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Small radial Moore graphs of radius 3

Martin Knor

Slovak University of Technology in Bratislava
Faculty of Civil Engineering
Department of Mathematics
Radlinského 11, 813 68, Bratislava
Slovakia
knor@math.sk

Abstract

A t-regular graph of radius s is *radial Moore* if it has diameter at most $s+1$ and $1+t+t(t-1)+\ldots+t(t-1)^{s-1}$ vertices. We construct radial Moore graphs of radius 3 and degrees $t = 3,5,7,9,10,\ldots,30$ with at least $t+1$ central vertices and at most $t+2$ orbits under the automorphism group.

1 Introduction

It is well-known that a graph of diameter s and degree not exceeding t can have at most

$$M_{s,t} = 1+t+t(t-1)+t(t-1)^2+\ldots+t(t-1)^{s-1}$$

vertices. The value $M_{s,t}$ is called *Moore bound* and regular graphs of degree t and diameter s with $M_{s,t}$ vertices are called *Moore graphs*.

Moore graphs would be ideal models (in terms of largest number of vertices) of interconnection networks if no other restrictions are considered. The problem is that they are extremely rare and exist only for diameter 1, or degree 2, or for diameter 2 and degree 2, 3, 7 and possibly also for degree 57; see [6] for diameters 2 and 3 and [1] and [4] for the general case.

To approach Moore bound for the remaining pairs (s,t), relaxations are needed for at least one of the three parameters involved: the degree, the diameter, and the number of vertices. The most frequently considered relaxation is on the number of vertices, leading to the well-known degree-diameter problem [10]. Let us recall that this problem has been motivated by applications in network design to maximize the number of nodes in a network, with limitations on the number t of links attached to each node (degree) and with the requirement that communication between any two nodes is delayed by processing in at most $s-1$ intermittent nodes. Another type of relaxation is to allow a few vertices whose degrees slightly exceed t, obtaining "near Moore graphs"; see [9]. However, one may relax also the requirement that the eccentricities of all vertices are s. In 1993 Znám in a personal communication suggested to study regular graphs of degree t with $M_{s,t}$ vertices, some of which have eccentricity s and the other have eccentricity at most $s+1$. Since such graphs have radius s (and diameter at most $s+1$), they are called *radial Moore graphs* (in earlier papers the notion *radially Moore graphs* was used as well).

Since the degree-diameter problem has also been widely considered for digraphs, we make a corresponding short digression in this place. A digraph with (in- and out-) degree at most t and diameter s can have at most

$$MD_{s,t} = 1 + t + t^2 + \ldots + t^s$$

vertices, and if it has $MD_{s,t}$ vertices then it is called *Moore digraph*. Also Moore digraphs are rare. They exist only for degree 1 or diameter 1; see [2]. A digraph of degree t, radius s (radius being $\min\{e(v);\ v \in V(D)\}$ where $e(v) = \max\{\mathrm{dist}(u,v), \mathrm{dist}(v,u);\ u \in V(D)\}$), diameter at most $s+1$ with $MD_{s,t}$ vertices is a *radial Moore digraph*. In [7] we proved that radial Moore digraphs exist for every pair (s,t), $s,t \geq 1$.

The situation for graphs appears to be much more complicated than the one for digraphs. Obviously, complete graphs are radial Moore graphs of radius 1. It is easy to construct radial Moore graphs of radius 2, and enumeration of all these graphs is considered in [3]. However, already the case of radius 3 seems to be complicated. In [8] we found radial Moore graphs of radius 3 and degree 3, 4, 5, 6 and 7. Here we extend the class of radial Moore graphs of radius 3. We present a construction, using which we found radial Moore graphs for odd degrees $3, 5, \ldots, 29$ and for even degrees $10, 12, \ldots, 30$. Moreover, for degrees $3, 5, 7, 9, 10, \ldots, 18$ we know all radial Moore graphs which are produced by our construction. Using a generalization of this construction we found radial Moore graphs for degrees 6 and 8, so that now we have radial Moore graphs of radius 3 for degrees $3, 4, \ldots, 30$. With the help of generalized de Bruijn graphs, recently Exoo et al proved that if $t \geq 22$ then there exists a radial Moore graph of radius 3 and degree t; see [5]. Hence, we have the following statement:

Theorem 1.1 *For every $t \geq 3$ there exists a radial Moore graph of radius* 3 *and degree t.*

As regards higher radii, we know that there exist radial Moore graphs of radius 4 and degrees 3, 4 and 5, and also radial Moore graphs of radius 5 and degree 3; see [5].

If a radial Moore graph has properties close to ideal properties of a Moore graph, then it should have as many central vertices as possible. (For other measures of how well a radial Moore graph approximates a Moore graph; see [3].) Radial Moore graphs which are not Moore graphs cannot be vertex transitive. For practical applications, however, it would be convenient to have graphs having a very small number of orbits under the automorphism group. In this paper we present a construction which in some cases produces radial Moore graphs of radius 3 and degree t having $M_{3,t} = t^3 - t^2 + t + 1$ vertices. Our graphs have at least $t+1$ central vertices and at most $t+2$ orbits under the automorphism group.

In the next section we describe the construction and summarize the results. We use standard notation. A graph with vertex set V and edge set E is denoted by (V, E). An edge from u to v is denoted by $[u, v]$. Eccentricity of a vertex $v \in V$ is $e(v) = \max_{u \in V} \operatorname{dist}(v, u)$, where by $\operatorname{dist}(v, u)$ we denote the distance from v to u. Then $\max_{v \in V} e(v)$ and $\min_{v \in V} e(v)$ are the diameter and radius, respectively.

2 Results

We start with a construction which, in some cases, yields radial Moore graphs of diameter 3 and degree t. Denote by $\mathbb{Z}_{t-1}^*$ the set of strings (of finite length), all elements of which are from $\mathbb{Z}_{t-1}$. The length of a string β will be denoted by $l(\beta)$. Then the vertex set is

$$V = \{\emptyset\} \cup \{a\beta;\ a \in \mathbb{Z}_t, \beta \in \mathbb{Z}_{t-1}^*, 0 \le l(\beta) \le 2\}.$$

Obviously, $|V| = 1 + t(1 + (t-1) + (t-1)^2) = M_{3,t}$. The edge set consists of E_1 and E_2.

$$\begin{aligned} E_1 = {} & \{[\emptyset, a];\ a \in \mathbb{Z}_t\} \\ & \cup \{[a\beta, a\beta b];\ a \in \mathbb{Z}_t, \beta \in \mathbb{Z}_{t-1}^*, 0 \le l(\beta) \le 1, b \in \mathbb{Z}_{t-1}\}. \end{aligned}$$

The edges of E_1 form a spanning tree of the graph; see Figure 1 for the case $t = 4$. Due to these edges, the radius is 3 and $\emptyset$ is the central vertex. Observe that all vertices $a\beta$, $l(\beta) = 2$, are incident with one edge of E_1 while the other vertices are incident with t edges of E_1. Hence, it remains to define edges connecting $a\beta$ and $a'\beta'$, where $l(\beta) = l(\beta') = 2$. Let $d = \lfloor t/2 \rfloor$. We set

$$E_2 = \cup_{i=1}^{d} \{[a\beta, (a+i)p_i(\beta)];\ a \in \mathbb{Z}_t, \beta \in \mathbb{Z}_{t-1}^*, l(\beta) = 2\},$$

where the addition is in $\mathbb{Z}_t$ and p_i is a permutation of $(t-1)^2$ elements b_1b_2, $b_1, b_2 \in \mathbb{Z}_{t-1}$. The permutations $p_1, p_2, \ldots, p_{d-1}$ can be arbitrary, so as the permutation p_d in the case when t is odd. If t is even, then p_d must be an involution since $a + d + d = a$ in $\mathbb{Z}_t$, and so the edge $[a\beta, (a+d)p_d(\beta)]$ must be the same as $[(a+d)p_d(\beta), ap_d(p_d(\beta))]$. Then all vertices $a\beta$, $l(\beta) = 2$, are incident with $t - 1$ edges of E_2, adjacent vertices being $(a+i)p_i(\beta)$ and $(a-i)p_i^{-1}(\beta)$, $1 \le i \le d$.

As already mentioned, $e(\emptyset) = 3$. But the center of our graph is larger.

Lemma 2.1 *Let $a \in \mathbb{Z}_t$. In graph $(V, E_1 \cup E_2)$ we have $e(a) = 3$.*

Proof: Due to the edges of E_1 we have $\operatorname{dist}(a, \emptyset) = 1$, $\operatorname{dist}(a, a'\beta') \le 3$ for every $a' \in \mathbb{Z}_t$, $\beta' \in \mathbb{Z}_{t-1}^*$, $0 \le l(\beta) \le 1$, and $\operatorname{dist}(a, a\beta) \le 2$ for $\beta \in \mathbb{Z}_{t-1}^*$, $l(\beta) \le 2$. Since the edges of E_2 are defined by permutations, for every $a' \in \mathbb{Z}_t$, $a' \ne a$, the edges between $a\beta$'s and $a'\beta'$'s, $l(\beta) = l(\beta') = 2$, form a perfect matching. Thus, for every a' and β', where $a' \in \mathbb{Z}_t$, $\beta' \in \mathbb{Z}_{t-1}^*$ and $l(\beta) = 2$, there is β such that $[a\beta, a'\beta'] \in E_2$, where $\beta \in \mathbb{Z}_{t-1}^*$ and $l(\beta) = 2$. Since $\operatorname{dist}(a, a\beta) = 2$, we have $\operatorname{dist}(a, a'\beta') = 3$, and so $\operatorname{dist}(a, v) \le 3$ for every $v \in V$. Consequently $e(a) = 3$. □

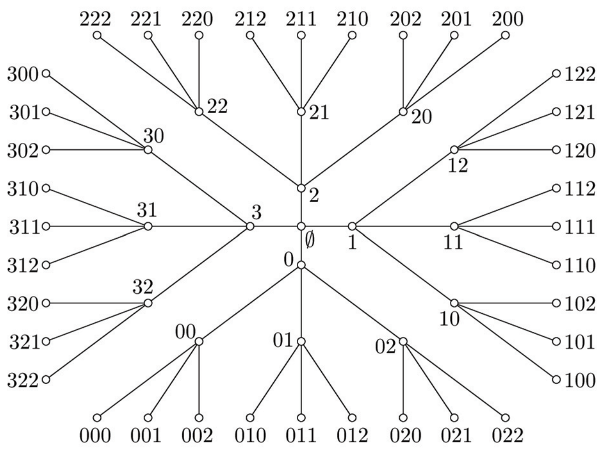

Figure 1: Edges of E_1 forming a spanning tree.

Corollary 2.2 *In the graph* $(V, E_1 \cup E_2)$ *there are at least* $t + 1$ *vertices with eccentricity* 3.

Since the eccentricities of adjacent vertices differ at most by 1, we have $e(ab) \leq 4$ for every $a \in \mathbb{Z}_t$ and $b \in \mathbb{Z}_{t-1}$. Hence, we have

Corollary 2.3 *The graph* $(V, E_1 \cup E_2)$ *is radial Moore if and only if* $dist(a\beta, a'\beta') \leq 4$ *for every* $a, a' \in \mathbb{Z}_t$ *and* $\beta, \beta' \in \mathbb{Z}^*_{t-1}$, $l(\beta) = l(\beta') = 2$.

Although in different notation, this general construction of graph $(V, E_1 \cup E_2)$ was used in [8] to find radial Moore graphs of radius 3 and degrees $3, 4, \ldots, 7$. Now we choose special permutations, which allow us to find radial Moore graphs of higher degrees.

Denote

$$S(b_1, i) = \{c_1;\ p_i(b_1b_2) = c_1c_2,\ b_2 \in \mathbb{Z}_{t-1}\}.$$

Our idea is to choose the permutations p_i so that for every b_1 and i, $b_1 \in \mathbb{Z}_{t-1}$ and $1 \leq i \leq d$, we have $S(b_1, i) = \mathbb{Z}_{t-1}$. Such a property means that if two pendant vertices of the tree (V, E_1) are close to each other, then by edges of E_2 they have neighbours which are far apart in (V, E_1). A natural candidate for this property is the involution $p(b_1b_2) = b_2b_1$. Unfortunately, if we choose all p_i's so that $p_i(b_1b_2) = b_2b_1$, then the resulting graph is not radial Moore. Hence, we "shift" the result a bit. Choose $c \in \mathbb{Z}_{t-1}$ and set $p_i(b_1b_2) = (b_2{+}c)b_1$. Then $S(b_1, i) = \mathbb{Z}_{t-1}$ as required, see Figure 2 for $c = 0$ and 1 in the case $t = 4$.

Denote by $G(t; c_1, c_2, \ldots, c_d)$ a graph of radius 3 and degree t on $M_{3,t}$ vertices described above, $d = \lfloor t/2 \rfloor$, in which $p_i(b_1b_2) = (b_2{+}c_i)b_1$, the addition being in $\mathbb{Z}_{t-1}$. (Recall that p_d must be an involution if t is even.) Although $G(t; c_1, c_2, \ldots, c_d)$

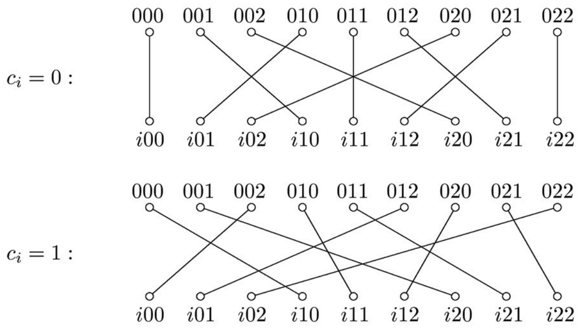

Figure 2: Permutations p_i, where $p_i(b_1b_2) = (b_2{+}c_i)b_2$, for $c_i = 0$ and 1.

has $t^3 - t^2 + t + 1$ vertices, its number of orbits under the automorphism group is small.

Proposition 2.4 *The graph $G(t; c_1, c_2, \ldots, c_d)$ has at most $t + 2$ orbits under the automorphism group.*

Proof: We use two automorphisms. Let $v \in V$. Define

$$\sigma(v) = \begin{cases} \emptyset & \text{if } v = \emptyset, \\ (a{+}1)\beta & \text{if } v = a\beta, \quad a \in \mathbb{Z}_t,\ \beta \in \mathbb{Z}_{t-1}^*,\ 0 \le l(\beta) \le 2, \end{cases}$$

the addition being in $\mathbb{Z}_t$. Since

$$[\sigma(\emptyset), \sigma(a)] = [\emptyset, a{+}1] \in E_1,$$
$$[\sigma(a\beta), \sigma(a\beta b)] = [(a{+}1)\beta, (a{+}1)\beta b] \in E_1 \quad \text{where } 0 \le l(\beta) \le 1,$$
$$[\sigma(a\beta), \sigma((a{+}i)p_i(\beta))] = [(a{+}1)\beta, (a{+}1{+}i)p_i(\beta)] \in E_2 \quad \text{where } l(\beta) = 2,$$

σ maps edges to edges, i.e., it is an automorphism. Analogously define

$$\rho(v) = \begin{cases} \emptyset & \text{if } v = \emptyset, \\ a & \text{if } v = a, \quad a \in \mathbb{Z}_t, \\ a(b_1{+}1) & \text{if } v = ab_1, \quad a \in \mathbb{Z}_t,\ b_1 \in \mathbb{Z}_{t-1}, \\ a(b_1{+}1)(b_2{+}1) & \text{if } v = ab_1b_2, \quad a \in \mathbb{Z}_t,\ b_1, b_2 \in \mathbb{Z}_{t-1}, \end{cases}$$

the addition being in $\mathbb{Z}_{t-1}$. Since

$$[\rho(\emptyset), \rho(a)] = [\emptyset, a] \in E_1,$$
$$[\rho(a), \rho(ab_1)] = [a, a(b_1{+}1)] \in E_1,$$
$$[\rho(ab_1), \rho(ab_1b_2)] = [a(b_1{+}1), a(b_1{+}1)(b_2{+}1)] \in E_1,$$
$$[\rho(ab_1b_2), \rho(a(b_2{+}c_i)b_1)] = [a(b_1{+}1)(b_2{+}1), a(b_2{+}1{+}c_i)(b_1{+}1)] \in E_2,$$

ρ is an automorphism of $G(c_1, c_2, \ldots, c_d)$.

The automorphism σ decomposes V into $1 + 1 + (t-1) + (t-1)^2$ orbits represented by $\emptyset$, 0, $0b_1$ and $0b_1b_2$, where $b_1, b_2 \in \mathbb{Z}_{t-1}$. Consequently, ρ merges the vertices $00, 01, \ldots, 0(t-2)$ as well as $00b_2, 01(b_2+1), \ldots, 0(t-2)(b_2+t-2)$. Thus, under σ and ρ we have $1 + 1 + 1 + (t-1) = t + 2$ orbits represented by $\emptyset$, 0, 00 and $00b_2$, where $b_2 \in \mathbb{Z}_{t-1}$. □

Observe that in $(V, E_1 \cup E_2)$, where V, E_1 and E_2 are defined as at the beginning of this section, σ (defined as in the previous proof) is an automorphism for arbitrary choice of p_i. By Corollary 2.3, it then suffices to check the eccentricities of vertices $0b_1b_2$, $b_1, b_2 \in \mathbb{Z}_{t-1}$. However, by Corollary 2.3 and Proposition 2.4, in $G(t; c_1, c_2, \ldots, c_d)$ it suffices to check the eccentricities of $00b_2$, $b_2 \in \mathbb{Z}_{t-1}$. Using this observation we show that the graph $G(5; 1, 3)$ of degree 5 on 106 vertices is radial Moore.

Proposition 2.5 *$G(5; 1, 3)$ is a radial Moore graph.*

Proof: Let v be a vertex such that $v = xy_1y_2$, where $x \in \mathbb{Z}_5$ and $y_1, y_2 \in \mathbb{Z}_4$. Denote by $N_2(v)$ the set of vertices which are at distance at most 2 from v. In Table 1 we have the vertices $u \in N_2(v)$ which are at distance 3 from $\emptyset$. The columns correspond to the first coordinate of u (that from $\mathbb{Z}_5$) and the entries correspond to other two coordinates of u (those from $\mathbb{Z}_4$). Observe that the first entry in columns $x+1, x+2, \ldots, x+4$ corresponds to a vertex adjacent to v, so that if z_1z_2 is the first entry in the column $x+i$, $1 \le i \le 4$, then $(x+i)z_1 \in N_2(v)$.

Table 1: Vertices of $N_2(xy_1y_2)$ at distance 2 from $\emptyset$.

$x+0$	$x+1$	$x+2$	$x+3$	$x+4$
y_10	$(y_2+1)y_1$	$(y_2+3)y_1$	$y_2(y_1+1)$	$y_2(y_1+3)$
y_11	$y_1(y_2+2)$	$(y_1+1)(y_2+1)$	$(y_1+3)(y_2+1)$	$y_1(y_2+2)$
y_12	$(y_1+1)(y_2+1)$	$(y_1+1)(y_2+3)$	$(y_1+1)(y_2+3)$	$(y_1+3)(y_2+3)$
y_13	$(y_1+2)y_2$	$(y_1+3)(y_2+1)$	$(y_1+3)(y_2+3)$	$(y_1+2)y_2$

By Corollary 2.3, it suffices to show that $\mathrm{dist}(ab_1b_2, a'b_1'b_2') \le 4$ for every $a, a' \in \mathbb{Z}_5$ and $b_1, b_2, b_1', b_2' \in \mathbb{Z}_4$. Denote $I = N_2(ab_1b_2) \cap N_2(a'b_1'b_2')$. We show that $I \ne \emptyset$ which implies $\mathrm{dist}(ab_1b_2, a'b_1'b_2') \le 4$.

By Proposition 2.4 (recall that it is enough to consider orbits under σ) we can assume that $a = 0$ and $a' = 0$, $a' = 1$ or $a' = 2$. This gives three cases.

Case 1. $a' = 0$. In this case $0 \in N_2(0b_1b_2)$ and $0 \in N_2(0b_1'b_2')$, that is, $0 \in I$. Therefore, $\mathrm{dist}(0b_1b_2, 0b_1'b_2') \le 4$.

Case 2. $a' = 1$. Comparing the column $x+0$ for ab_1b_2 with the column $x+4$ for $a'b_1'b_2'$ we see that $0z_1z_2 \in I$ for some $z_1, z_2 \in \mathbb{Z}_4$ if $b_1 = b_2'$, $b_1 = b_1'$, $b_1 = b_1'+3$ or $b_1 = b_1'+2$; see Table 1. By the last three subcases, $\mathrm{dist}(0b_1b_2, 1b_1'b_2') \le 4$ if $b_1 \ne b_1'+1$.

Further, from the first row of Table 1 we deduce that $2z_1 \in I$ if $b_2{+}3 = b'_2{+}1$, $3z_1 \in I$ if $b_2 = b'_2{+}3$, and $4z_1 \in I$ if $b_2 = b'_2$, $z_1 \in \mathbb{Z}_4$. This gives $\mathrm{dist}(0b_1b_2, 1b'_1b'_2) \le 4$ if $b_2 \ne b'_2{+}1$. Hence, it suffices to consider $\mathrm{dist}(0b_1b_2, 1(b_1{+}3)(b_2{+}3))$. But $2(b_1{+}3)(b_2{+}1) \in N_2(0b_1b_2)$ and $2(b_1{+}3)(b_2{+}1) \in N_2(1(b_1{+}3)(b_2{+}3))$. So $\mathrm{dist}(0b_1b_2, 1b'_1b'_2) \le 4$ for every $b_1, b_2, b'_1, b'_2 \in \mathbb{Z}_4$.

Case 3. $a' = 2$. Then $0z_1z_2 \in I$ if $b_1 \in \{(b'_1{+}3), (b'_1{+}1)\}$; see Table 1. Thus, $\mathrm{dist}(0b_1b_2, 1b'_1b'_2) \le 4$ if $b_1 \ne b'_1$ and $b_1 \ne b'_1{+}2$. Further, $1z_1 \in I$ if $b_2{+}1 = b'_2$ and $3z_1 \in I$ if $b_2 = b'_2{+}1$. This gives $\mathrm{dist}(0b_1b_2, 1b'_1b'_2) \le 4$ if $b_2 \ne b'_2$ and $b_2 \ne b'_2{+}2$. So we reduced the problem to $\mathrm{dist}(0b_1b_2, 2b'_1b'_2)$, where $b'_1 \in \{b_1, b_1{+}2\}$ and $b'_2 \in \{b_2, b_2{+}2\}$. Since $1b_1(b_2{+}2) \in N_2(0b_1b_2) \cap N(2b_1b_2)$, we have $\mathrm{dist}(0b_1b_2, 2b_1b_2) \le 4$; further, as $3(b_1{+}3)(b_2{+}1) \in N_2(0b_1b_2) \cap N(2(b_1{+}2)b_2)$, we have $\mathrm{dist}(0b_1b_2, 2(b_1{+}2)b_2) \le 4$; since $3(b_1{+}1)(b_2{+}3) \in N_2(0b_1b_2) \cap N(2b_1(b_2{+}2))$, we have $\mathrm{dist}(0b_1b_2, 2b_1(b_2{+}2)) \le 4$; and as $1b_1(b_2{+}2) \in N_2(0b_1b_2) \cap N(2(b_1{+}2)(b_2{+}2))$, we have $\mathrm{dist}(0b_1b_2, 2(b_1{+}2)(b_2{+}2)) \le 4$. So $\mathrm{dist}(0b_1b_2, 1b'_1b'_2) \le 4$ for every $b_1, b_2, b'_1, b'_2 \in \mathbb{Z}_4$. □

We remark that analogously one can prove that also other graphs $G(t; c_1, c_2, \dots, c_d)$ are radial Moore. The problem is that if t is bigger, then we have to solve more pairs separately in each case $1 \le a' \le d$. Therefore, we let this work for a computer.

At the moment we do not know under which conditions $G(t; c_1, c_2, \dots, c_d)$ and $G(t; c'_1, c'_2, \dots, c'_d)$ are isomorphic graphs. However, applying relabelling τ:

$$\tau(v) = \begin{cases} \emptyset & \text{if } v = \emptyset, \\ a & \text{if } v = a, \quad a \in \mathbb{Z}_t \\ a(q{\cdot}b_1) & \text{if } v = ab_1, \quad a \in \mathbb{Z}_t,\ b_1 \in \mathbb{Z}_{t-1}, \\ a(q{\cdot}b_1)(q{\cdot}b_2) & \text{if } v = ab_1b_2, \quad a \in \mathbb{Z}_t,\ b_1, b_2 \in \mathbb{Z}_{t-1}, \end{cases}$$

where q is coprime with $t{-}1$, we obtain $G(t; qc_1, qc_2, \dots, qc_d)$. So the graphs $G(t; qc_1, qc_2, \dots, qc_d)$ and $G(t; c_1, c_2, \dots, c_d)$ are isomorphic. This fact was used when computing the number of realizations $G(t; c_1, c_2, \dots, c_d)$ which give radial Moore graphs.

It is easy to check that the graph $G(t; 0, 0, \dots, 0)$ is not radial Moore. (If $\mathrm{dist}(aaa, x) \le 2$ in $G(t; 0, 0, \dots, 0)$, then either $x = a$ or $x = aa$ or $x = aay_2$ or $x = y_1aa$ or $x = y_1a$, where $y_1 \in \mathbb{Z}_t$ and $y_2 \in \mathbb{Z}_{t-1}$. Hence, $\mathrm{dist}(000, 111) > 4$ in $G(t; 0, 0, \dots, 0)$.) Not only this. Our experiments show that only a very few graphs $G(t; c_1, c_2, \dots, c_d)$ with $c_i = 0$ for some i, $1 \le i \le d$, are radial Moore. Therefore we focused our attention on graphs $G(t; c_1, c_2, \dots, c_d)$ for which $c_i \ne 0$. The unique exception is c_d in the case when t is even. Since p_d must be an involution in this case, we have $p_d(p_d(b_1b_2) = p_d((b_2{+}c_d)b_1) = (b_1{+}c_d)(b_2{+}c_d) = b_1b_2$ which gives $c_d = 0$.

In Table 2, by n we denote the number of d-tuples $c_1, c_2, \dots, c_d$ for which $G(t; c_1, c_2, \dots, c_d)$ is a radial Moore graph, $c_i \in \mathbb{Z}_{t-1}$. By n_0 we denote the number of those d-tuples producing radial Moore graphs $G(t; c_1, c_2, \dots, c_d)$, for which $c_i \in \mathbb{Z}_{t-1} \setminus \{0\}$ (with the exception of c_d in the case when t is even). Observe that to find n, we have to check $(t{-}1)^{\lfloor (t-1)/2 \rfloor}$ graphs $G(t; c_1, c_2, \dots, c_d)$. On the other hand, to find n_0 we have to check $(t{-}2)^{\lfloor (t-1)/2 \rfloor}$ graphs $G(t; c_1, c_2, \dots, c_d)$. Further, $\lim_{t\to\infty} [(t{-}1)/(t{-}2)]^{\lfloor (t-1)/2 \rfloor} = \sqrt{e} \doteq 1.65$ and $f(t) = [(t{-}1)/(t{-}2)]^{\lfloor (t-1)/2 \rfloor}$ is a decreasing function for odd, as well as for even numbers. Hence, if 0 occurs equally often as the other values of $\mathbb{Z}_{t-1}$ among $c_1, c_2, \ldots, c_d$ if t is odd (among $c_1, c_2, \ldots, c_{d-1}$ if t is even) in radial Moore graphs $G(t; c_1, c_2, \ldots, c_d)$, then we should have $n \geq n_0\sqrt{e}$, which is not the case; see Table 2.

Table 2: Number of d-tuples $c_1, c_2, \ldots, c_d$ for which $G(t; c_1, c_2, \ldots, c_d)$ is radial Moore.

t	n	n_0	p_0
3	1	1	100
5	4	4	44.44
7	2	2	1.6
9	4	4	0.17
10	6	6	0.14
11	124	124	0.21
12	10	10	0.01
13	816	772	0.044
14	132	132	0.0042
15	6 828	6 612	0.011
16	504	504	0.00048
17	86, 144	83, 224	0.0032
18	6 736	6 720	0.00016

In Table 2 we also present the proportion p_0 (in percents) of the number of sequences $c_1, c_2, \ldots, c_d$ giving radial Moore graphs (and included in n_0) to the number of all considered d-tuples $c_1, c_2, \ldots, c_d$. Since there are no solutions for degrees 4, 6 and 8, we omit these cases. As one can see, our construction is much more prolific for odd degrees than for the even ones, but in both cases a significant proportion of considered d-tuples $c_1, c_2, \ldots, c_d$ give radial Moore graphs.

In Table 3, for every $t \in \{3, 5, 7, 9, 10, \ldots, 30\}$ we present one d-tuple $c_1, c_2, \ldots, c_d$ such that $G(t; c_1, c_2, \ldots, c_d)$ is a radial Moore graph. Our intention was to choose d-tuples which "look similarly". Nevertheless, we cannot spot any regularity there and so we are not able to prove that for arbitrary t, $t \geq 9$, there is a graph $G(t; c_1, c_2, \ldots, c_d)$ which is radial Moore. A computer representation of the graphs presented in Table 3 is available at `http://www.math.sk/knor/moore`.

Finally, let us consider a generalization of our construction. Denote by $G'(t; e_1, e_2, \ldots, e_d)$ a graph of radius 3 and degree t on $M_{3,t}$ vertices with vertex set V and edge set $E_1 \cup E_2$, in which $p_i(b_1b_2) = b_2b_1 \oplus e_i$, where b_2b_1 is considered as two-digit $(t-1)$-ary number, $e_i \in \mathbb{Z}_{(t-1)^2}$ and $\oplus$ is addition in $\mathbb{Z}_{(t-1)^2}$. Observe that for every $e_i \in \mathbb{Z}_{(t-1)^2}$, $b_1 \in \mathbb{Z}_{t-1}$ and $1 \leq i \leq d$, we have $S(b_1, i) = \mathbb{Z}_{t-1}$. Moreover, the graph $G'(t; (t-1)c_1, (t-1)c_2, \ldots, (t-1)c_d)$ is isomorphic with $G(t; c_1, c_2, \ldots, c_d)$, $c_i \in \mathbb{Z}_{t-1}$. The problem is that Proposition 2.4 is not valid for $G'(t; e_1, e_2, \ldots, e_d)$ as ρ is not an automorphism of this graph. Although there are no radial Moore graphs $G(4; c_1, 0)$, $G(6; c_1, c_2, 0)$ and $G(8; c_1, c_2, c_3, 0)$, there are radial Moore graphs $G'(6; c_1, c_2, 0)$ and $G'(8; c_1, c_2, c_3, 0)$. For instance, $G'(6; 5, 19, 0)$ and $G'(8; 1, 31, 20, 0)$ are radial Moore graphs. There is no radial Moore graph $G'(4; c_1, 0)$. Nevertheless, having in mind the results of [8], we can summarize that for every degree t, $3 \leq t \leq 30$, there is a radial Moore graph of degree t with radius 3 which, together with the results of [5], gives Theorem 1.1.

Table 3: Values $c_1, c_2, \ldots, c_d$ for which $G(t; c_1, c_2, \ldots, c_d)$ is radial Moore.

t	$M_{3,t}$	$c_1, c_2, \ldots, c_d$
3	22	1
5	106	$1, 3$
7	302	$1, 3, 1$
9	658	$1, 3, 6, 4$
10	911	$1, 7, 6, 4, 0$
11	1 222	$1, 3, 4, 8, 1$
12	1 597	$1, 3, 4, 2, 5, 0$
13	2 042	$1, 3, 7, 10, 11, 6$
14	2 563	$1, 3, 7, 4, 2, 8, 0$
15	3 166	$1, 3, 4, 7, 12, 9, 1$
16	3 857	$1, 3, 6, 7, 5, 13, 4, 0$
17	4 642	$1, 3, 7, 2, 12, 10, 11, 8$
18	5 527	$1, 3, 12, 15, 10, 8, 6, 13, 0$
19	6 518	$1, 3, 4, 7, 13, 8, 12, 10, 1$
20	7 621	$1, 7, 8, 2, 6, 3, 5, 4, 10, 0$
21	8 842	$1, 3, 7, 2, 14, 15, 8, 6, 11, 10$
22	$10, 187$	$1, 7, 8, 2, 6, 9, 5, 10, 4, 3, 0$
23	$11, 662$	$1, 3, 4, 7, 14, 5, 3, 1, 12, 16, 20$
24	$13, 273$	$1, 7, 8, 2, 6, 9, 11, 10, 3, 19, 5, 0$
25	$15, 026$	$1, 3, 7, 2, 16, 17, 4, 6, 5, 13, 10, 15$
26	$16, 927$	$1, 7, 8, 2, 6, 9, 11, 10, 5, 13, 4, 11, 0$
27	$18, 982$	$1, 3, 4, 7, 15, 5, 8, 13, 10, 6, 17, 15, 1$
28	$21, 197$	$1, 7, 8, 2, 6, 9, 11, 10, 5, 13, 15, 3, 3, 0$
29	$23, 578$	$1, 3, 7, 2, 16, 17, 4, 6, 5, 13, 19, 8, 14, 18$
30	$26, 131$	$1, 7, 8, 2, 6, 9, 11, 10, 5, 13, 5, 3, 15, 12, 0$

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