AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 127–132

A note on 4-regular distance magic graphs

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Abstract

Let $G = (V, E)$ be a graph on n vertices. A bijection $f : V \to \{1, 2, \dots, n\}$ is called a *distance magic labeling* of G if there exists an integer k such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the set of all vertices adjacent to v. The constant k is the *magic constant* of f and any graph which admits a distance magic labeling is a *distance magic graph*. In this paper we solve some of the problems posted in a recent survey paper on distance magic graph labelings by Arumugam et al. We classify all orders n for which a 4-regular distance magic graph exists and by this we also show that there exists a distance magic graph with $k = 2^t$ for every integer $t \geq 6$.

1 Introduction and definitions

Throughout this paper we consider simple undirected graphs. The concept of magic labeling was introduced already in the 1960s. It became very popular shortly after 2000 and a great number of papers was published in which various types of magic labelings were introduced. For a comprehensive list see the dynamic survey [5].

* Research for this article was partially supported by the institutional project MSM6198910027.

In this paper we focus on distance magic labeling. Let $G = (V, E)$ be a graph of order n. The *distance magic labeling* of G is a bijection $f : V \to \{1, 2, \ldots, n\}$ for which there exists an integer k such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the set of all vertices adjacent to v. The sum $\sum_{u \in N(v)} f(u)$ is the *weight* $w_f(v)$ of the vertex v and the constant k is called the *magic constant* of f. If a graph admits a distance magic labeling, it is called a *distance magic graph.* The same concept was introduced independently by several authors and was called a *1-vertex magic vertex* labeling (see [4], [6]) or a *sigma* labeling (Σ-labeling). Nowadays the terminology seems to have settled on the "distance magic labeling", see e.g. [7]. A recent survey on distance magic labelings is [2].

Besides pure mathematical interest, distance magic labelings can be used while scheduling fair incomplete tournaments. Suppose we want to schedule a one-divisional tournament, but do not have enough time to play the complete round robin tournament. We want to schedule a *fair incomplete round robin tournament* with the following properties:

1. Every team plays the same number of opponents.
2. The difficulty of the tournament for each team mimics the difficulty of the complete round robin tournament.

Condition 2 can be justified as follows. If we know the strength of each team based on team standings in the previous year, the teams can be ranked from 1 to n. The sum of rankings of all opponents of the i-th ranked team T_i is $R_i = n(n+1)/2 - i$. We observe that the sums of rankings $R_1, R_2, \ldots, R_n$ form an arithmetic progression with difference one. Therefore, we want the sums of rankings of opponents for respective teams in our incomplete tournament to form such a progression as well. In general, we want to find a tournament of n teams with each team playing g games in which the sums of rankings of opponents of the i-th ranked team T_i is $R_i^* = n(n+1)/2 - i - k$ for some integer k.

Obviously, this is equivalent to finding a set of games that are left out of the complete tournament with the property that the sum of rankings of opponents in the $n - g - 1$ left out games is equal to the constant k for every team i.

In the language of distance magic labelings, constructing such a set of left out games is nothing else than finding a distance magic labeling of an r-regular graph on n vertices for a given value $r = n - g - 1$ with the magic constant k. For details we refer to [4] where the question of existence of regular distance magic graphs on an even number of vertices was settled completely. In short one can say that regular distance magic graphs of even order exist whenever the (somewhat obvious) necessary conditions are satisfied:

Proposition 1.1 ([4]). *For n even, an r-regular distance magic graph exists if and only if* $2 \le r \le n - 2$, $r \equiv 0 \pmod 2$ *and either* $n \equiv 0 \pmod 4$ *or* $n \equiv r + 2 \equiv 2 \pmod 4$.

On the other hand, when the order of a regular graph is odd, the answer is more complicated. For certain orders and certain regularity such graphs do not exist and no general construction that would provide distance magic graphs of all odd orders is known, see [3]. The survey paper [2] provides a list of open problems, partially collected from various papers, partially never published before. In the next section we address the following three (numbered according to [2]).

Problem 10.5. Characterize 4-regular distance magic graphs.

Problem 10.6. Does there exist a distance magic graph whose magic constant is a power of 2?

Problem 10.7. Does there exist an r-regular distance magic graph with n vertices where n is odd and r is a power of 2?

While the latter two problems we solve completely, for the first problem we characterize all orders for which a distance magic graph exists.

In a distance magic labeling the label of each vertex u contributes to the weights of $\deg(u)$ neighbors. Now in an r-regular distance magic graph the sum of vertex weights is $rn(n+1)/2$ and equals kn. Thus, the magic constant is determined uniquely as $k = r(n+1)/2$. One can observe that by providing a 4-regular distance magic graph of order $2^t - 1$ for a positive integer t, Problems 10.6. and 10.7. are settled immediately since the corresponding magic constant is a power of 2.

2 Recursive construction of 4-regular distance magic graphs

The following lemma provides a tool for a characterization of all orders of 4-regular distance magic graphs. It is based on a recursive approach.

Lemma 2.1. *If there exists a 4-regular distance magic graph on m vertices with a subgraph C_4 such that the sum of each pair of opposite (i.e., non-adjacent in C_4) vertices is $m+1$, then there exists a 4-regular distance magic graph on n vertices for every integer $n \geq m$ with the same parity as m.*

Proof. There exists a 4-regular distance magic graph on n vertices for all even $n \geq 6$ by a result proven in [4]. The claim follows trivially.

For odd orders we proceed by induction. It is enough to show that whenever we have a graph G with a distance magic labeling f satisfying the assumptions of the lemma, we can always construct a 4-regular graph G' on $m+2$ vertices with a distance magic labeling f' which contains a cycle C_4 such that the sum of opposite (non-adjacent in C_4) vertices is $m+3$ in f'.

Suppose G is a distance magic 4-regular graph of odd order m. By C we denote the cycle v_1, v_2, v_3, v_4 in G with the sum of labels of opposite vertices $m+1$, that is, $f(v_1) + f(v_3) = f(v_2) + f(v_4) = m+1$. Let f be some distance magic labeling of G. In Figure 1 the cycle is drawn in full lines.

For odd m we show that there exists a distance magic labeling of G', where G' arises form G by removing the edges of C, adding two new vertices x, y and eight edges of two different cycles C_4: v_1, x, v_3, y and v_2, x, v_4, y. In Figure 1 they are drawn in dotted and dashed lines, respectively. It is easy to observe that G' is connected and 4-regular on $m+2$ vertices.

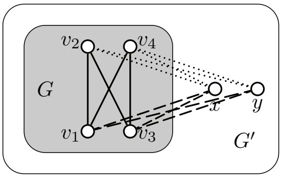

Figure 1: Adding two vertices to a 4-regular distance magic graph on m vertices.

Consider the following labeling f' of G'.

$$f'(v) = \begin{cases} f(v)+1 & \text{for } v \in V(G), \\ 1 & \text{for } v = x, \\ m+2 & \text{for } v = y. \end{cases}$$

Clearly f' is a bijection $V(G') \to \{1, 2, \dots, m+2\}$. Now we show that the weight $w(v)$ of every vertex v in G' is the same. For x and y we have

$$\begin{aligned} w_{f'}(x) = w_{f'}(y) &= f'(v_1) + f'(v_2) + f'(v_3) + f'(v_4) \\ &= f(v_1) + f(v_2) + f(v_3) + f(v_4) + 4 = 2(m+3). \end{aligned}$$

For v_1, v_2, v_3, v_4 we always replace two neighbors with the sum $m+1$ by two vertices x and y with the sum $m+3$ and increase the labels of each of the two remaining neighbors by 1. Thus

$$\begin{aligned} w_{f'}(v_1) &= w_{f'}(v_2) = w_{f'}(v_3) = w_{f'}(v_4) \\ &= w_f(v_1) - (m+1) + (m+3) + 2 = w_f(v_1) + 4 = 2(m+3). \end{aligned}$$

For all remaining vertices v in G' we increase the weight by 4, so $w_{f'}(v) = w_f(v)+4 = 2(m+1)+4 = 2(m+3)$. Thus f' is a distance magic labeling of G'. Also both new cycles v_1, x, v_3, y and v_2, x, v_4, y have the sum of opposite vertices $(m+2)+1$ and the claim follows by induction. □

There are no 4-regular distance magic graphs of odd order n for $n \leq 5$. All distance magic graphs of order 7 are examined in [1] and from the examination it follows that such graphs are never 4-regular. A computer assisted brute force search shows that there exists no 4-regular distance magic graph of odd order $n = 9, 11, 13, 15$. Now we prove that for all remaining odd orders there exist 4-regular distance magic graphs.

Theorem 2.2. *There exists a 4-regular distance magic graph on an odd number of vertices n if and only if $n \geq 17$.*

Proof. For $n \leq 15$ it was argued that no such labeling exists. A distance magic graph on 17 vertices is in Figure 2. The cycle C_4 required by Lemma 2.1 is highlighted by dashed lines and for $n > 17$ the result follows. □

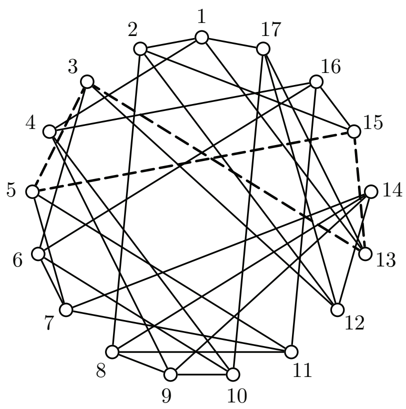

Figure 2: A 4-regular distance magic graph on 17 vertices with a highlighted cycle C_4.

The previous theorem gives a general answer to Problem 10.5. Notice that we did not provide a full list of all 4-regular distance magic graphs, but we have fully characterized the number of vertices for which such graphs exist. In fact a brute force search showed there are 16 different (labeled) 4-regular distance magic graphs on 17 vertices.

Corollary 2.3. *There exists a* 4*-regular distance magic graph with magic constant* $k = 2^t$ *for every* $t \geq 6$*.*

Proof. Let G be any 4-regular distance magic graph on 17 vertices. For every t such that $2^{t-1} \geq 17$ at least one 4-regular distance magic graph on $2^{t-1} - 1$ vertices with the magic constant $k = 4((2^{t-1} - 1) + 1)/2 = 2^t$ can be found by Theorem 2.2. Thus there exists a 4-regular distance magic graph with magic constant $k = 2^t$ for every $t \geq 6$. The smallest example is on $n = 31$ vertices with magic constant $k = 64$ (see Figure 3). □

Corollary 2.3 is an answer to Problems 10.6. and 10.7. given in [2]. At the same time we disprove the conjecture in [1] stating that there exist no distance magic graphs with a magic constant being a power of 2.

Notice that when scheduling incomplete tournaments it is convenient to obtain connected graphs. The construction in Theorem 2.2 always yields a connected graph. We always obtain a connected distance magic graph. In fact it is 4-edge-connected, since the graph in Figure 2 can be decomposed into two hamiltonian cycles (1, 4, 16, 6, 7, 14, 12, 3, 13, 17, 10, 9, 8, 11, 5, 15, 2) and (1, 17, 12, 2, 8, 14, 9, 4, 10, 6, 3, 5, 7, 11, 16, 15, 13) and one can check that by the construction given in Lemma 2.1 the 4-edge-connectivity is preserved.

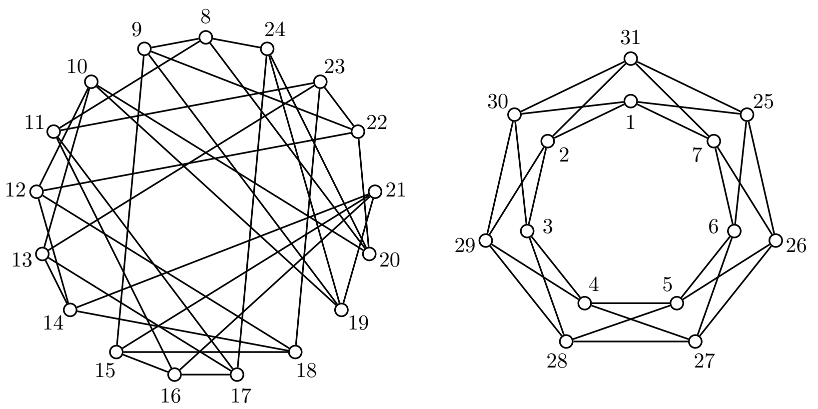

Figure 3: A 4-regular distance magic graph on 31 vertices with magic constant $k = 64$.

3 Conclusion

We expect that a similar approach as in the proof of Lemma 2.1 can be used for other regularities r as well. Unfortunately, considering only odd orders (all even orders are solved by Proposition 1.1 in [4]) the order of graphs to find for the "basis step" increases along with the value r. In general for odd n we need to find some r-regular graph on at least $r+3$ vertices and show that for all smaller orders no such graph exists which makes the complete classification by hand impossible for higher r.

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(Received 27 July 2011; revised 18 May 2012)