AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 141–149

On the total edge irregularity strength of zigzag graphs*

Ali Ahmad

College of computer & information system
Jazan University, Jazan
Kingdom of Saudi Arabia
ahmadsms@gmail.com

Muhammad Kamran Siddiqui   Deeba Afzal

Abdus Salam School of Mathematical Sciences
GC University, Lahore
Pakistan
{kamransiddiqui75,deebafzal}@gmail.com

Abstract

An edge irregular total k-labeling of a graph G is a labeling of the vertices and edges with labels $1, 2, \ldots, k$ such that the weights of any two different edges are distinct, where the weight of an edge is the sum of the label of the edge itself and the labels of its two end vertices. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength, $\mathrm{tes}(G)$. In this paper we determine the exact values of the total edge irregularity strength of zigzag graphs.

1 Introduction

In [6], the authors defined the notion of an edge irregular total k-labeling of a graph $G = (V, E)$ to be a labeling of the vertices and edges of G, $\psi : V \cup E \rightarrow \{1, 2, \ldots, k\}$, such that the *edge weights* $wt_\psi(uv) = \psi(u) + \psi(uv) + \psi(v)$ are different for all edges, i.e. $wt_\psi(uv) \neq wt_\psi(u'v')$ for all edges $uv, u'v' \in E$ with $uv \neq u'v'$. They also defined the *total edge irregularity strength of* G, $\mathrm{tes}(G)$, as the minimum k for which the graph G has an edge irregular total k-labeling.

The total edge irregularity strength is an invariant analogous to irregular assignments and the irregularity strength of a graph G introduced by Chartrand et al. [9]

* This research work was supported by the Higher Education Commission Pakistan.

and studied by numerous authors; see [7, 11, 12, 14]. An *irregular assignment* is a k-labeling of the edges $\varphi : E \to \{1, 2, \ldots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$.

The corresponding problem where only adjacent vertices are required to have different weights, i.e. the weights form a proper vertex colouring of the graph, was introduced by Karoński et al. in [17]. They conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$, such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different. The current record is that 16 labels suffice; see [1].

In [6], a lower bound is given on the total edge irregularity strength of a graph:

$$\operatorname{tes}(G) \geq \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G.

Ivančo and Jendroľ [13] posed the following conjecture:

Conjecture 1. [13] *Let G be an arbitrary graph different from K_5. Then*

$$\operatorname{tes}(G) = \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}. \tag{2}$$

Conjecture 1 has been verified for: trees [13]; for complete graphs and complete bipartite graphs [15, 16]; for the Cartesian product of two paths $P_n \Box P_m$ [18]; for the corona product of a path with certain graphs [19]; for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ [8]; for hexagonal grids [5]; for toroidal grids [10]; and for the strong product of two paths [4].

Motivated by the papers [2, 3, 11, 18], we investigate the total edge irregularity strength of the zigzag graphs Z_n^m.

The graphs considered here will be finite. For $n, m \geq 2$ we denote the zigzag graph by Z_n^m, defined in the planar map labeled as in Figure 1 with m rows and n columns. The symbols $V(Z_n^m)$ and $E(Z_n^m)$ will denote the vertex set and the edge set of zigzag graph Z_n^m. Moreover,

$$V(Z_n^m) = \{a_{i,j} : 1 \leq i \leq n;\ 1 \leq j \leq m\} \text{ with } |V(Z_n^m)| = mn$$

and

$$\begin{aligned} E(Z_n^m) &= \{a_{i,j}a_{i+1,j} : 1 \leq i \leq n-1;\ 1 \leq j \leq m\} \\ &\quad \cup \{a_{i,j}a_{i-1,j+1} : 2 \leq i \leq n;\ 1 \leq j \leq m-1\} \\ &\qquad \text{with } |E(Z_n^m)| = 2mn - 2m - n + 1. \end{aligned}$$

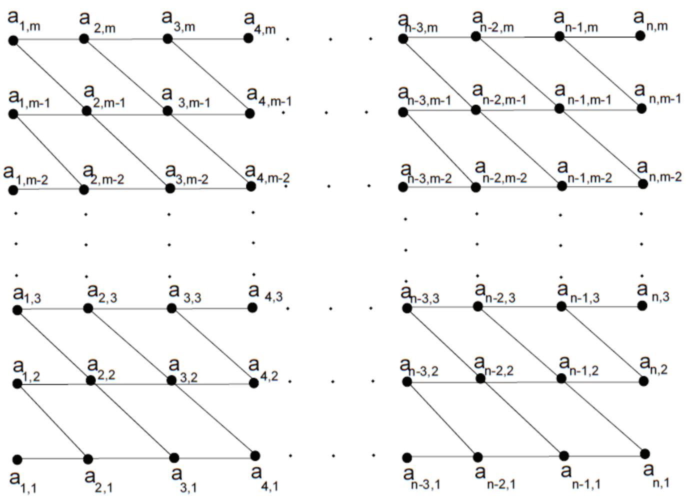

Figure 1: The graph of Z_n^m

In this paper, we deal with zigzag graph Z_n^m for all $n, m \geq 2$. In [6], it is proved that $\text{tes}(G) \geq \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$. Since the maximum degree $\triangle(Z_n^m) = 4$, this implies that $\text{tes}(G) \geq \left\lceil\frac{|E(G)|+2}{3}\right\rceil$. To show that $\left\lceil\frac{2mn-2m-n+3}{3}\right\rceil$ is an upper bound for $\text{tes}(Z_n^m)$, we describe an edge irregular total $\left\lceil\frac{2mn-2m-n+3}{3}\right\rceil$-labeling for Z_n^m.

For convenience, we split the edge set of Z_n^m into mutually disjoint subsets A_i and B_i, where

$$\begin{aligned} A_i &= \{a_{i,j}a_{i+1,j} : 1 \leq i \leq n-1;\ 1 \leq j \leq m\},\ \ |A_i| = m, \\ B_i &= \{a_{i,j}a_{i-1,j+1} : 2 \leq i \leq n;\ 1 \leq j \leq m-1\},\ \ |B_i| = m-1. \end{aligned}$$

Clearly $E(Z_n^m) = \bigcup_{i=1}^{n-1} A_i \cup \bigcup_{i=2}^{n} B_i$.

First we discuss the total edge irregularity strength of Z_n^m for a small case. Moreover, it is easy to verify that $\text{tes}(Z_2^2) = 2$.

Lemma 1. $\text{tes}(Z_3^2) = 3$.

Proof. The existence of an optimal labeling ξ proves the above equality. For this, first we define the vertex labels:

$$\xi(a_{1,1}) = \xi(a_{2,1}) = \xi(a_{3,1}) = 1;$$
$$\xi(a_{1,2}) = \xi(a_{2,2}) = \xi(a_{3,2}) = 3.$$

Now we define the edge labels as follows:

$$\xi(a_{1,1}a_{2,1}) = \xi(a_{2,1}a_{1,2}) = \xi(a_{1,2}a_{2,2}) = 1;$$
$$\xi(a_{2,1}a_{3,1}) = \xi(a_{3,1}a_{2,2}) = \xi(a_{2,2}a_{3,2}) = 2.$$

It is easy to see that ξ is an edge irregular total labeling having the required property. □

Theorem 1. *Let $m \geq 3$. Then the total edge irregularity strength of zigzag graph Z_3^m is $\lceil \frac{4m}{3} \rceil$.*

Proof. From (1), we have $\text{tes}(Z_3^m) \geq \lceil \frac{4m}{3} \rceil$. To prove the equality we define the vertex labeling ϕ_1 with $k = \lceil \frac{4m}{3} \rceil$ as follows:

$$\phi_1(a_{i,j}) = \begin{cases} j, & \text{if } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq \lceil \frac{m}{2} \rceil; \\ k - m + j, & \text{if } 1 \leq i \leq 3 \text{ and } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m. \end{cases}$$

Observe that, under the vertex labeling ϕ_1, the weights of the edges:

(i) from the sets A_1, A_2 receive the even integers from 2 to $2\lceil \frac{m}{2} \rceil$ for $1 \leq j \leq \lceil \frac{m}{2} \rceil$;

(ii) from the sets A_1, A_2 receive the even integers from $2k - 2m + 2\lceil \frac{m}{2} \rceil + 2$ to $2k$ for $\lceil \frac{m}{2} \rceil + 1 \leq j \leq m$;

(iii) from the sets B_2, B_3 receive the odd integers from 3 to $2\lceil \frac{m}{2} \rceil - 1$ for $1 \leq j \leq \lceil \frac{m}{2} \rceil - 1$;

(iv) from the sets B_2, B_3 receive the integer $k - m + 1 + 2\lceil \frac{m}{2} \rceil$ for $j = \lceil \frac{m}{2} \rceil$;

(v) from the sets B_2, B_3 receive the odd integers from $2k - 2m + 2\lceil \frac{m}{2} \rceil + 3$ to $2k - 1$ for $\lceil \frac{m}{2} \rceil + 1 \leq j \leq m - 1$.

To complete the labeling to a total one, we label the edges of the graph Z_n^m as follows:

$$\phi_1(A_i) = \begin{cases} 2j - 1, & \text{if } i = 1 \text{ and } 1 \leq j \leq \lceil \frac{m}{2} \rceil \\ 2(m + j - k) - 1, & \text{if } 1 = 1 \text{ and } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m \\ 2j, & \text{if } i = 2 \text{ and } 1 \leq j \leq \lceil \frac{m}{2} \rceil \\ 2(m + j - k), & \text{if } i = 2 \text{ and } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m \end{cases}$$

$$\phi_1(B_i) = \begin{cases} 2j, & \text{if } i = 2 \text{ and } 1 \leq j \leq \lceil \frac{m}{2} \rceil - 1 \\ 2\lceil \frac{m}{2} \rceil + m - k & \text{if } i = 2 \text{ and } j = \lceil \frac{m}{2} \rceil \\ 2(m + j - k), & \text{if } i = 2 \text{ and } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m - 1 \\ 2j + 1, & \text{if } i = 3 \text{ and } 1 \leq j \leq \lceil \frac{m}{2} \rceil - 1 \\ 2\lceil \frac{m}{2} \rceil + m - k + 1 & \text{if } i = 3 \text{ and } j = \lceil \frac{m}{2} \rceil \\ 2(m + j - k) + 1, & \text{if } i = 3 \text{ and } \lceil \frac{m}{2} \rceil + 1 \leq j \leq m - 1. \end{cases}$$

Observe that under the labeling ϕ_1, the total weights of the edges are defined as follows:

(i) from the set A_1, receive the consecutive integers with difference four from 3 to $4\lceil \frac{m}{2} \rceil - 1$ for $1 \le j \le \lceil \frac{m}{2} \rceil$;

(ii) from the set A_2, receive the consecutive integers with difference four from 4 to $4\lceil \frac{m}{2} \rceil$ for $1 \le j \le \lceil \frac{m}{2} \rceil$;

(iii) from the set B_2, receive the consecutive integers with difference four from 5 to $4\lceil \frac{m}{2} \rceil - 3$ for $1 \le j \le \lceil \frac{m}{2} \rceil - 1$;

(iv) from the set B_3, receive the consecutive integers with difference four from 6 to $4\lceil \frac{m}{2} \rceil - 2$ for $1 \le j \le \lceil \frac{m}{2} \rceil - 1$;

(v) from the set B_2, receive the constant integer $4\lceil \frac{m}{2} \rceil + 1$ for $j = \lceil \frac{m}{2} \rceil$;

(vi) from the set B_3, receive the constant integer $4\lceil \frac{m}{2} \rceil + 2$ for $j = \lceil \frac{m}{2} \rceil$;

(vii) from the set A_1, receive the consecutive integers with difference four from $4\lceil \frac{m}{2} \rceil + 3$ to $4m - 1$ for $\lceil \frac{m}{2} \rceil + 1 \le j \le m$;

(viii) from the set A_2, receive the consecutive integers with difference four from $4\lceil \frac{m}{2} \rceil + 4$ to $4m$ for $\lceil \frac{m}{2} \rceil + 1 \le j \le m$;

(ix) from the set B_2, receive the consecutive integers with difference four from $4\lceil \frac{m}{2} \rceil + 5$ to $4m - 3$ for $\lceil \frac{m}{2} \rceil + 1 \le j \le m - 1$;

(x) from the set B_3, receive the consecutive integers with difference four from $4\lceil \frac{m}{2} \rceil + 6$ to $4m - 2$ for $\lceil \frac{m}{2} \rceil + 1 \le j \le m - 1$.

Now it is easy to see that all vertex and edge labels are at most k and the edge weights are pairwise distinct. Thus the resulting labeling is the desired edge irregular k-labeling. □

Theorem 2. *Let $m \ge 2$ and $n \ge 4$ be two integers; then the total edge irregularity strength of the zigzag graph Z_n^m is $\left\lceil \frac{2mn-2m-n+3}{3} \right\rceil$.*

Proof. From (1), we have $\text{tes}(Z_n^m) \ge \lceil \frac{2mn-2m-n+3}{3} \rceil$. To prove the equality, let $k = \left\lceil \frac{2mn-2m-n+3}{3} \right\rceil$ and $1 \le j \le m$; we define the labeling ϕ_2 as follows:

$$\phi_2(a_{i,j}) = \begin{cases} j, & \text{if } i = 1, 2; \\ m(i-2) + j, & \text{if } 3 \le i \le \lceil \frac{n}{2} \rceil; \\ k - m(n-1-i) - 1 + j, & \text{if } \lceil \frac{n}{2} \rceil + 1 \le i \le n-2; \\ k, & \text{if } i = n-1, n. \end{cases}$$

Observe that under the vertex labeling ϕ_2, the weights of the edges:

(i) from the set A_1 receive the even integers from 2 to $2m$;

(ii) from the set A_2 receive the consecutive integers with difference two from $m+2$ to $3m$;

(iii) from the set A_i receive the consecutive integers with difference two from $2m(i-1)-m+2$ to $2m(i-1)+m$ for $3 \leq i \leq \lceil \frac{n}{2} \rceil - 1$;

(iv) from the set $A_{\lceil \frac{n}{2} \rceil}$ receive the consecutive integers with difference two from $k+m(2\lceil \frac{n}{2} \rceil - n)+1$ to $k+m(2\lceil \frac{n}{2} \rceil - n+2)-1$;

(v) from the set A_i receive the consecutive integers with difference two from $2k-2m(n-i-1)+m$ to $2k-2m(n-i-1)+3m-2$ for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n-3$;

(vi) from the set A_{n-2} receive the consecutive integers from $2k-m$ to $2k-1$;

(vii) from the set A_{n-1} receive the constant integers $2k$;

(viii) from the set B_2 receive the odd integers from 3 to $2m-1$;

(ix) from the set B_3 receive the consecutive integers with difference two from $m+3$ to $3m-1$;

(x) from the set B_i receive the consecutive integers with difference two from $2m(i-2)-m+3$ to $2m(i-2)+m-1$ for $4 \leq i \leq \lceil \frac{n}{2} \rceil$;

(xi) from the set $B_{\lceil \frac{n}{2} \rceil + 1}$ receive the consecutive integers with difference two from $k+m(2\lceil \frac{n}{2} \rceil - n)+2$ to $k+m(2\lceil \frac{n}{2} \rceil - n+2)-2$;

(xii) from the set B_i receive the consecutive integers with difference two from $2k-2m(n-i)+m+1$ to $2k-2m(n-i)+3m-3$ for $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n-2$;

(xiii) from the set B_{n-1} receive the consecutive integers from $2k-m+1$ to $2k-1$;

(xiv) from the set B_n receive the constant integer $2k$.

Now we define the edge labels of the graph Z_n^m as follows:

$$\phi_2(A_i) = \begin{cases} 1, & \text{if } i=1 \\ m-(i-2), & \text{if } 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ m(n-2)-k-\lceil \frac{n}{2} \rceil + 3, & \text{if } i = \lceil \frac{n}{2} \rceil \\ m(2n-5)-2k+4-i, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-3 \\ m(2n-5)-n-2k+5+j, & \text{if } i=n-2 \text{ and } j \in [1,m] \\ m(2n-4)-n-2k+3+2j, & \text{if } i=n-1 \text{ and } j \in [1,m] \end{cases}$$

$$\phi_2(B_i) = \begin{cases} 1, & \text{if } i = 2 \\ m - (i-3), & \text{if } 3 \le i \le \lceil \frac{n}{2} \rceil \\ m(n-2) - k - \lceil \frac{n}{2} \rceil + 3, & \text{if } i = \lceil \frac{n}{2} \rceil + 1 \\ m(2n-5) - 2k + 4 - i, & \text{if } \lceil \frac{n}{2} \rceil + 2 \le i \le n-2 \\ m(2n-5) - n - 2k + 5 + j, & \text{if } i = n-1 \text{ and } j \in [1, m-1] \\ m(2n-4) - n - 2k + 4 + 2j, & \text{if } i = n \text{ and } j \in [1, m-1] \end{cases}$$

Observe that under the labeling ϕ_2, the total weights of the edges are define as follows:

(i) from the set A_1 receive the consecutive integers with difference two from 3 to $2m+1$;

(ii) from the set A_2 receive the consecutive integers with difference two from $2m+2$ to $4m$;

(iii) from the set A_i receive the consecutive integers with difference two from $2m(i-1) - 2m + 4$ to $(2m-1)i + 2$ for $3 \le i \le \lceil \frac{n}{2} \rceil - 1$;

(iv) from the set $A_{\lceil \frac{n}{2} \rceil}$ receive the consecutive integers with difference two from $(2m-1)\lceil \frac{n}{2} \rceil - 2m + 4$ to $(2m-1)\lceil \frac{n}{2} \rceil + 2$;

(v) from the set A_i receive the consecutive integers with difference two from $(2m-1)i - 2m + 4$ to $(2m-1)i + 2$ for $\lceil \frac{n}{2} \rceil + 1 \le i \le n-3$;

(vi) from the set A_{n-2} receive the consecutive integers with difference two from $(2m-1)n - 6m + 6$ to $(2m-1)n - 4m + 4$;

(vii) from the set A_{n-1} receive the consecutive integers with difference two from $m(2n-4) - n + 5$ to $m(2n-2) - n + 3$;

(viii) from the set B_2 receive the consecutive integers with difference two from 4 to $2m$;

(ix) from the set B_3 receive the consecutive integers with difference two from $2m+3$ to $4m-1$;

(x) from the set B_i receive the consecutive integers with difference two from $(2m-1)i - 4m + 6$ to $(2m-1)i - 2m + 2$ for $4 \le i \le \lceil \frac{n}{2} \rceil$;

(xi) from the set $B_{\lceil \frac{n}{2} \rceil + 1}$ receive the consecutive integers with difference two from $(2m-1)\lceil \frac{n}{2} \rceil - 2m + 5$ to $(2m-1)\lceil \frac{n}{2} \rceil + 1$;

(xii) from the set B_i receive the consecutive integers with difference two from $(2m-1)i - 4m + 5$ to $(2m-1)i - 2m + 1$ for $\lceil \frac{n}{2} \rceil + 2 \le i \le n-2$;

(xiii) from the set B_{n-1} receive the consecutive integers difference two from $2m(n-3)-n+7$ to $2m(n-2)-n+3$;

(xiv) from the set B_n receive the consecutive integers difference two from $2m(n-2)-n+6$ to $2m(n-1)-n+2$.

Now, it is not hard to see that all vertex and edge labels are at most k and the edge weights are pairwise distinct. In fact, our total labeling has been chosen in such a way that the edge-weights of the edges form a consecutive sequence of integers from 3 to $m(2n-2)-n+3$. Thus, the resulting labeling is the desired edge irregular k-labeling. □

Acknowledgements

The authors wish to thank the anonymous referees for their valuable comments.

References

[1] L. Addario-Berry, K. Dalal and B.A. Reed, Degree constrained subgraphs, *Discrete Appl. Math.* **156** (2008), 1168–1174.

[2] A. Ahmad and M. Bača, Edge irregular total labeling of certain family of graphs, *AKCE J. Graphs. Combin.* **6** no. 1 (2009), 21–29.

[3] A. Ahmad and M. Bača, Total edge irregularity strength of a categorical product of two paths, *Ars Combin.* (to appear).

[4] A. Ahmad, M. Bača, Y. Bashir and M.K. Siddiqui, Total edge irregularity strength of strong product of two paths, *Ars Combin.* (to appear).

[5] O. Al-Mushayt, Ali Ahmad and M.K. Siddiqui, On the total edge irregularity strength of hexagonal grid graphs, *Australas. J. Combin.* **53** (2012), 263–271.

[6] M. Bača, S. Jendroľ, M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378–1388.

[7] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory* **45** (2004), 241–254.

[8] S. Brandt, J. Miškuf and D. Rautenbach, On a conjecture about edge irregular total labellings, *J. Graph Theory* **57** (2008), 333–343.

[9] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 187–192.

[10] T. Chunling, L. Xiaohui, Y. Yuansheng and W. Liping, Irregular total labellings of $C_m \square C_n$, *Utilitas Math.* **81** (2010), 3–13.

[11] J.H. Dimitz, D.K. Garnick and A. Gyárfás, On the irregularity strength of the $m \times n$ grid, *J. Graph Theory* **16** (1992), 355–374.

[12] A. Frieze, R.J. Gould, M. Karonski and F. Pfender, On graph irregularity strength, *J. Graph Theory* **41** (2002), 120–137.

[13] J. Ivančo and S. Jendroľ, Total edge irregularity strength of trees, *Discussiones Math. Graph Theory* **26** (2006), 449–456.

[14] S. Jendroľ, M. Tkáč and Z. Tuza, The irregularity strength and cost of the union of cliques, *Discrete Math.* **150** (1996), 179–186.

[15] S. Jendroľ, J.Miškuf and R. Soták, Total edge irregularity strength of complete and complete bipartite graphs, *Electron. Notes Discrete Math.* **28** (2007), 281–285.

[16] S. Jendroľ, J. Miškuf and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Math.* **310** (2010), 400–407.

[17] M. Karoński, T. Luczak and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory Ser. B* **91** (2004), 151–157.

[18] J. Miškuf and S. Jendroľ, On total edge irregularity strength of the grids, *Tatra Mt. Math. Publ.* **36** (2007), 147–151.

[19] Nurdin, A.N.M. Salman and E.T. Baskoro, The total edge-irregular strengths of the corona product of paths with some graphs, *J. Combin. Math. Combin. Comput.* **65** (2008), 163–175.

(Received 8 Oct 2011; revised 5 July 2012)