AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 279–287

On (a, d)-distance antimagic graphs

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Abstract

Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. For any vertex $v \in V$, the neighbor sum $\sum_{u \in N(v)} f(u)$ is called the weight of the vertex v and is denoted by $w(v)$. If $w(v) = k$, (a constant) for all $v \in V$, then f is called a distance magic labeling with magic constant k. If the set of vertex weights forms an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (n - 1)d\}$, then f is called an (a, d)-distance antimagic labeling and a graph which admits such a labeling is called an (a, d)-distance antimagic graph. In this paper we present several results on (a, d)-distance antimagic graphs.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

The concept of distance magic labeling of a graph has been motivated by the construction of magic squares. A magic square of order n is an $n \times n$ array whose entries are an arrangement of the integers $\{1, 2, \ldots, n^2\}$, in which all elements in any row, any column, the main diagonal or the main back-diagonal, add to the same sum r. Now if we take a complete n partite graph with parts $V_1, V_2, \ldots, V_n$ with $|V_i| = n$, $1 \leq i \leq n$, and label the vertices of V_i with the integers in the i^{th} row of the magic square, we find that the sum of the labels of all the vertices in the neighborhood of each vertex is the same and is equal to $r(n - 1)$. This observation is a motivation for the concept of distance magic labelings, which was first introduced by Vilfred [8] under the name sigma labelings.

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Definition 1.1. [6] Distance magic labeling of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k such that $\sum_{u \in N(v)} f(u) = k$ for every $v \in V$. The constant k is called the magic constant of the labeling f.

The sum $\sum_{u \in N(v)} f(u)$ is called the weight of the vertex v and is denoted by $w(v)$.

Acharya et al. [1] further studied this concept under the name of neighbourhood magic graphs. The same concept was introduced by Miller et al. [6] under the name 1-vertex magic vertex labeling. Sugeng et al. [7] introduced the term distance magic labeling for this concept and obtained further results on this concept. The existence of regular distance magic graphs and fair incomplete tournaments have been studied by Fronček [5]. For a recent survey and open problems on distance magic graphs we refer to Arumugam et al. [2].

In this paper we introduce the concept of an (a, d)-distance antimagic labeling, which arises in a natural way from distance magic labelings. Throughout this paper we assume that G is a graph without isolated vertices. If $f : V \to \{1, 2, \ldots, n\}$ is a bijection, then we identify the vertex v with its label $f(v)$. Also in all the figures we represent the vertex label in the usual font and the weight of a vertex in bold font.

2 (a, d)-distance antimagic graphs

Let G be a distance magic graph of order n with labeling f and magic constant k. Then $\sum\limits_{u \in N_{G^c}(v)} f(u) = \frac{n(n+1)}{2} - k - f(v)$, and hence the set of all vertex weights in the complement G^c is $\{\frac{n(n+1)}{2} - k - i : 1 \le i \le n\}$, which is an arithmetic progression with first term $a = \frac{n(n+1)}{2} - k - n$ and common difference $d = 1$. This observation motivates the following definition.

Definition 2.1. A graph G is said to be (a, d)-distance antimagic if there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ such that the set of all vertex weights is $\{a, a + d, a + 2d, \ldots, a + (n-1)d\}$, where a and d are fixed integers with $d \ge 0$ and any graph which admits such a labeling is called an (a, d)-distance antimagic graph.

Thus if G is distance magic, then G^c is $(a, 1)$-distance antimagic. The converse is not true. For example, the graph G given in Figure 1 has an $(a, 1)$-distance antimagic labeling. However its complement is the cycle C_6, which is not distance magic [3].

Observation 2.2. If G is an (a, d)-distance antimagic graph of order n with $d > 0$, then $N(x) \neq N(y)$ for any two distinct vertices $x, y \in V(G)$.

Lemma 2.3. *If G is an (a, d)-distance antimagic graph of order n, then*

$$d \le \frac{2n\Delta - \Delta(\Delta - 1) - \delta(\delta + 1)}{2(n-1)}.$$

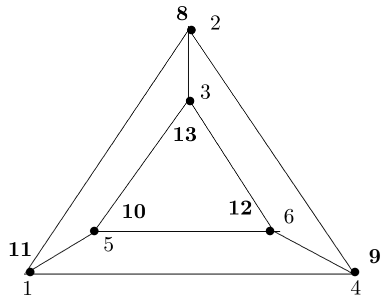

Figure 1: $(8,1)$-distance antimagic labeling of G

Proof. Since the minimum possible vertex weight is at least $(1+2+\cdots+\delta)$, it follows that $a \geq \frac{\delta(\delta+1)}{2}$. Also since the maximum possible weight is at most $n+(n-1)+\cdots+(n-\Delta+1)$, we have $a+(n-1)d \leq n\Delta - \frac{\Delta(\Delta-1)}{2}$. Hence $d \leq \frac{2n\Delta-\Delta(\Delta-1)-\delta(\delta+1)}{2(n-1)}$. □

Corollary 2.4. *If G is an r-regular (a,d)-distance antimagic graph of order n with $r \geq 2$, then $d < r$.*

Corollary 2.5. *If G is a 3-regular (a,d)-distance antimagic graph of order n, then $a = n+2$ and $d = 1$.*

Proof. It follows from Corollary 2.4 that $d = 1$ or 2. Further $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = 3(1+2+\cdots+n)$. Thus $\frac{n}{2}(2a+(n-1)d) = \frac{3n(n+1)}{2}$. If $d = 2$, then $a = \frac{n+5}{2}$. As G is 3-regular, n is even and hence a is not an integer, which is a contradiction. Hence $d = 1$ and $a = n+2$ □

Corollary 2.6. *If G is a 2-regular (a,d)-distance antimagic graph of order n, then $a = \frac{n+3}{2}$ and $d = 1$.*

Proof. It follows from Corollary 2.4 that $d = 1$. Also $a+(a+1)+(a+2)+\cdots+(a+(n-1)) = 2(1+2+\cdots+n)$. Thus $\frac{n}{2}(2a+(n-1)) = n(n+1)$. Hence $a = \frac{n+3}{2}$. □

Theorem 2.7. *The cycle C_n is (a,d)-distance antimagic if and only if n is odd and $d = 1$.*

Proof. Let $C_n = (v_1, v_2, \ldots v_n, v_1)$ be (a,d)-distance antimagic. Then by Corollary 2.6, $a = \frac{n+3}{2}$, and hence n is odd. Also it follows from Corollary 2.4 that $d = 1$. Conversely, suppose n is odd. We now produce an $(a,1)$-distance antimagic labeling f of C_n.

Case (i) $n \equiv 3 \pmod 4$.

Let $n = 4r-1$, $r \geq 1$. Since r and n are relatively prime, the function $h : V(G) \to \{0,1,\ldots,n-1\}$ defined by $h(v_i) = r(i-1)+1 \pmod n$, is a bijection. Now $f : V(G) \to \{1,2,\ldots,n\}$ defined by

$$f(v_i) = \begin{cases} n & \text{if } h(v_i) = 0 \\ h(v_i) & \text{if } h(v_i) \neq 0 \end{cases}$$

is also a bijection. Also the vertex weights are given by

$$w(v_i) = \begin{cases} 2r(i-1)+2 \pmod{n} & \text{if } i \text{ is even and } i \neq n-3, n-1 \\ 2r(i-1)+2 \pmod{n} + n & \text{if } i \text{ is odd and } i \neq n. \end{cases}$$

$w(v_{n-3}) = n, w(v_{n-1}) = n+1$, and $w(v_n) = \frac{n^2-1}{2} + 2 \pmod{n}$.

It is clear that the weights of the vertices form an arithmetic progression with first term $a = \frac{n+3}{2}$ and common difference $d = 1$.

Case (ii) $n \equiv 1 \pmod{4}$.

Let $n = 4r+1$, $r \geq 1$. First we define $g : V(G) \to \{0, 1, \ldots, n-1\}$ by $g(v_i) = r(i-1)+1 \pmod{n}$. Now define $f : V(G) \to \{1, 2, \ldots, n\}$ by

$$f(v_i) = \begin{cases} n & \text{if } g(v_i) = 0 \\ g(v_i) & \text{if } g(v_i) \neq 0. \end{cases}$$

Then the vertex weights are given by

$$w(v_i) = \begin{cases} n+2-(\frac{i-1}{2}) & \text{if } i \text{ is odd} \\ \frac{n-1}{2}(i-1)+2-(\frac{i-4}{2})n & \text{if } i \text{ is even and } i \neq 2. \end{cases}$$

and $w(v_2) = \frac{n+3}{2}$.

The weights of the vertices form an arithmetic progression with first term $a = \frac{n+3}{2}$ and common difference $d = 1$. □

Example 2.8. For the cycles C_7 and C_9, $(a, 1)$-distance antimagic labelings are given in Figure 2.

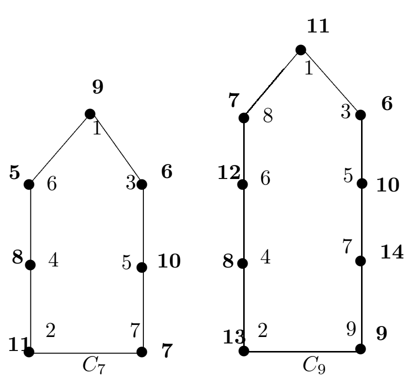

Figure 2: A $(5, 1)$-distance antimagic labeling of C_7 and a $(6, 1)$-distance antimagic labeling of C_9

Theorem 2.7 gives a characterization of connected 2-regular (a, d)-distance antimagic graphs. However there are disconnected 2-regular graphs which admit an (a, d)-distance antimagic labeling. A $(9, 1)$-distance antimagic labeling of $3C_5$ is given in Figure 3.

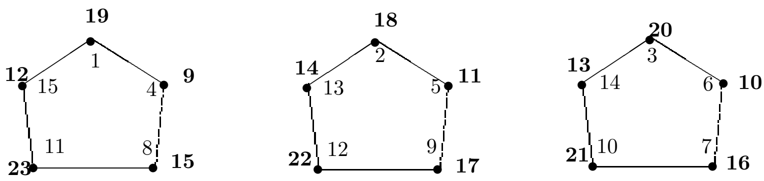

Figure 3: A $(9,1)$-distance antimagic labeling of $3C_5$

Hence the following problem naturally arises.

Problem 2.9. Characterize disconnected, 2-regular (a,d)-distance antimagic graphs.

Proposition 2.10. If $a, d \geq 2$, then there exists no (a,d)-distance antimagic labeling for the path P_n and for the cycle C_n.

Proof. Since $\Delta = 2$, the maximum possible vertex weight is $2n-1$. If P_n (C_n) admits an (a,d)-distance antimagic labeling for $a, d \geq 2$, then the maximum vertex weight is $a + (n-1)d \geq 2n$, a contradiction. □

Proposition 2.11. *There is no $(1,d)$-distance antimagic labeling for P_n, when $n \geq 3$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. Suppose P_n admits a $(1,d)$-distance antimagic labeling f. It follows from Lemma 2.3 that $d \leq 2$.

Case (i) $d = 1$.

Since $a = d = 1$, we may assume that $f(v_2) = 1$ and $f(v_{n-1}) = 2$, so that $w(v_1) = 1$ and $w(v_n) = 2$. Now the vertex v_i with weight 3 must be adjacent to both v_2 and v_{n-1} and hence $n = 5$ and $w(v_3) = 3$. Now there is no possibility of getting 4 as a vertex weight.

Case (ii) $d = 2$.

Since the vertex weights are $1, 3, \ldots, 2n-1$, we may assume that $f(v_2) = 1$ and $f(v_{n-1}) = x$ where x is an odd integer and $x \in \{2, 3, \ldots, n\}$. Now $\sum_{i=1}^{n} w(v_i) = 1 + 3 + \cdots + 2n - 1 = n^2$. Also $\sum_{i=1}^{n} w(v_i) = \sum_{i=1}^{n} \deg(v_i) f(v_i) = 2\sum_{i=1}^{n} f(v_i) - (f(v_1) + f(v_n)) = n^2 + n - (f(v_1) + f(v_n))$. Hence $f(v_1) + f(v_n) = n$. Since $f(v_2) = 1$, to get 3 as a vertex weight, we must have $f(v_4) = 2$. Now to get 5 as a vertex weight, we must have $f(v_6) = 3$. Continuing this process we get $f(v_{2i}) = i$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Hence $f(v_1) > \frac{n}{2}$ and $f(v_n) \geq \frac{n}{2}$. Thus $f(v_1) + f(v_n) > n$, which is a contradiction.

Hence there is no $(1,d)$-distance antimagic labeling for P_n, when $n \geq 3$. □

Problem 2.12. Does there exist an $(a,1)$-distance antimagic labeling for the path P_n with $a \geq 2$?

Theorem 2.13. *A graph G is $(1,1)$-distance antimagic if and only if every component of G is K_2.*

Proof. If every component of G is K_2, then any bijection $f : V(G) \to \{1,2,\ldots,n\}$ gives a $(1,1)$-distance antimagic labeling of G. Conversely, suppose G is $(1,1)$-distance antimagic and let f be a $(1,1)$-distance antimagic labeling of G. Then $\sum_{i=1}^{n} \deg(v_i) f(v_i) = \sum_{i=1}^{n} w(v_i) = \frac{n(n+1)}{2}$. Hence it follows that $\deg(v_i) = 1$ for each v_i, so that every component of G is K_2. □

Definition 2.14. The Cartesian product of G and H, written $G\Box H$, is the graph with vertex set $V(G\Box H) = \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}$ and edge set $E(G\Box H) = \{(u,v)(u',v') : u = u' \text{ and } vv' \in E(H) \text{ or } v = v' \text{ and } uu' \in E(G)\}$.

Theorem 2.15. *The graph $C_n\Box K_2$ is $(n+2,1)$-distance antimagic.*

Proof. Let $(v_1, v_2, \ldots, v_n, v_1)$ and $(u_1, u_2, \ldots, u_n, u_1)$ be the two disjoint cycles in $C_n\Box K_2$ with $v_iu_i \in E(C_n\Box K_2)$.

Case (i) $n = 2k+1$.

Define $f : V(C_n\Box K_2) \to \{1,2,\ldots,4k+2\}$ by

$$f(v_i) = \begin{cases} i & \text{if } 1 \le i \le 2k \\ 2k+2 & \text{if } i = 2k+1 \end{cases}$$

and

$$f(u_i) = \begin{cases} 4k+2-i & \text{if } 1 \le i \le 2k-1 \\ 2k+1 & \text{if } i = 2k \\ 4k+2 & \text{if } i = 2k+1. \end{cases}$$

Clearly f is a bijection. Now,

$$w(v_i) = \begin{cases} 6k+5 & \text{if } i = 1 \\ 4k+2+i & \text{if } 2 \le i \le 2k-1 \\ 6k+2 & \text{if } i = 2k \\ 6k+3 & \text{if } i = 2k+1. \end{cases}$$

$$w(u_i) = \begin{cases} 8k+3 & \text{if } i = 1 \\ 8k+4-i & \text{if } 2 \le i \le 2k-2 \\ 6k+4 & \text{if } i = 2k-1 \\ 8k+5 & \text{if } i = 2k \\ 8k+4 & \text{if } i = 2k+1. \end{cases}$$

Hence the vertex weights form an arithmetic progression with $a = n+2$ and $d = 1$.

Case (ii) $n = 2k$.

Define $f : V(C_n \square K_2) \rightarrow \{1, 2, \ldots, 4k\}$ by

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4k-2 & \text{if } i = 2 \\ i-1 & \text{if } 3 \leq i \leq 2k \end{cases}$$

and

$$f(u_i) = \begin{cases} 2k & \text{if } i = 1 \\ 4k-1 & \text{if } i = 2 \\ 2k+1 & \text{if } i = 3 \\ 4k+1-i & \text{if } 4 \leq i \leq 2k-1 \\ 4k & \text{if } i = 2k. \end{cases}$$

Clearly f is a bijection. Now,

$$w(v_i) = \begin{cases} 8k-3 & \text{if } i = 1 \\ 4k+2 & \text{if } i = 2 \\ 6k+2 & \text{if } i = 3 \\ 4k-1+i & \text{if } 4 \leq i \leq 2k-1 \\ 6k-1 & \text{if } i = 2k \end{cases}$$

and

$$w(u_i) = \begin{cases} 8k & \text{if } i = 1 \\ 8k-1 & \text{if } i = 2 \\ 8k-2 & \text{if } i = 3 \\ 6k & \text{if } i = 4 \\ 8k+1-i & \text{if } 5 \leq i \leq 2k-2 \\ 8k+1 & \text{if } i = 2k-1 \\ 6k+1 & \text{if } i = 2k. \end{cases}$$

Hence the vertex weights form an arithmetic progression with $a = n + 2$ and $d = 1$. □

Example 2.16. For the graphs $C_7 \square K_2$ and $C_8 \square K_2$, $(a, 1)$-distance antimagic labelings are given in Figures 4(a) and 4(b).

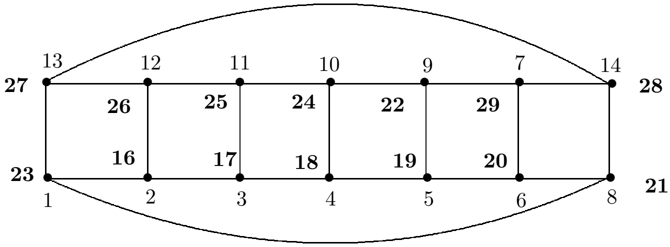

Figure 4(a): A (16, 1)-distance antimagic labeling of $C_7 \square K_2$

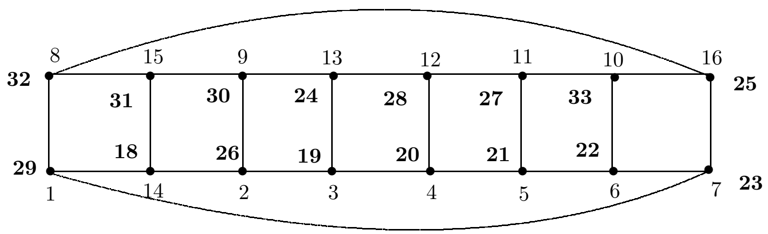

Figure 4(b): A (18,1)-distance antimagic labeling of $C_8 \Box K_2$

Definition 2.17. The graph obtained from the cycle $C_{2n} = (v_1, v_2, \ldots, v_{2n}, v_1)$ by adding a perfect matching consisting of the edges v_1v_{n+1} and v_iv_{2n+2-i}, where $2 \leq i \leq n$, is called an M-augmentation of C_n and is denoted by C_{2n}^+.

Theorem 2.18. *The graph C_{2n}^+ is $(2n+2, 1)$-distance antimagic.*

Proof. Define $f : V(C_{2n}^+) \to \{1, 2, \ldots, 2n\}$ by $f(v_i) = i$. Then

$$w(v_i) = \begin{cases} 3n+3 & \text{if } i = 1 \\ 2n+3 & \text{if } i = n+1 \\ 2n+2 & \text{if } i = 2n \\ 2n+2+i & \text{if } 2 \leq i \leq 2n-1, i \neq n+1. \end{cases}$$

Clearly the vertex weights form an arithmetic progression with $a = 2n+2$ and $d = 1$. □

Example 2.19. A $(10, 1)$-distance antimagic labeling of C_8^+ for the case $n = 4$ is given in Figure 5.

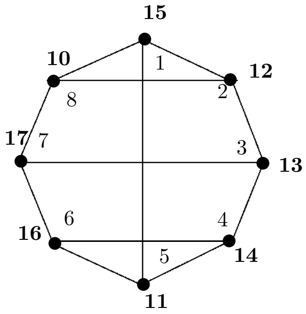

Figure 5: A $(10, 1)$-distance antimagic labeling of C_8^+

3 Conclusion and scope

In this paper we have introduced the concept of an (a, d)-distance antimagic labeling, which arises in a natural way. We have obtained a few basic results on graphs which admit such a labeling. This concept is wide open for further investigation. One can investigate whether standard families of graphs are (a, d)-distance antimagic. Finding necessary or sufficient condition for a graph to admit an (a, d)-distance antimagic labeling and study of (a, d)-distance antimagic graphs with reference to various graph operations are other possible directions for future research.

Acknowledgements

The authors are thankful to the Department of Science and Technology, New Delhi, Government of India for its support through n-CARDMATH Project SR/S4/MS:427/07. The authors are thankful to the referees for their helpful suggestions.

References

[1] B.D. Acharya, S.B. Rao, T. Singh and V. Parameswaran, Neighborhood magic graphs, presented at: *Nat. Conf. Graph Theory, Combin. and Algorithms*, 2004.

[2] S. Arumugam, D. Frončel and N. Kamatchi, Distance Magic Graphs—A Survey, *J. Indones. Math. Soc.* Special Ed. (2011), 11–26.

[3] S. Arumugam, N. Kamatchi and P. Kovář, Distance magic graphs, *Util. Math.* (to appear).

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, CRC, 4th ed., 2005.

[5] D. Frončel, P. Kovář and T. Kovářová, Fair incomplete tournaments, *Bull. Inst. Combin. Appl.* **48** (2006), 31–33.

[6] M. Miller, C. Rodger and R. Simanjuntak, Distance magic labelings of graphs, *Australas. J. Combin.* **28** (2003), 305–315.

[7] K.A. Sugeng, D. Frončel, M. Miller, J. Ryan and J. Walker, On distance magic labeling of graphs, *J. Combin. Math. Combin. Comput.* **71** (2009), 39–48.

[8] V. Vilfred, Σ*-labelled graph and Circulant Graphs*, Ph.D. Thesis, University of Kerala, Trivandrum, India, 1994.

(Received 18 Feb 2012; revised 28 May 2012)