AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 289–301

On decomposing complete tripartite graphs into 5-cycles

Sharareh Alipour E.S. Mahmoodian* E. Mollaahmadi

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11155–9415
Tehran,
I.R. IRAN

Abstract

The problem of finding necessary and sufficient conditions to decompose a complete tripartite graph $K(r, s, t)$ into 5-cycles was first considered by Mahmoodian and Mirzakhani (1995). They stated some necessary conditions and conjectured that these conditions are also sufficient. Since then, many cases of the problem have been solved by various authors; however the case when the partite sets $r \le s \le t$ have odd and distinct sizes remains open. A necessary condition is $t \le 3r$. Billington and Cavenagh (2011) have shown that when r, s, and t are all odd and $r \le s \le t \le \kappa r$, where $\kappa \approx 1.0806$, then the conjectured necessary conditions for decomposing are also sufficient. We extend this result further to the cases where $\kappa \approx \frac{5}{3}$.

1 Introduction

A graph with vertex set V is said to be a *complete n-partite* graph, if V may be partitioned into n disjoint non-empty sets $V_1, V_2, \dots, V_n$, such that there exists exactly one edge between vertices from different partite sets, and no other edges. If $|V_i| = a_i$ for $1 \le i \le n$, then the graph is denoted by $K(a_1, a_2, \dots, a_n)$. The complete tripartite graph having three parts of sizes r, s, and t will be denoted by $K(r, s, t)$. This graph has all vertices of even degree provided that r, s, and t all have the same parity. The problem of decomposing complete tripartite graphs into 5-cycles was first considered by Mahmoodian and Mirzakhani [4].

Theorem A. ([4]) *If the complete tripartite graph $K(r, s, t)$, where $r \le s \le t$, can be decomposed into 5-cycles, then the following conditions are satisfied:*

* The research of this author is partially supported by a grant from the INSF.

- *r, s, and t are either all even or all odd;*
- $5|(rs + rt + st)$*;*
- $t \leq 4rs/(r + s)$*.*

From Theorem A we have:

Corollary A. ([3]) *If the complete tripartite graph $K(r, s, t)$ (with $r \leq s \leq t$) can be decomposed into 5-cycles, then $t \leq 3r, s \leq 3r$, and $t \leq 2s$.*

Conjecture 1. ([4]) *Three conditions given in Theorem A are also sufficient.*

Mahmoodian and Mirzakhani proved this conjecture in the case when two parts have equal sizes and satisfy the necessary conditions with one exception, $K(5x, 5x, z)$, where z is not a multiple of 5. Cavenagh and Billington in [3] extended this result, using different method and showed that when two parts have same sizes or when both r and s are divisible by 10 and the conditions in Theorem A hold then a decomposition into 5-cycles exists. They also completed the solution to the existence problem for a 5-cycle decomposition (satisfying the necessary conditions) when all the parts have even sizes [2].

Theorem B. ([3]) *The necessary conditions of Theorem A are sufficient in the case when two partite sets have equal size or in the case when r and s are divisible by 10.*

Theorem C. ([2]) *The necessary conditions of Theorem A are sufficient in the case when two partite sets have equal size or in the case when all partite sets have even size.*

The following theorem is from [3].

Theorem D. ([3]) *If the graphs $K(r_i, s_j, t_k)$ admit a decomposition into 5-cycles for each i, j, and k, $1 \leq i, j, k \leq m$, then the graph $K(r_1+r_2+\cdots+r_m,\ s_1+s_2+\cdots+s_m,\ t_1 + t_2 + \cdots + t_m)$ also admits a decomposition into 5-cycles.*

Recently, Billington and Cavenagh considered the cases when r, s, and t have similar sizes asymptotically. They showed that if r, s, and t are all odd and satisfy the necessary conditions in Theorem A and if $100 \leq r \leq s \leq t \leq \kappa r$ (where κ is approximately 1.0806), then $K(r, s, t)$ has a decomposition into 5-cycles.

Theorem E. ([1]) *Let $\kappa = -\frac{95}{16} + \frac{3}{16}\sqrt{1401} \approx 1.0806$. Let r, s, and t be odd numbers such that $rs + st + rt$ is divisible by 5 and $100 \leq r \leq s \leq t \leq \kappa r$. Then $K(r, s, t)$ has a decomposition into 5-cycles.*

In this paper we show decomposition of some small cases that were not known before and prove that if r, s, and t are all odd, and $107 \leq r \leq s \leq t \leq \kappa r$ (where κ is approximately $\frac{5}{3}$), then necessary conditions in Theorem A are sufficient.

2 Decomposing into 5-cycles using Latin representation

A method for decomposing a complete tripartite graph was developed in [3] which in fact extends the idea of Latin square. It is easy to see that a Latin square of order m is equivalent to decomposition of the complete tripartite graph $K(m,m,m)$ into triangles. *A Latin representation of complete tripartite graph* $K(r,s,t)$ is a Latin rectangle L of order $r \times s$ based on t elements together with L_1 which contains a set of $t-s$ entries at the end of each row of L, and L_2 containing a set of $t-r$ entries at the end of each column of L. Each entry from the set $T = \{1,2,\ldots,t\}$ occurs once in each of the r rows and once in each of the s columns.
Each entry of the Latin rectangle L represents a triangle (See Figure 1) and each entry of L_1 represents a single edge from the partite set of size r to the partite set of size t. Similarly each entry of L_2 represents a single edge from the partite set of size s to the partite set of size t. So a Latin representation of $K(r,s,t)$ is in fact equivalent to a decomposition of $K(r,s,t)$ into $r \times s$ triangles and $r \times (t-s) + s \times (t-r)$ single edges.

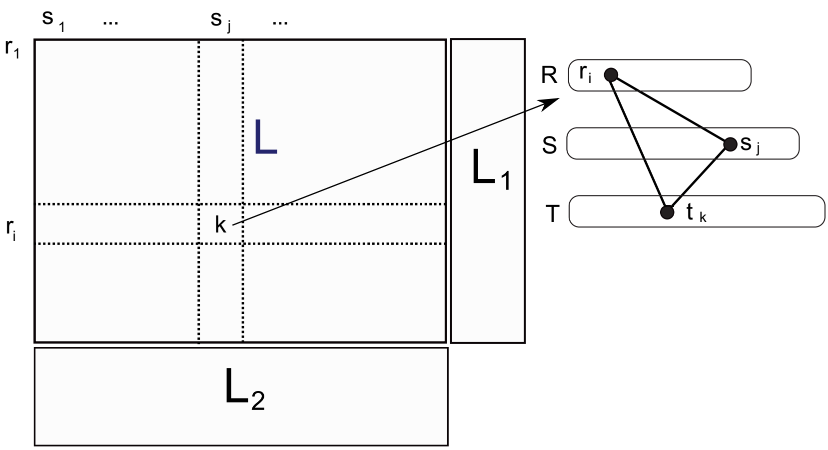

Figure 1: Each entry of the Latin rectangle L represents a triangle

Cavenagh and Billington [3] introduced a useful type of graphical trade which is defined as follow:

Definition . ([3]) *Let M be a Latin representation of the complete tripartite graph $K(r,s,t)$. A trade is a set of entries in M, corresponding to a set of triangles and edges in $K(r,s,t)$ which can be decomposed into 5-cycles.*

Trades of different types are used in [3] to decompose complete tripartite graphs into 5-cycles. Actually given a complete tripartite graph, it is sufficient to find a set of trades that covers every entry in Latin representation, making sure that no trades

overlap. Now it is sufficient to solve a jigsaw puzzle. This idea was used in [3] and [2] to show that necessary conditions are sufficient in the case where two partite sets have equal sizes or in the case where all partite sets have even sizes.
All used trades in [3] and [2] were classified into two types. In the first type, entries from both inside and outside of Latin rectangle are used. In this type of trade we exchange a set of triangles and a set of edges with a set of 5-cycles. In the second type, no entries from outside of the Latin rectangle are used and in this case we exchange a set of triangles with a set of 5-cycles.

2.1 A new trade and some small cases

In this section a new trade is introduced.

Definition. *Let M be a Latin representation of $K(r,s,t)$. A new trade is made of n triangles and $2n$ edges which can be decomposed into n 5-cycles. It has n entries from a row of Latin rectangle L and $2n$ entries from L_1 and/or L_2 (See Figure 2).*

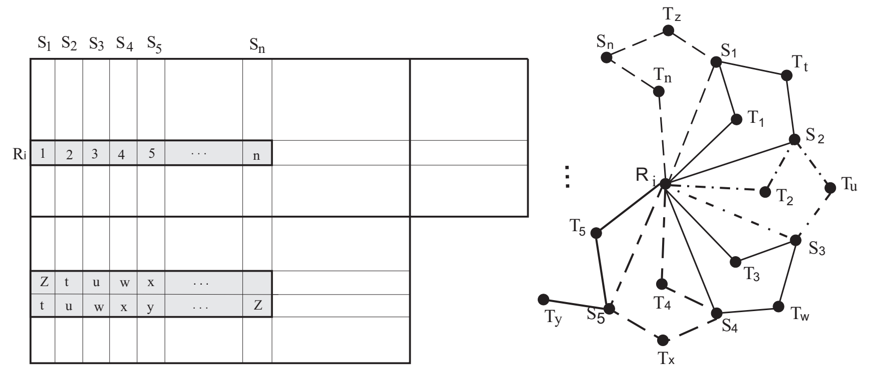

Figure 2: A trade with n triangles and $2n$ edges

The graph $K(7,17,19)$ was the smallest case satisfying the necessary conditions of Theorem A for which no 5-cycle decomposition was known. Using the new and old trades we can cover Latin representation of $K(7,17,19)$ and some other cases so that we decompose them into 5-cycles. Note that in Figure 3 and other similar figures in each cell the lower entry represents a vertex from part T and the upper entry is the label of the used trade.

Theorem 1. $K(11,15,25)$*,* $K(13,15,25)$*,* $K(15,17,25)$*, and* $K(15,19,25)$ *are decomposable into 5-cycles.*

Proof. Similar to $K(7,17,19)$, Latin representations of these cases can be covered by the old and new trades. Latin representation of these cases are given in the Appendices. □

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	20	1	19	19	31	31	7	34	8	8	40	40	44	44	9	9	9	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1	24	2	28	22	11	34	8	8	8	41	41	45	45	11	11	22	2	2
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1
3	24	2	28	3	32	32	12	12	37	37	42	42	46	46	9	9	12	3	3
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2
4	25	25	3	29	4	16	16	35	38	38	18	18	17	17	48	48	20	4	4
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3
5	26	26	29	4	23	5	35	23	13	13	43	43	47	47	49	49	13	5	5
	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4
6	27	27	30	30	5	33	6	10	10	10	10	10	10	10	10	10	10	6	6
	6	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5
7	7	14	21	15	33	6	36	36	39	39	14	14	15	15	50	50	21	7	7
	7	8	9	10	11	12	13	14	15	16	17	18	19	1	2	3	4	5	6
8	20	14	28	28	31	31	34	34	37	37	14	14	44	44	16	16	20		
	8	18	18	18	18	18	18	18	18	18	18	9	9	9	9	9	9		
9	20	14	28	28	31	31	34	34	37	37	14	14	44	44	16	16	20		
	9	9	19	19	19	19	19	19	19	19	19	19	10	10	10	10	8		
10	24	24	21	15	32	32	35	35	38	38	40	40	15	15	48	48	21		
	10	10	10	1	1	1	1	1	1	1	1	1	1	11	11	11	11		
11	24	24	21	15	32	32	35	35	38	38	40	40	15	15	48	48	21		
	11	11	11	11	2	2	2	2	2	2	2	2	2	2	12	12	10		
12	25	25	29	29	22	11	36	36	19	19	41	41	45	45	11	11	22		
	12	12	12	12	12	3	3	3	3	3	3	3	3	3	3	13	13		
13	25	25	29	29	22	11	36	36	19	19	41	41	45	45	11	11	22		
	13	13	13	13	13	13	4	4	4	4	4	4	4	4	4	4	12		
14	26	26	18	18	33	33	12	12	39	39	42	42	46	46	49	49	12		
	14	14	14	14	14	14	14	5	5	5	5	5	5	5	5	5	5		
15	26	26	18	18	33	33	12	12	39	39	42	42	46	46	49	49	12		
	15	15	15	15	15	15	15	15	6	6	6	6	6	6	6	6	14		
16	27	27	30	30	23	17	17	23	13	13	43	43	47	47	50	50	13		
	16	16	16	16	16	16	16	16	16	7	7	7	7	7	7	7	7		
17	27	27	30	30	23	17	17	23	13	13	43	43	47	47	50	50	13		
	17	17	17	17	17	17	17	17	17	17	8	8	8	8	8	8	16		
18	1	2	3	4	5	6	7	10	10	10	10	10	10	10	10	10	10		
	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
19	1	2	3	4	5	6	7	10	10	10	10	10	10	10	10	10	10		
	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	6		

Figure 3: A covered Latin representation of $K(7,17,19)$ by trades

3 New results

In this section by applying previous theorems and the new trade we extend the earlier results. We prove our main results in three theorems. First we need two lemmas and the following introductory result which is straightforward.

Proposition 1. *Let $K(r,s,t)$ be a complete tripartite graph which satisfies the conditions of Theorem A such that r, s, and t are odd, and let $r \equiv r' \pmod{10}$, $s \equiv s' \pmod{10}$, and $t \equiv t' \pmod{10}$, where $1 \le r', s', t' \le 9$. Then the multiset $\{r', s', t'\}$ is equal to one of the following multisets:*

$\{1,1,7\}$, $\{1,3,3\}$, $\{3,9,9\}$, $\{7,7,9\}$, $\{1,5,5\}$, $\{3,5,5\}$, $\{5,5,5\}$, $\{7,5,5\}$, $\{9,5,5\}$.

Lemma 1. *An odd integer a which is divisible by 5 can be written as:* $a = a_1 + \cdots + a_m$, *where* $a_i \in \{15, 25\}$ *for* $i = 1, 2, \ldots, m$ *if and only if* $15m \leq a \leq 25m$.

Proof. Let $a = a_1 + \cdots + a_m$ where $a_i \in \{15, 25\}$, for $i = 1, 2, \ldots, m$. Then we have $15m \leq a \leq 25m$.
On the other hand if $15m \leq a \leq 25m$, then it can easily be shown that a may be written as $a = 15k_1 + 25k_2$ where $k_1 + k_2 = m$. In fact since a is odd, m is an odd integer and $k_1 = \frac{25m - a}{10}$, $k_2 = \frac{a - 15m}{10}$. Since $a - 15m$, and $25m - a$ are divisible by 10 so nonnegative integers k_1 and k_2 exist. □

Lemma 2. *An odd integer a can be written as:* $a = a_1 + a_2 + \cdots + a_m$, *where* $a_1 \in \{11, 13, 15, 17, 19\}$ *and* $a_i \in \{15, 25\}$, *for* $i = 2, 3, \ldots, m$ *if and only if* $15m - 4 \leq a \leq 25m - 6$.

Proof. Let $a = a_1 + \cdots + a_m$, where $a_1 \in \{11, 13, 15, 17, 19\}$ and $a_i \in \{15, 25\}$ for $i = 2, 3, \ldots, m$, then obviously $15m - 4 \leq a \leq 25m - 6$.
If $15m - 4 \leq a \leq 25m - 6$, then there exists $a_1 \in \{11, 13, 15, 17, 19\}$ such that $a - a_1$ is multiple of 5 and $15m' \leq a - a_1 \leq 25m'$ for $m' = m - 1$. Now we can proceed similar to Lemma 1. □

Theorem 2. *Let $K(r, s, t)$ be a complete tripartite graph such that*

- *r, s, and t are all odd and divisible by 5;*
- $75 \leq r \leq s \leq t \leq \frac{5}{3}r - 50$.

Then the necessary conditions of Theorem A are sufficient.

Proof. Since $t \leq \frac{5}{3}r - 50$, we have $\frac{r}{15} - \frac{t}{25} \geq 2$. So there exists an odd number m so that $\frac{t}{25} \leq m \leq \frac{r}{15}$. This implies that $15m \leq r \leq 25m$, $15m \leq s \leq 25m$, and $15m \leq t \leq 25m$. Thus by Lemma 1, each of r, s, and t can be written as: $r = r_1 + r_2 + \cdots + r_m$, $s = s_1 + \cdots + s_m$, and $t = t_1 + \cdots + t_m$, where for $i, j.k \in \{1, 2, \ldots m\}$, r_i, s_j, and $t_k \in \{15, 25\}$. Now by Theorem C and Theorem D the assertion follows. □

Theorem 3. *Let $K(r, s, t)$ be a complete tripartite graph such that*

- *r, s, and t are odd and exactly one of them is not divisible by 5; and*
- $86 \leq r \leq s \leq t \leq \frac{5}{3}r - 57$.

Then the necessary conditions of Theorem A are sufficient.

Proof. Suppose r is not divisible by 5. Since $t \leq \frac{5}{3}r - 57$, we have $\frac{r+4}{15} - \frac{t+6}{25} \geq 2$. Let $a = \max\{\frac{r+6}{25}, \frac{s}{25}, \frac{t}{25}\}$ and $b = \min\{\frac{r+4}{15}, \frac{s}{15}, \frac{t}{15}\}$. It can be checked that by the assumption of theorem we have $b - a \geq 2$. So there is an odd number m between a and b. Such m satisfies all of the following inequalities: $\frac{r+6}{25} \leq m \leq \frac{r+4}{15}$, $15m - 4 \leq r \leq 25m - 6$, $15m \leq s \leq 25m$, and $15m \leq t \leq 25m$. Thus by Lemma 1 and Lemma 2 there exist r_i, s_j, and t_k, $1 \leq i, j, k \leq m$ so that $r = r_1 + r_2 + \cdots + r_m$, $s = s_1 + s_2 + \cdots + s_m$, and $t = t_1 + t_2 + \cdots + t_m$; where all r_i, s_j, and t_k's have sizes equal to 15 or 25 except r_1, and $r_1 \in \{11, 13, 15, 17, 19\}$. Then by Proposition 1, Theorem D, Theorem 1, and Theorem C, the assertion follows.

For the cases where either s or t is not divisible by 5 we may employ a similar argument. □

Theorem 4. *Let $K(r, s, t)$ be a complete tripartite graph such that the conditions of Theorem A are satisfied and $96 \leq r \leq s \leq t \leq \frac{5}{3}r - 46$; r,s, and t are odd and none of them is divisible by 5. Then $K(r, s, t)$ has 5-cycle decomposition.*

Proof. Let $r \equiv r_1 (\text{mod } 10)$, $s \equiv s_1 (\text{mod } 10)$, and $t \equiv t_1 (\text{mod } 10)$, where $10 < r_1, s_1, t_1 < 20$. Let $a = \max\{\frac{r+25-r_1}{25}, \frac{s+25-s_1}{25}, \frac{t+25-t_1}{25}\}$ and $b = \min\{\frac{r+15-r_1}{15}, \frac{s+15-s_1}{15}, \frac{t+15-t_1}{15}\}$. It can be checked that by the assumption of theorem we have $b - a \geq 2$. So there is an odd number m between a and b. Such m satisfies the following inequalities: $15m - 4 \leq r \leq s \leq t \leq 25m - 6$. Thus by Lemma 2 there exist r_i, s_j, and t_k, $2 \leq i, j, k \leq m$ so that $r = r_1 + r_2 + \cdots + r_m$, $s = s_1 + s_2 + \cdots + s_m$, and $t = t_1 + t_2 + \cdots + t_m$; where all r_i, s_j, and t_k's, $2 \leq i, j, k \leq m$ have sizes equal to 15 or 25, while $r_1, s_1, t_1 \in \{11, 13, 15, 17, 19\}$.

Now by Proposition 1, Theorem D, Theorem 1, and Theorem C, the assertion follows. We should note that in decomposing $K(r_1 + r_2 + \cdots + r_m, s_1 + s_2 + \cdots + s_m, t_1 + t_2 + \cdots + t_m)$ into 5-cycles in Theorem D, one way is to let L be a Latin square with rows, columns and symbols indexed by $\{1, \ldots, m\}$ and demand a 5-cycle decomposition of $K(r_i, s_j, t_k)$ for each $(i, j, k) \in L$. In appealing to this theorem we can choose a Latin square such that $(1, 1, 1) \in L$ which guarantees that for each $(i, j, k) \in L \setminus (1, 1, 1)$ we have that at least two of r_i, s_j, t_k are in the set $\{15, 25\}$. □

Example 1. *For an example of Theorem 4 consider the case where we want to have a 5-cycle decomposition for $K(123, 131, 163)$ (See Figure 4). Consider $m = 7$, then*
$123 = 13 + 15 + 15 + 15 + 15 + 25 + 25,$
$131 = 11 + 15 + 15 + 15 + 25 + 25 + 25,$ *and*
$163 = 13 + 25 + 25 + 25 + 25 + 25 + 25.$

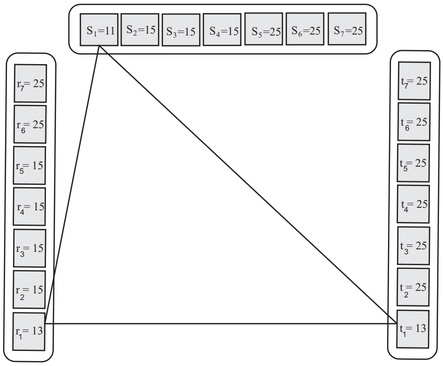

Figure 4: Decomposition of $K(123, 131, 163)$ using Theorem 4.

Acknowledgement

Referees deserve thanks for careful reading and many useful comments. We also thank Saeid Khaninejad for writing down a computer program to check the answers of cases mentioned in Theorem 1.

4 Appendices

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
01	26 1	26 2	1 3	3 4	4 5	45 6	5 7	46 8	6 9	50 10	52 11	53 12	53 13	72 14	74 15	1 16	1 17	3 18	3 19	4 20	4 21	5 22	5 23	6 24	6 25
02	61 3	61 1	33 2	7 6	3 4	4 5	6 9	5 7	46 8	51 12	50 10	54 11	54 15	73 13	72 14	7 16	7 17	3 18	3 19	4 20	4 21	5 22	5 23	6 24	6 25
03	2 2	32 3	61 1	8 5	9 6	44 4	32 8	10 9	11 7	52 11	51 12	55 10	55 14	74 15	73 13	2 16	2 17	8 18	8 19	9 20	9 21	10 22	10 23	11 24	11 25
04	27 13	56 14	31 15	36 2	9 1	14 3	13 4	36 5	7 6	67 7	71 8	71 9	12 11	75 12	76 10	7 16	7 17	12 18	12 19	9 20	9 21	14 22	14 23	13 24	13 25
05	29 15	57 13	15 14	40 1	40 3	2 2	16 6	13 4	59 5	71 9	67 7	67 8	34 10	12 11	75 12	2 16	2 17	12 18	12 19	16 20	16 21	15 22	15 23	13 24	13 25
06	28 14	31 15	27 13	1 3	33 2	14 1	28 5	16 6	11 4	71 8	71 9	30 7	35 12	76 10	21 11	1 16	1 17	21 18	21 19	16 20	16 21	14 22	14 23	11 24	11 25
07	17 10	62 11	35 12	37 13	38 14	41 15	41 1	47 2	47 3	44 4	23 5	22 6	69 7	24 8	10 9	17 16	17 17	24 18	24 19	23 20	23 21	10 22	10 23	22 24	22 25
08	18 12	17 10	62 11	39 15	37 13	42 14	42 3	48 1	48 2	58 6	60 4	22 5	25 9	69 7	24 8	17 16	17 17	24 18	24 19	18 20	18 21	25 22	25 23	22 24	22 25
09	62 11	18 12	34 10	38 14	39 15	43 13	43 2	49 3	49 1	8 5	45 6	20 4	19 8	25 9	69 7	20 16	20 17	8 18	8 19	18 20	18 21	25 22	25 23	19 24	19 25
10	30 7	19 8	15 9	63 10	63 11	63 12	65 13	65 14	65 15	68 1	68 2	68 3	20 4	23 5	21 6	20 16	20 17	21 18	21 19	23 20	23 21	15 22	15 23	19 24	19 25
11	60 4	59 5	58 6	64 7	64 8	64 9	66 10	66 11	66 12	57 13	56 14	29 15	70 1	70 2	70 3	56 16	56 17	57 18	57 19	60 20	60 21	58 22	58 23	59 24	59 25
12	61 6	61 4	61 5	63 9	63 7	63 8	66 12	66 10	66 11	68 3	68 1	68 2	70 3	70 1	70 2										
13	61 5	61 6	61 4	63 8	63 9	63 7	66 11	66 12	66 10	68 2	68 3	68 1	70 2	70 3	70 1										
14	62 9	62 7	62 8	64 12	64 10	64 11	65 15	65 13	65 14	67 15	67 13	67 14	69 6	69 4	69 5										
15	62 8	62 9	62 7	64 11	64 12	64 10	65 14	65 15	65 13	67 14	67 15	67 13	69 5	69 6	69 4										
16	30 16	56 16	35 16	37 16	37 16	41 16	41 16	47 16	47 16	50 16	50 16	30 16	35 16	72 16	72 16										
17	30 17	56 17	35 17	37 17	37 17	41 17	41 17	47 17	47 17	50 17	50 17	30 17	35 17	72 17	72 17										
18	29 18	57 18	34 18	38 18	38 18	42 18	42 18	48 18	48 18	51 18	51 18	29 18	34 18	73 18	73 18										
19	29 19	57 19	34 19	38 19	38 19	42 19	42 19	48 19	48 19	51 19	51 19	29 19	34 19	73 19	73 19										
20	28 20	31 20	31 20	39 20	39 20	44 20	28 20	49 20	49 20	44 20	60 20	53 20	53 20	74 20	74 20										
21	28 21	31 21	31 21	39 21	39 21	44 21	28 21	49 21	49 21	44 21	60 21	53 21	53 21	74 21	74 21										
22	27 22	32 22	27 22	40 22	40 22	45 22	32 22	46 22	46 22	58 22	45 22	54 22	54 22	75 22	75 22										
23	27 23	32 23	27 23	40 23	40 23	45 23	32 23	46 23	46 23	58 23	45 23	54 23	54 23	75 23	75 23										
24	26 24	26 24	33 24	36 24	33 24	43 24	43 24	36 24	59 24	52 24	52 24	55 24	55 24	76 24	76 24										
25	26 25	26 25	33 25	36 25	33 25	43 25	43 25	36 25	59 25	52 25	52 25	55 25	55 25	76 25	76 25										

Figure 5: A covered Latin representation of $K(11, 15, 25)$ by trades

9 1	10 2	11 3	68 4	69 5	35 6	56 7	56 8	2 9	14 10	75 11	76 12	65 13	66 14	65 15	9 16	9 17	10 18	10 19	11 20	11 21	35 22	35 23	14 24	14 25
9 3	10 1	11 2	12 6	68 4	69 5	57 9	57 7	2 8	74 12	14 10	77 11	67 15	67 13	66 14	9 16	9 17	10 18	10 19	11 20	11 21	12 22	12 23	14 24	14 25
15 2	16 3	17 1	71 5	13 6	70 4	58 8	58 9	44 7	75 11	74 12	23 10	84 14	84 15	84 13	15 16	15 17	16 18	16 19	17 20	17 21	13 22	13 23	23 24	23 25
15 13	16 14	41 15	4 1	4 2	18 3	70 4	72 5	22 6	59 7	59 8	64 9	23 10	78 11	79 12	15 16	15 17	16 18	16 19	18 20	18 21	22 22	22 23	23 24	23 25
5 15	5 13	45 14	19 3	20 1	18 2	22 6	73 4	72 5	60 9	62 7	61 8	76 12	24 10	78 11	19 16	19 17	20 18	20 19	18 20	18 21	22 22	22 23	24 24	24 25
45 14	5 15	17 13	4 2	19 3	20 1	71 5	25 6	73 4	61 8	60 9	63 7	77 11	79 12	24 10	19 16	19 17	20 18	20 19	17 20	17 21	25 22	25 23	24 24	24 25
37 10	40 11	21 12	42 13	6 14	6 15	8 1	8 2	82 3	1 4	31 5	25 6	62 7	27 8	64 9	21 16	21 17	31 18	31 19	27 20	27 21	25 22	25 23	37 24	37 25
46 12	37 10	21 11	26 15	13 13	6 14	8 3	8 1	8 2	1 6	1 4	83 5	2 9	63 7	27 8	21 16	21 17	26 18	26 19	27 20	27 21	13 22	13 23	37 24	37 25
40 11	46 12	41 10	6 14	6 15	42 13	82 2	82 3	43 1	83 5	83 6	43 4	2 8	2 9	44 7	44 16	44 17	43 18	43 19	42 20	42 21	41 22	41 23	40 24	40 25
47 7	47 8	50 9	50 10	53 11	53 12	5 13	38 14	5 15	32 1	34 2	33 3	3 4	3 5	36 6	33 16	33 17	32 18	32 19	34 20	34 21	36 22	36 23	38 24	38 25
48 9	48 7	51 8	12 12	54 10	54 11	7 15	38 13	51 14	33 3	31 1	34 2	1 6	1 4	3 5	33 16	33 17	31 18	31 19	34 20	34 21	12 22	12 23	38 24	38 25
49 8	49 9	52 7	52 11	55 12	55 10	7 14	7 15	39 13	30 2	29 3	32 1	3 5	35 6	3 4	30 16	30 17	32 18	32 19	29 20	29 21	35 22	35 23	39 24	39 25
80 4	80 5	80 6	81 7	81 8	81 9	7 10	7 11	39 12	30 13	29 14	26 15	4 1	4 2	36 3	30 16	30 17	26 18	26 19	29 20	29 21	36 22	36 23	39 24	39 25
80 6	80 4	80 5	81 9	81 7	81 8	82 12	82 10	82 11	83 15	83 13	83 14	84 3	84 1	84 2										
80 5	80 6	80 4	81 8	81 9	81 7	82 11	82 12	82 10	83 14	83 15	83 13	84 2	84 3	84 1										
47 16	47 16	50 16	50 16	53 16	53 16	56 16	56 16	44 16	59 16	59 16	63 16	65 16	63 16	65 16										
47 17	47 17	50 17	50 17	53 17	53 17	56 17	56 17	44 17	59 17	59 17	63 17	65 17	63 17	65 17										
48 18	48 18	52 18	52 18	54 18	54 18	57 18	57 18	43 18	60 18	60 18	64 18	67 18	67 18	64 18										
48 19	48 19	52 19	52 19	54 19	54 19	57 19	57 19	43 19	60 19	60 19	64 19	67 19	67 19	64 19										
49 20	49 20	51 20	42 20	55 20	55 20	58 20	58 20	51 20	61 20	62 20	61 20	62 20	66 20	66 20										
49 21	49 21	51 21	42 21	55 21	55 21	58 21	58 21	51 21	61 21	62 21	61 21	62 21	66 21	66 21										
46 22	46 22	41 22	68 22	68 22	70 22	70 22	72 22	72 22	74 22	74 22	76 22	76 22	78 22	78 22										
46 23	46 23	41 23	68 23	68 23	70 23	70 23	72 23	72 23	74 23	74 23	76 23	76 23	78 23	78 23										
45 24	40 24	45 24	71 24	69 24	69 24	71 24	73 24	73 24	75 24	75 24	77 24	77 24	79 24	79 24										
45 25	40 25	45 25	71 25	69 25	69 25	71 25	73 25	73 25	75 25	75 25	77 25	77 25	79 25	79 25										

Figure 6: A covered Latin representation of $K(13, 15, 25)$ by trades

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	60 1	60 2	63 3	22 4	22 5	63 6	20 7	11 8	11 9	56 10	34 11	35 12	5 13	5 14	5 15	24 16	25 19	24 17	24 18	25 20	25 21	34 22	34 23	35 24	35 25
2	61 3	61 1	64 2	66 6	66 4	64 5	20 9	20 7	20 8	35 12	56 10	34 11	4 15	4 13	5 14	24 17	25 20	24 18	24 16	25 21	25 19	34 22	34 23	35 24	35 25
3	62 2	62 3	65 1	67 5	67 6	65 4	12 8	12 9	20 7	36 11	37 12	57 10	5 14	3 15	3 13	24 18	25 21	24 16	24 17	25 19	25 20	36 22	36 23	37 24	37 25
4	16 19	16 20	17 21	68 1	68 2	21 3	69 4	69 5	72 6	72 7	75 8	75 9	57 10	36 11	37 12	26 13	27 16	26 14	26 15	27 17	27 18	36 22	36 23	37 24	37 25
5	17 21	17 19	17 20	21 3	21 1	21 2	70 6	70 4	73 5	73 9	76 7	76 8	38 12	58 10	39 11	26 14	27 17	26 15	26 13	27 18	27 16	38 22	38 23	39 24	39 25
6	13 20	13 21	17 19	53 2	53 3	21 1	71 5	71 6	74 4	74 8	77 9	77 7	39 11	38 12	58 10	26 15	27 18	26 13	26 14	27 16	27 17	38 22	38 23	39 24	39 25
7	19 16	52 17	52 18	1 19	2 20	2 21	29 1	83 2	82 3	22 4	22 5	9 6	28 7	3 8	3 9	2 10	1 13	28 11	28 12	29 14	29 15	82 22	82 23	83 24	83 25
8	19 18	19 16	18 17	1 21	1 19	2 20	85 3	29 1	84 2	18 6	22 4	18 5	4 9	28 7	3 8	2 11	1 14	28 12	28 10	29 15	29 13	84 22	84 23	85 24	85 25
9	19 17	19 18	18 16	47 20	47 21	54 19	87 2	86 3	29 1	18 5	9 6	9 4	4 8	4 9	28 7	59 12	59 15	28 10	28 11	29 13	29 14	86 22	86 23	87 24	87 25
10	30 13	14 14	14 15	31 16	40 17	41 18	54 19	6 20	6 21	10 1	9 2	9 3	78 4	7 5	7 6	7 7	6 10	30 8	30 9	31 11	31 12	40 22	40 23	41 24	41 25
11	14 15	30 13	14 14	41 18	31 16	40 17	13 21	55 19	6 20	10 3	10 1	8 2	8 6	78 4	7 5	7 8	6 12	30 9	30 7	31 11	31 10	40 22	40 23	41 24	41 25
12	51 14	14 15	30 13	42 17	43 18	31 16	13 20	13 21	55 19	10 2	10 3	8 1	8 5	8 6	79 4	79 9	51 11	30 7	30 8	31 10	31 12	42 22	42 23	43 24	43 25
13	16 10	15 11	15 12	42 13	43 14	15 15	12 16	11 17	11 18	49 19	33 20	32 21	50 1	84 2	82 3	80 4	80 7	32 5	32 6	33 8	33 9	42 22	42 23	43 24	43 25
14	16 12	16 10	15 11	44 15	45 13	15 14	12 18	12 16	11 17	32 21	49 19	33 20	86 3	50 1	83 2	81 5	81 8	32 6	32 4	33 9	33 7	44 22	44 23	45 24	45 25
15	46 11	46 12	23 10	44 14	45 15	23 13	48 17	48 18	23 16	33 20	32 21	23 19	87 2	85 3	23 1	23 6	23 9	32 4	32 5	33 7	33 8	44 22	44 23	45 24	45 25
16	51 4	52 4	52 4	53 10	53 10	54 10	54 10	55 10	55 10	56 16	56 16	57 16	57 16	58 16	58 16	59 1	59 1								
17	51 5	52 5	52 5	53 11	53 11	54 11	54 11	55 11	55 11	56 17	56 17	57 17	57 17	58 17	58 17	59 2	59 2								
18	46 6	46 6	23 6	47 12	47 12	23 12	48 12	48 12	23 12	49 18	49 18	23 18	50 18	50 18	23 18	23 3	23 3								
19	46 7	46 7	23 7	47 7	47 7	23 7	48 13	48 13	23 13	49 13	49 13	23 13	50 19	50 19	23 19	23 19	23 6								
20	60 8	60 8	63 8	68 8	68 8	63 8	69 14	69 14	72 14	72 14	75 14	75 14	78 20	78 20	79 20	79 20	51 5								
21	60 9	60 9	63 9	68 9	68 9	63 9	69 15	69 15	72 15	72 15	75 15	75 15	78 21	78 21	79 21	79 21	51 4								
22	61 22	61 22	64 22	67 22	67 22	64 22	70 22	70 22	73 22	73 22	76 22	76 22	86 22	84 22	82 22	80 22	80 22								
23	61 23	61 23	64 23	67 23	67 23	64 23	70 23	70 23	73 23	73 23	76 23	76 23	86 23	84 23	82 23	80 23	80 23								
24	62 24	62 24	65 24	66 24	66 24	65 24	71 24	71 24	74 24	74 24	77 24	77 24	87 24	85 24	83 24	81 24	81 24								
25	62 25	62 25	65 25	66 25	66 25	65 25	71 25	71 25	74 25	74 25	77 25	77 25	87 25	85 25	83 25	81 25	81 25								

Figure 7: A covered Latin representation of $K(15, 17, 25)$ by trades

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	3 1	77 2	3 3	4 4	78 5	4 6	5 7	72 8	5 9	6 10	7 11	7 12	8 13	9 14	9 15	10 16	12 17	12 18	12 19	72 20	72 21	77 22	77 23	78 24	78 25
2	3 3	3 1	77 2	4 6	4 4	78 5	5 9	5 7	72 8	6 12	6 10	7 11	8 15	8 13	9 14	10 18	10 16	12 17	12 21	72 19	72 20	77 22	77 23	78 24	78 25
3	79 2	3 3	30 1	80 5	4 6	30 4	72 8	5 9	30 7	6 11	6 12	30 10	8 14	8 15	30 13	36 17	36 18	30 16	30 20	72 21	72 19	79 22	79 23	80 24	80 25
4	11 19	13 20	13 21	39 1	79 2	81 3	73 4	19 5	20 6	14 7	7 8	7 9	15 10	9 11	9 12	17 13	27 14	27 15	10 16	73 17	73 18	79 22	79 23	81 24	81 25
5	11 21	11 19	13 20	82 3	39 1	81 2	73 6	19 4	19 5	14 9	14 7	14 8	15 12	15 10	15 11	17 15	18 13	18 14	10 18	73 16	73 17	82 22	82 23	81 24	81 25
6	37 20	13 21	13 19	80 2	82 3	40 1	73 5	20 6	20 4	34 8	34 9	14 7	35 11	35 12	15 10	45 14	45 15	46 13	37 17	73 18	73 16	82 22	82 23	80 24	80 25
7	1 16	1 17	1 18	2 19	21 20	21 21	23 1	23 2	24 3	74 4	89 5	90 6	47 7	2 8	2 9	50 10	51 11	49 12	46 13	74 14	74 15	89 22	89 23	90 24	90 25
8	1 18	38 16	1 17	2 21	21 19	21 20	41 3	41 1	24 2	92 6	74 4	91 5	28 9	47 7	2 8	49 12	50 10	52 11	52 15	74 13	74 14	91 22	91 23	92 24	92 25
9	31 17	31 18	38 16	32 20	32 21	21 19	33 2	33 3	40 1	93 5	94 6	74 4	28 8	28 9	48 7	51 11	48 12	53 10	53 14	74 15	74 13	93 22	93 23	94 24	94 25
10	11 13	11 14	83 15	25 16	25 17	26 18	26 19	23 20	26 21	42 1	43 2	42 3	84 4	91 5	90 6	28 8	27 7	27 9	75 10	75 11	75 12	83 22	83 23	84 24	84 25
11	22 15	17 13	83 14	16 18	25 16	26 17	26 21	22 19	22 20	24 3	16 1	16 2	94 6	84 4	89 5	28 9	18 8	27 7	75 12	75 10	75 11	83 22	83 23	84 24	84 25
12	22 14	17 15	17 13	16 17	25 18	25 16	23 20	23 21	22 19	24 2	24 3	16 1	93 5	92 6	86 4	85 7	18 9	18 8	75 11	75 12	75 10	85 22	85 23	86 24	86 25
13	54 10	54 11	55 12	55 13	56 14	56 15	57 16	57 17	58 18	58 19	59 20	59 21	76 1	44 2	86 3	29 4	29 5	29 6	85 7	76 8	76 9	85 22	85 23	86 24	86 25
14	60 12	60 10	61 11	61 15	62 13	62 14	63 18	63 16	64 17	64 21	65 19	65 20	88 3	76 1	44 2	29 6	19 4	19 5	87 9	76 7	76 8	87 22	87 23	88 24	88 25
15	66 11	66 12	67 10	67 14	68 15	68 13	69 17	69 18	70 16	70 20	71 21	71 19	43 2	88 3	76 1	29 5	20 6	20 4	87 8	76 9	76 7	87 22	87 23	88 24	88 25
16	37 4	38 4	38 4	39 10	39 10	40 10	41 10	41 10	40 10	42 16	43 16	42 16	43 16	44 16	44 16	45 1	45 1	46 1	46 1						
17	37 5	38 5	38 5	39 11	39 11	40 11	41 11	41 11	40 11	42 17	43 17	42 17	43 17	44 17	44 17	45 2	45 2	46 2	46 2						
18	31 6	31 6	30 6	32 12	32 12	30 12	33 12	33 12	30 12	34 18	34 18	30 18	35 18	35 18	30 18	36 3	36 3	30 3	30 3						
19	31 7	31 7	30 7	32 7	32 7	30 7	33 13	33 13	30 13	34 13	34 13	30 13	35 19	35 19	30 19	36 19	36 19	30 19	30 6						
20	54 8	54 8	55 8	55 8	56 8	56 8	57 14	57 14	58 14	58 14	59 14	59 14	47 20	47 20	48 20	49 20	48 20	49 20	37 5						
21	54 9	54 9	55 9	55 9	56 9	56 9	57 15	57 15	58 15	58 15	59 15	59 15	47 21	47 21	48 21	49 21	48 21	49 21	37 4						
22	60 22	60 22	61 22	61 22	62 22	62 22	63 22	63 22	64 22	64 22	65 22	65 22	93 22	91 22	89 22	50 22	50 22	52 22	52 22						
23	60 23	60 23	61 23	61 23	62 23	62 23	63 23	63 23	64 23	64 23	65 23	65 23	93 23	91 23	89 23	50 23	50 23	52 23	52 23						
24	66 24	66 24	67 24	67 24	68 24	68 24	69 24	69 24	70 24	70 24	71 24	71 24	94 24	92 24	90 24	51 24	51 24	53 24	53 24						
25	66 25	66 25	67 25	67 25	68 25	68 25	69 25	69 25	70 25	70 25	71 25	71 25	94 25	92 25	90 25	51 25	51 25	53 25	53 25						

Figure 8: A covered Latin representation of $K(15, 19, 25)$ by trades

References

[1] E. J. Billington and N. J. Cavenagh, Decomposing complete tripartite graphs into 5-cycles when the partite sets have similar size, *Aequationes Math.* 82(3) (2011), 277–289.

[2] N. J. Cavenagh, Further decompositions of complete tripartite graphs into 5-cycles, *Discrete Math.* 256(1-2) (2002), 55–81.

[3] N. J. Cavenagh and E. J. Billington, On decomposing complete tripartite graphs into 5-cycles, *Australas. J. Combin.* 22 (2000), 41–62.

[4] E. S. Mahmoodian and M. Mirzakhani, Decomposition of complete tripartite graphs into 5-cycles, In *Combinatorics advances (Tehran, 1994)*, vol. 329 of *Math. Appl.*, pp. 235–241. Kluwer Acad. Publ., Dordrecht, 1995.

(Received 16 Mar 2012; revised 3 Aug 2012)