3,43\}$

Table 1

What we are calculating in $L(\mathcal{T}^{1,0}_{a,b,c},t)$ reduces to counting the number of integer triples (x,y,z) satisfying

$$\begin{cases} ax+by+cz=0 \\ (3dbr-acs)x-(3dar+bcs)y+(a^2+b^2)sz\ge 0 \\ [ac(s-3r)-3db(r+s)]x+[3da(r+s)+bc(s-3r)]y+(3r-s)(a^2+b^2)z\ge 0 \\ [3db(r-s)-ac(3r+s)]x+[3da(s-r)-bc(3r+s)]y+(a^2+b^2)(3r+s)z \\ \qquad\qquad \le 2(a^2+b^2)d^2t, \quad t\in\mathbb{N}, \end{cases}$$

with r and s as in Theorem 1.1. It turns out that this can be simplified to

$$\begin{cases} ax+by+cz=0 \\ r(bx-ay)+sdz\ge 0 \\ (r+s)(ay-bx)+(s-3r)dz\ge 0 \\ (r-s)(bx-ay)+(3r+s)dz\le 2dt \end{cases}, \ t\in\mathbb{N}.$$

It is natural then to expect a formula for $L(\mathcal{T}^{1,0}_{a,b,c},t)$ in terms of r, s, a, b, c, d and t.

2 A fundamental domain in $\mathcal{P}_{a,b,c}$ and the coefficient c_0

In what follows we are going to assume that $\gcd(a,b,c) = 1$ and use the notation introduced in Theorem 1.1. We notice that from the relations (3) and (4), we obtain

$$\begin{aligned} \frac{r+s}{2}\overrightarrow{\zeta} - r\overrightarrow{\eta} &= d(-b,a,0), \\ \frac{\zeta_1+\varsigma_1}{2}\overrightarrow{\zeta} - \zeta_1\overrightarrow{\eta} &= d(0,-c,b), \\ \text{and} \quad \frac{\zeta_2+\varsigma_2}{2}\overrightarrow{\zeta} - \zeta_2\overrightarrow{\eta} &= d(c,0,-a). \end{aligned} \tag{5}$$

It is clear that the vectors $\overrightarrow{u} = \frac{1}{\gcd(a,b)}(-b,a,0)$, $\overrightarrow{v} = \frac{1}{\gcd(a,c)}(-c,0,a)$ and $\overrightarrow{w} = \frac{1}{\gcd(b,c)}(0,-c,b)$ correspond to points in $\mathcal{P}_{a,b,c}$. We first show that $\mathcal{P}_{a,b,c}$ is generated by $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ (a point P in $\mathcal{P}_{a,b,c}$ is identified by its position vector $\overrightarrow{OP}$ as usual).

Lemma 2.1. *With the above notation, we have*

$$\mathcal{P}_{a,b,c} = \{m\overrightarrow{u} + n\overrightarrow{v} + p\overrightarrow{w} \mid m,n,p \in \mathbb{Z}\}.$$

Proof. If $(x,y,z) \in \mathcal{P}_{a,b,c}$, we have $ax+by+cz = 0$. Because $\omega := \gcd(a,b)$ and $\gcd(a,b,c) = 1$ we need to have $z = \omega z'$ with $z' \in \mathbb{Z}$. Also, the existence of integers k and l such that $ka + \ell b = \omega$ is ensured by the fact $\omega = \gcd(a,b)$. This means that we have

$$(x,y,z) - z'[\gcd(a,c)k\overrightarrow{v} + \gcd(b,c)\ell\overrightarrow{w}] = (\alpha',\beta',0) \in \mathcal{P}_{a,b,c}.$$

Since $a\alpha' + b\beta' = 0$ we see that $(\alpha',\beta',0) = \lambda\overrightarrow{u}$ for some $\lambda \in \mathbb{Z}^3$. This shows the inclusion $\subset$ in the equality claimed, and the inclusion $\supset$ is obvious. ■

If we look at the proof of the above lemma we see that it is not necessary to have three vectors to generate $\mathcal{P}_{a,b,c}$, but only $\overrightarrow{u}$ and $\overrightarrow{\tau} := \gcd(a,c)k\overrightarrow{v} + \gcd(b,c)\ell\overrightarrow{w}$, where $ka + \ell b = \gcd(a,b)$, are enough. Since there are infinitely many pairs (k,l) satisfying this equality, let us take the solution that minimizes k so that $k > 0$. For computational purposes, we have the following more useful result.

Lemma 2.2. *With the above definition of* $\overrightarrow{\tau}$ *we have*

(i) $\mathcal{P}_{a,b,c} = \{k\overrightarrow{u} + \ell\overrightarrow{\tau} \mid k,\ell \in \mathbb{Z}\}$,

(ii) $\overrightarrow{u} = \frac{\widetilde{r}+\widetilde{s}}{2d}\overrightarrow{\zeta} - \frac{\widetilde{r}}{d}\overrightarrow{\eta}$, *where* $r = \omega\widetilde{r}$ *and* $s = \omega\widetilde{s}$ *(Theorem 1.1), and*

$$\overrightarrow{\tau} = \frac{\alpha}{d}\overrightarrow{\zeta} + \frac{\beta}{d}\overrightarrow{\eta}, \quad \textit{for some } \alpha,\beta \in \mathbb{Z}. \tag{6}$$

Proof. The first part follows from Lemma 2.1 and (ii) is a consequence of the equalities (5). ■

So, $\overrightarrow{u}$ and $\overrightarrow{\tau}$ form a fundamental domain for $\mathcal{P}_{a,b,c}$. The area of the parallelogram formed by these vectors is given by $|\overrightarrow{u} \times \overrightarrow{\tau}|$.

Lemma 2.3.

(i) The area of a fundamental domain for $\mathcal{P}_{a,b,c}$ is equal to $d\sqrt{3}$.

(ii) The integers α and β in (6) satisfy the relation

$$|\frac{\widetilde{r}+\widetilde{s}}{2}\beta+\widetilde{r}\alpha|=d. \tag{7}$$

Proof. (i) We observe that $\overrightarrow{u}\times\overrightarrow{v}=\frac{a}{\omega\gcd(a,c)}(a\overrightarrow{i}+b\overrightarrow{j}+c\overrightarrow{k})$ and similarly $\overrightarrow{u}\times\overrightarrow{w}=\frac{b}{\omega\gcd(b,c)}(a\overrightarrow{i}+b\overrightarrow{j}+c\overrightarrow{k})$. Hence the area of the parallelogram determined by $\overrightarrow{u}$ and $\overrightarrow{\tau}$ is equal to

$$|\overrightarrow{u}\times\overrightarrow{\tau}|=|\frac{1}{\omega}(ak+bl)(a\overrightarrow{i}+b\overrightarrow{j}+c\overrightarrow{k})|=\sqrt{a^2+b^2+c^2}=d\sqrt{3}.$$

The second part follows from (i) and the equality (6), if we take into account that $\overrightarrow{\zeta}\times\overrightarrow{\zeta}=\overrightarrow{\eta}\times\overrightarrow{\eta}=0$, $\overrightarrow{\zeta}\times\overrightarrow{\eta}=-\overrightarrow{\eta}\times\overrightarrow{\zeta}$, and $|\overrightarrow{\zeta}\times\overrightarrow{\eta}|=d^2\sqrt{3}$:

$$d\sqrt{3}=|\overrightarrow{u}\times\overrightarrow{\tau}|=|\frac{\frac{\widetilde{r}+\widetilde{s}}{2}\beta+\widetilde{r}\alpha}{d^2}\overrightarrow{\zeta}\times\overrightarrow{\eta}|=\frac{|\frac{\widetilde{r}+\widetilde{s}}{2}\beta+\widetilde{r}\alpha|}{d^2}d^2\sqrt{3}\Rightarrow(7).$$

Proposition 2.4. *The coefficient c_0 in the Ehrhart polynomial associated with an equilateral triangle $\mathcal{T}_{a,b,c}^{m,n}$ described by Theorem 1.1 is given by*

$$c_0=\frac{d(m^2-mn+n^2)}{2}.$$

Proof. By the general theory of the Ehrhart polynomial, c_0 is equal to the area of the triangle normalized by the area of a fundamental domain of the lattice $\mathcal{P}_{a,b,c}$. Since the area of the triangle $\mathcal{T}_{a,b,c}^{m,n}$ is equal to $\frac{2d^2(m^2-mn+n^2)\sqrt{3}}{4}$, using Lemma 2.3, we obtain $c_0=\frac{2d^2(m^2-mn+n^2)\sqrt{3}}{4d\sqrt{3}}=\frac{d(m^2-mn+n^2)}{2}$. ■

3 The coefficient c_1

Recall that OPQ is given by $O=(0,0,0)$, $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$. If we denote by $\kappa_1=\gcd(p_1,p_2,p_3)$, $\kappa_2=\gcd(q_1,q_2,q_3)$ and $\kappa_3=\gcd(p_1-q_1,p_2-q_2,p_3-q_3)$ then we simply have

$$c_1=\kappa_1+\kappa_2+\kappa_2.$$

This is a good formula if we have the coordinates of the equilateral triangle. We will provide another formula just in terms of r and s that appear in Theorem 1.1.

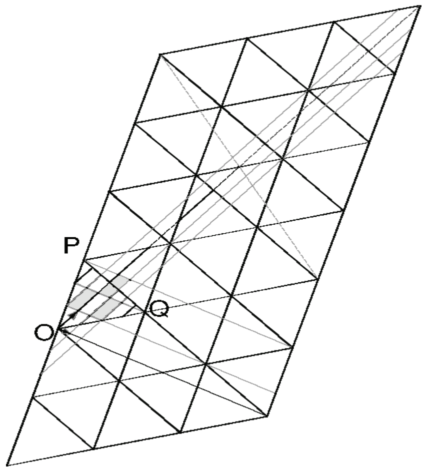

Figure 3: $\overrightarrow{u} = \frac{\widetilde{r}+\widetilde{s}}{2d}\overrightarrow{\zeta} - \frac{\widetilde{r}}{d}\overrightarrow{\eta}$, $\overrightarrow{\tau} = \frac{\alpha}{d}\overrightarrow{\zeta} + \frac{\beta}{d}\overrightarrow{\eta}$

We consider first the case $m = 1$ and $n = 0$. From Lemma 2.1, we see (Figure 2) that the problem at hand is to count the points of the lattice, that are determined by $\overrightarrow{u}$ and $\overrightarrow{\tau}$, on the sides of the triangle $\mathcal{T}^{1,0}_{a,b,c}$. Clearly we have 3 points, the vertices of the triangle, and in most cases these are all of the lattice points on the sides ($c_1 = 3$). The points on the sides can be (see Figure 2) on the side $\overline{OP}$, $\overline{PQ}$ or $\overline{OQ}$. We note that the points on the side $\overline{OP}$, are characterized by the existence of integers k, ℓ, λ and μ such that

$$k\overrightarrow{u} + \ell\overrightarrow{\tau} = (k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d})\overrightarrow{\zeta} + (-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})\overrightarrow{\eta} = (k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d})\overrightarrow{\zeta} + \mu\overrightarrow{\eta}. \quad (8)$$

Similarly, for points on the side $\overline{OQ}$, the equality above changes to

$$k\overrightarrow{u} + \ell\overrightarrow{\tau} = (k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d})\overrightarrow{\zeta} + (-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})\overrightarrow{\eta} = \lambda\overrightarrow{\zeta} + (-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})\overrightarrow{\eta}. \quad (9)$$

For the side $\overline{PQ}$, we have the characterization

$$k\frac{\widetilde{s}-\widetilde{r}}{2d} + \ell\frac{\alpha+\beta}{d} \in \mathbb{Z}. \quad (10)$$

Lemma 3.1. *(i) The coefficient* $(-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})$ *is an integer if* $\ell = \mu\frac{\widetilde{r}+\widetilde{s}}{2} + \lambda\widetilde{r}$ *and* $k = \lambda\beta - \mu\alpha$ *for every* $\lambda, \mu \in \mathbb{Z}^3$.

(ii) For the given values of ℓ *and* k *above the other coefficient in (8) becomes*

$$k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d} = \pm\lambda \in \mathbb{Z}.$$

(iii) If $\gcd(\widetilde{r}, \beta) = 1$, *then there are no points of the lattice in the interior of* $\overline{OP}$.

PROOF. Using (7), we have

$$(-k\frac{\widetilde{r}}{d}+\ell\frac{\beta}{d}) \ = \ (-\lambda\beta+\mu\alpha)\frac{\widetilde{r}}{d}+(\mu\frac{\widetilde{r}+\widetilde{s}}{2}+\lambda\widetilde{r})\frac{\beta}{d} \ = \ \pm\mu.$$

Similarly, one checks that (ii) is true.

We observe that (7) implies in particular that $\gcd(\widetilde{r},\beta)$ divides d. Therefore, if $\gcd(\widetilde{r},\beta)=1$, the values in (i) for k and ℓ give all of the solutions of the Diophantine equation $(-k\frac{\widetilde{r}}{d}+\ell\frac{\beta}{d}) = \mu \in \mathbb{Z}$. (We refer the reader to a basic text on linear Diophantine equations such as [8].) Part (ii) implies that there are no lattice points on the side $\overline{OP}$ other than the endpoints. ■

Clearly, a similar lemma is true for the sides $\overline{OQ}$ and $\overline{PQ}$.

So, we have proved the following proposition.

PROPOSITION 3.2. *If* $\gcd(\widetilde{r},\beta)=\gcd(\frac{\widetilde{r}+\widetilde{s}}{2},\alpha)=\gcd(\frac{\widetilde{r}-\widetilde{s}}{2},\frac{\alpha+\beta}{2})=1$*, then there are no lattice points on the sides of* $\mathcal{T}^{1,0}_{a,b,c}$ *other than the vertices. The Ehrhart polynomial in this case is*

$$L(\mathcal{T}^{1,0}_{a,b,c},t)=\frac{dt^2+3t}{2}+1,\ t\in\mathbb{N}.$$

It is natural to ask whether all sides of the equilateral triangle $\mathcal{T}^{1,0}_{a,b,c}$ may simultaneously contain lattice points in their interiors and how many can there be? The next lemma answers these questions.

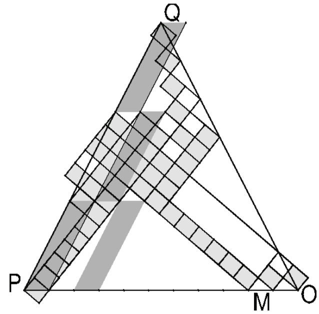

Figure 4: All sides may contain lattice points

LEMMA 3.3. *(i) If a side contains lattice points in its interior, the intersection of that side with* $\mathcal{P}_{a,b,c}$ *is the set of points which divide that side into* d' *equal parts, where* d' *divides* d.

(ii) It is possible that all of the sides of a minimal triangle $\mathcal{T}^{1,0}_{a,b,c}$ *have lattice points of* $\mathcal{P}_{a,b,c}$ *in their interiors. If* d_1 *and* d_2 *are the corresponding numbers as in part (i), for two sides, then* $\gcd(d_1,d_2)=1$.

Proof. (i) We refer to Figure 3 in this proof. Without loss of generality, let us assume that the closest point to O in $\mathcal{P}_{a,b,c}$ on the side $\overline{OP}$, is M. Then $\overrightarrow{OM}$ has integer coordinates. Then so does $k\overrightarrow{OM}$ for all $k \in \mathbb{Z}$. Therefore, for some k we must have $k\overrightarrow{OM} = \overrightarrow{OP}$, otherwise we can construct some point in $\mathcal{P}_{a,b,c}$, on the side $\overline{OP}$, closer to O than M, using the idea of division with non-zero remainder. Therefore, we have shown the first part of (i). For the second part we observe that the Diophantine equation $\frac{2d^2}{k^2} = |\overrightarrow{OM}|^2 \in \mathbb{Z}$ is possible only if k divides d. This shows (i).

For (ii), we include here the first example we found of such a triangle,

$$\Delta_3 = \{(0,0,0),(220,539,-539),(747,12,-267)\},$$

corresponding to $d = 561 = 3(11)(17)$, $a = 245$, $b = 613$ and $c = 713$. There are respectively 2, 10 and 16 lattice points interior to its sides. The associated Ehrhart polynomial is

$$L(\Delta_3,t) = \frac{561t^2+31t}{2} + 1,\ t \in \mathbb{N}.$$

For the second part of (ii), let us assume by way of contradiction, that $\gcd(d_1,d_2) = \delta > 1$. Then there exist integers d_1' and d_2' such that $d_1 = \delta d_1'$ and $d_2 = \delta d_2'$. From this we get that $1 > \frac{1}{\delta} = \frac{d_1'}{d_1} = \frac{d_2'}{d_2}$. Without loss of generality we can assume that the side $\overline{OP}$ is divided into d_1 equal parts and the side $\overline{OQ}$ into d_2 equal parts. Then by taking the division points on the sides corresponding to d_1' and d_2' respectively, but different from O, we can construct an equilateral triangle in $\mathcal{P}_{a,b,c}$, with side lengths $\frac{d\sqrt{2}}{\delta}$ which are strictly smaller than those of $\mathcal{T}^{1,0}_{a,b,c}$. Theorem 1.1 shows that this is not possible. The contradiction shows that $\gcd(d_1,d_2) = 1$. ■

Let us now look at the case $\gcd(\widetilde{r},\beta) = \nu > 1$. Then Lemma 3.1 easily changes into the following.

Lemma 3.4.

(i) If $\gcd(\widetilde{r},\beta) = \nu$, *the coefficient* $(-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})$ *is an integer if* $\ell = \mu\frac{\widetilde{r}+\widetilde{s}}{2} + \lambda\frac{\widetilde{r}}{\nu}$ *and* $k = \lambda\frac{\beta}{\nu} - \mu\alpha$ *for every* $\lambda, \mu \in \mathbb{Z}^3$.

(ii) For the given values of ℓ *and* k *above the other coefficient in (8) becomes*

$$k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d} = \pm\frac{\lambda}{\nu} \in \mathbb{Z}.$$

(iii) If $\gcd(\widetilde{r},\beta) = \nu$, *then there are* $\nu - 1$ *points of the lattice in the interior of the side* $\overline{OP}$.

Proof. The calculations we have done in showing Lemma 3.1 are valid here with the substitution $\frac{\lambda}{\nu}$ instead of λ. Part (iii) follows from parts (i) and (ii). ■

Finally, we can put all these things together into the following generalization of Proposition 3.2:

Theorem 3.5. *The Ehrhart polynomial for $\mathcal{T}^{1,0}_{a,b,c}$ (given by Theorem 1.1) is*

$$L(\mathcal{T}^{1,0}_{a,b,c}, t) = \frac{dt^2 + c_1 t}{2} + 1, \ t \in \mathbb{N}, \ \textit{with}$$

$$c_1 = \gcd(\widetilde{r}, \beta) + \gcd(\frac{\widetilde{r}+\widetilde{s}}{2}, \alpha) + \gcd(\frac{\widetilde{r}-\widetilde{s}}{2}, \alpha+\beta), \ \textit{where}$$

$r = \gcd(a,b)\widetilde{r}$, $s = \gcd(a,b)\widetilde{s}$, (α, β) is a particular solution of the Diophantine equation

$$\frac{\widetilde{r}+\widetilde{s}}{2}\beta + \widetilde{r}\alpha = d$$

and $3d^2 = a^2 + b^2 + c^2$.

We observe that this theorem does not depend on the particular solution (α, β) of the Diophantine equation $\frac{\widetilde{r}+\widetilde{s}}{2}\beta + \widetilde{r}\alpha = d$ (or equivalently $\frac{-\widetilde{r}+\widetilde{s}}{2}\beta + \widetilde{r}(\alpha+\beta) = d$).

4 The general case

We fix two integers m and n with $\gcd(m,n) = 1$ in Theorem 1.1. In this case, from (3), we have $\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta}$ and $\overrightarrow{OQ} = n\overrightarrow{\zeta} + (m-n)\overrightarrow{\eta}$. We can solve for $\overrightarrow{\zeta}$ and $\overrightarrow{\eta}$:

$$\overrightarrow{\zeta} = \frac{(m-n)\overrightarrow{OP} + n\overrightarrow{OQ}}{m^2 - mn + n^2} \ \textit{and} \ \overrightarrow{\eta} = \frac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m^2 - mn + n^2}.$$

Then, the equations (8) and (9) change into

$$\begin{array}{l} k\overrightarrow{u} + \ell\overrightarrow{\tau} = (k\frac{\widetilde{r}+\widetilde{s}}{2d} + \ell\frac{\alpha}{d})\overrightarrow{\zeta} + (-k\frac{\widetilde{r}}{d} + \ell\frac{\beta}{d})\overrightarrow{\eta} = \\ \left(k\dfrac{m(\widetilde{r}+\widetilde{s}) + n(\widetilde{r}-\widetilde{s})}{2d(m^2-mn+n^2)} + \ell\dfrac{m\alpha - n(\alpha+\beta)}{d(m^2-mn+n^2)}\right)\overrightarrow{OP} + \\ \left(k\dfrac{n(\widetilde{r}+\widetilde{s}) - 2m\widetilde{r}}{2d(m^2-mn+n^2)} + \ell\dfrac{m\beta + n\alpha}{d(m^2-mn+n^2)}\right)\overrightarrow{OQ} := \lambda\overrightarrow{OP} + \mu\overrightarrow{OQ}. \end{array} \tag{11}$$

Here we have something similar to the case $m = 1$ and $n = 0$.

Lemma 4.1. *(i) The coefficient λ in (11) is an integer, and equal to $\widetilde{t}(m-n)$, if*

$k = \beta(m^2 - mn + n^2)\widetilde{t} + [m\alpha - n(\alpha+\beta)]t$, $\ell = \widetilde{r}(m^2 - mn + n^2)\widetilde{t} - (m\frac{\widetilde{r}+\widetilde{s}}{2} + n\frac{\widetilde{r}-\widetilde{s}}{2})t$,
where t and $\widetilde{t}$ are arbitrary integers. In this case, the coefficient μ is equal to $n\widetilde{t} - t$.

(ii) The coefficient μ in (11) is an integer, and equal to $\widetilde{t}m$, if

$k = -\alpha(m^2 - mn + n^2)\widetilde{t} + (m\beta + n\alpha)t$, $\ell = \frac{\widetilde{r}+\widetilde{s}}{2}(m^2 - mn + n^2)\widetilde{t} + (m\widetilde{r} - n\frac{\widetilde{r}+\widetilde{s}}{2})t$,
where t and $\widetilde{t}$ are arbitrary integers. In this case, the coefficient λ is equal to $t - n\widetilde{t}$.

(iii) The value $\mu+\lambda$ *is an integer, and equal to* $\widetilde{t}m$, *if*

$k=\beta(m^2-mn+n^2)\widetilde{t}+[m(\alpha+\beta)-n\beta]t$, $\ell=\widetilde{r}(m^2-mn+n^2)\widetilde{t}-(m\frac{\widetilde{s}-\widetilde{r}}{2}+n\widetilde{r})t$, *where* t *and* $\widetilde{t}$ *are arbitrary integers. In this case, the coefficient* μ *is equal to* $n\widetilde{t}-t$.

(iv) If $\gcd[m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2}, m\alpha-n(\alpha+\beta)]=1$, *then there are no points of the lattice in the interior of* $\overline{OQ}$.

(v) If $\gcd(m\widetilde{r}-n\frac{\widetilde{r}+\widetilde{s}}{2}, m\beta+n\alpha)=1$, *then there are no points of the lattice in the interior of* $\overline{OP}$.

(vi) If $\gcd(m\frac{\widetilde{s}-\widetilde{r}}{2}+n\widetilde{r}, m(\alpha+\beta)-n\beta)=1$, *then there are no points of the lattice in the interior of* $\overline{PQ}$.

The proof of this lemma is similar to the proof of Lemma 3.1. As before, the hypothesis in cases (iv), (v), and (vi) in Lemma 4.1 can be relaxed.

Lemma 4.2. *(i) If* $\gcd(m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2}, m\alpha-n(\alpha+\beta))=\nu$, *then the coefficient* λ *is still an integer if* $k=\beta(m^2-mn+n^2)\widetilde{t}+(m\alpha-n(\alpha+\beta))\frac{t}{\nu}$ *and* $\ell=\widetilde{r}(m^2-mn+n^2)\widetilde{t}-(m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2})\frac{t}{\nu}$, *where* t *and* $\widetilde{t}$ *are arbitrary integers. In this case, the coefficient* μ *is equal to* $n\widetilde{t}-\frac{t}{\nu}$, *and this gives* $\nu-1$ *points of the lattice in the interior of* $\overline{OQ}$.

(ii) If $\gcd(m\widetilde{r}-n\frac{\widetilde{r}+\widetilde{s}}{2}, m\beta+n\alpha)=\nu$, *then the coefficient* μ *in (11) is an integer equal to* $\widetilde{t}m$, *if* $k=-\alpha(m^2-mn+n^2)\widetilde{t}+(m\beta+n\alpha)\frac{t}{\nu}$, $\ell=\frac{\widetilde{r}+\widetilde{s}}{2}(m^2-mn+n^2)\widetilde{t}+(m\widetilde{r}-n\frac{\widetilde{r}+\widetilde{s}}{2})\frac{t}{\nu}$, *where* t *and* $\widetilde{t}$ *are arbitrary integers. The coefficient* λ *is equal to* $\frac{t}{\nu}-n\widetilde{t}$.

(iii) If $\gcd(m\frac{\widetilde{s}-\widetilde{r}}{2}+n\widetilde{r}, m(\alpha+\beta)-n\beta)=\nu$, *then the value of* $\mu+\lambda$ *is an integer, and equal to* $\widetilde{t}m$, *if* $k=\beta(m^2-mn+n^2)\widetilde{t}+(m(\alpha+\beta)-n\beta)\frac{t}{\nu}$, $\ell=\widetilde{r}(m^2-mn+n^2)\widetilde{t}-(m\frac{\widetilde{s}-\widetilde{r}}{2}+n\widetilde{r})\frac{t}{\nu}$, *where* t *and* $\widetilde{t}$ *are arbitrary integers. In this case, the coefficient* μ *is equal to* $n\widetilde{t}-\frac{t}{\nu}$.

(iv) The value ν *in each of the above cases is always a divisor of* d.

Proof. The calculations we have done in showing Lemma 4.1 are valid here with the substitution $\frac{t}{\nu}$ instead of t. For the part (iv), we note that since $\gcd(m,n)=1$ we have $mx+ny=1$ for some integers x and y. Let us consider on (i). One can check the identities

$$\alpha\left(m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2}\right)-(m\alpha-n(\alpha+\beta))\frac{\widetilde{r}+\widetilde{s}}{2}=nd, \text{ and}$$

$$(\alpha+\beta)\left(m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2}\right)+(m\alpha-n(\alpha+\beta))\frac{\widetilde{r}-\widetilde{s}}{2}=md.$$

These two equalities can be combined to get

$$((x+y)\alpha+x\beta)\left(m\frac{\widetilde{r}+\widetilde{s}}{2}+n\frac{\widetilde{r}-\widetilde{s}}{2}\right)+(m\alpha-n(\alpha+\beta))(x\frac{\widetilde{r}-\widetilde{s}}{2}-y\frac{\widetilde{r}+\widetilde{s}}{2})=d,$$

which implies the claim that ν must divide d. ■

The following theorem allows one to compute the Ehrhart polynomial for an equilateral triangle in $\mathbb{Z}^3$.

Theorem 4.3. *The Ehrhart polynomial of an equilateral triangle $\mathcal{T}_{a,b,c}^{m,n}$ (given by Theorem 1.1) with* $\gcd(m,n)=1$, *is given by*

$$L(\mathcal{T}_{a,b,c}^{m,n},t)=\frac{d(m^2-mn+n^2)t^2+c_1t}{2}+1,\ t\in\mathbb{N},\ \textit{where}$$

$$c_1=\gcd(m\tfrac{\widetilde{r}+\widetilde{s}}{2}+n\tfrac{\widetilde{r}-\widetilde{s}}{2},m\alpha-n(\alpha+\beta))+\gcd(m\widetilde{r}-n\tfrac{\widetilde{r}+\widetilde{s}}{2},m\beta+n\alpha)\\ +\gcd(m\tfrac{\widetilde{s}-\widetilde{r}}{2}+n\widetilde{r},m(\alpha+\beta)-n\beta),$$

and $(\alpha,\ \beta)$ is a particular solution of the Diophantine equation

$$\frac{\widetilde{r}+\widetilde{s}}{2}\beta+\widetilde{r}\alpha=d,$$

d, $\widetilde{r}$, $\widetilde{s}$ being defined in Theorem 1.1.

For the triangle Δ_0 in the Introduction, we see that there is only one option of three coprime divisors n_i of 105 which give $15=n_1+n_2+n_3$. Then we must have $d=105$. By looking into all solutions of $a^2+b^2+c^2=3(105)^2$, one can find all the corresponding triangles $\mathcal{T}_{a,b,c}^{1,0}$ and then the coefficient c_1. It turns out that $c_1\in\{3,5,9,17\}$ and so there is no Δ_0 in $\mathbb{Z}^3$ with the required property.

5 The case of $a=b$ and some further questions

If the equation of the plane (2) has the property that $a=b$ ($2a^2+c^2=3d^2$), then the parametrization of the equilateral triangles given by Theorem 1.1 simplifies to: $\mathcal{T}_{a,b,c}^{m,n}:=\triangle OPQ$ with P, Q in $\mathcal{P}_{a,b,c}$, is equilateral if and only if for some integers m, n,

$$\overrightarrow{OP}=m\overrightarrow{\zeta}-n\overrightarrow{\eta},\ \overrightarrow{OQ}=n\overrightarrow{\zeta}+(m-n)\overrightarrow{\eta},\text{with }\overrightarrow{\zeta}=(\zeta_1,\zeta_2,\zeta_3),\ \overrightarrow{\eta}=(\eta_1,\eta_2,\eta_3), \tag{12}$$

$$\begin{cases}\zeta_1=-\dfrac{d+c}{2}\\[2ex] \zeta_2=\dfrac{d-c}{2}\\[2ex] \zeta_3=a\end{cases},\ \textit{and}\ \begin{cases}\eta_1=\dfrac{d-c}{2}\\[2ex] \eta_2=-\dfrac{d+c}{2}\\[2ex] \eta_3=a\end{cases}. \tag{13}$$

We observe that $\widetilde{r} = \widetilde{s} = 1$ and so we can choose $\alpha = d$ and $\beta = 0$ to satisfy (7). Assuming as before that $\gcd(m, n) = 1$, then the Ehrhart polynomial reduces to the simple formula

$$L(\mathcal{T}^{m,n}_{a,a,c}, t) = \tfrac{1}{2}d(m^2 - mn + n^2)t^2 + [\gcd(m, d) + \gcd(n, d) + \gcd(m - n, d)]t + 1, \tag{14}$$

$t \in \mathbb{N}$.

In [6] we have characterized the primitive triples $(a, c, d) \in \mathbb{N}^3$ satisfying $2a^2+c^2 = 3d^2$. This was done in a manner similar to the way that Pythagorean triples are usually described with a one-to-one correspondence to a special set of pairs of natural numbers.

Theorem 5.1. *Suppose that k and ℓ are positive integers with k odd and $\gcd(k, \ell) = 1$. Then a, c and d given by*

$$d = 2\ell^2 + k^2$$

$$\text{with} \quad \begin{cases} a = |2\ell^2 + 2k\ell - k^2|, \ c = |k^2 + 4k\ell - 2\ell^2|, \ \text{if } k \not\equiv \ell \pmod 3, \\ a = |2\ell^2 - 2k\ell - k^2|, \ c = |k^2 - 4k\ell - 2\ell^2|, \ \text{if } k \not\equiv -\ell \pmod 3, \end{cases}$$

constitute a positive primitive solution of $2a^2 + c^2 = 3d^2$.

Conversely, with the exception of the trivial solution $a = c = d = 1$, every positive primitive solution for $2a^2 + c^2 = 3d^2$ appears in the way described above for some l and k.

In particular, if $d > 3$ is a prime of the form $8m + 1$ or $8m - 5$ $(m \in \mathbb{N})$, we can find (see [3]) k and ℓ as in Theorem 5.1. Hence we have shown the following corollary of our investigation:

Corollary 5.2. *Given a prime number p of the form $8m + 1$ or $8m - 5$ $(m \in \mathbb{N})$, there exits an equilateral triangle $\mathcal{T}^{1,0}_{a,a,c}$ in $\mathbb{Z}^3$ whose Ehrhart polynomial is*

$$L(\mathcal{T}^{1,0}_{a,a,c}, t) = \frac{(pt + 2)(t + 1)}{2}, \quad t \in \mathbb{N}.$$

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We thank Dr. Albert VanCleave who painstakingly corrected several versions of this paper. Also, we thank the referee who wrote one of most constructive reports we have seen in years. Besides the improvements suggested, he/she gave us the following line of further investigations. If we let $T_k = \text{conv}\{(0, 0, 0), (1, 0, 0), (1, k, 0)\}$ $(k \in \mathbb{N})$ (the convex closure of the three vertices), one of the most obvious triangles with no interior points and Ehrhart polynomial $E(t) = (kt + 2)(t + 1)/2$, then T_k is actually equivalent ($T_k B_k = \mathcal{T}^{1,0}_{a,a,c}$) to an equilateral triangle $\mathcal{T}^{1,0}_{a,a,c}$ as in Corollary 5.2, via

$$B_k = \begin{pmatrix} -(k + c)/2 & (k - c)/2 & a \\ 1 & -1 & 0 \\ 0 & \beta & \gamma \end{pmatrix} \in SL(3, \mathbb{Z}), \ \text{where } \beta a + \gamma c = 1,$$

for primes k which are 1 or 3 (mod 8). Can this fact be extended to other values of k? The more general question, in view of these examples, is whether or not two equilateral triangles having the same Ehrhart polynomial are equivalent (as above).

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