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Vertex identifying codes for the n-dimensional lattice

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Abstract

An r-identifying code on a graph G is a set $C \subset V(G)$ such that for every vertex in $V(G)$, the intersection of the radius-r closed neighborhood with C is nonempty and different. Here, we provide an overview on codes for the n-dimensional lattice, discussing the case of 1-identifying codes, constructing a sparse code for the 4-dimensional lattice as well as showing that for fixed n, the minimum density of an r-identifying code is $\Theta(1/r^{n-1})$.

1 Introduction

Vertex identifying codes were introduced by Karpovsky, Chakrabarty, and Levitin in [7] as a way to help with fault diagnosis in multiprocessor computer systems. Amongst the many results in that paper, an interesting result is that if $n = 2^k - 1$ for some integer k, we can find a code of optimal density for the n-dimensional lattice by using a Hamming code (see for instance [9] for a discussion of Hamming codes). Denote by $\mathcal{D}(G, r)$ the minimum possible density of an r-identifying code (if it exists) for a graph G. Let L_n denote the n-dimensional lattice. In [8], we present a slight generalization of the proof of Theorem 6 in [7] to get Theorem 1.

Theorem 1 ([8]) *Let D be a dominating set for the n-dimensional hypercube, then $\mathcal{D}(L_n, 1) \leq |D|/2^n$.*

The proof of this comes from replacing Hamming codes (which are already dominating sets) with the more general dominating sets to get bounds in the case that $n \neq 2^k - 1$. For small values of n, we use Table 6.1 of [4] to get good bounds in Figure 1.

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				$\mathcal{D}(L_1, 1)$	$=$	$1/2$		
				$\mathcal{D}(L_2, 1)$	$=$	$7/20$ [1]		
				$\mathcal{D}(L_3, 1)$	$=$	$1/4$		
$.2$	$=$	$1/5$	$\leq$	$\mathcal{D}(L_4, 1)$	$\leq$	$2/9$	$\approx$	$.222$ [Theorem 4]
$.167$	$\approx$	$1/6$	$\leq$	$\mathcal{D}(L_5, 1)$	$\leq$	$7/32$	$\approx$	$.219$ [Theorem 1]
$.143$	$\approx$	$1/7$	$\leq$	$\mathcal{D}(L_6, 1)$	$\leq$	$3/16$	$\approx$	$.188$ [Theorem 1]
				$\mathcal{D}(L_7, 1)$	$=$	$1/8$		
$.111$	$\approx$	$1/9$	$\leq$	$\mathcal{D}(L_8, 1)$	$\leq$	$1/8$	$=$	$.125$
$.1$	$=$	$1/10$	$\leq$	$\mathcal{D}(L_9, 1)$	$\leq$	$31/256$	$\approx$	$.121$ [Theorem 1]
$.091$	$\approx$	$1/11$	$\leq$	$\mathcal{D}(L_{10}, 1)$	$\leq$	$15/128$	$\approx$	$.117$ [Theorem 1]

Figure 1: A table of bounds of densities of codes for small values of n. All bounds not cited are due to [7].

The result for L_4 is proven in Section 2. For larger values of n we use this theorem in conjunction with a result of Kabatyanskiĭ and Panchenko[6] in which they show that there is a constant b such that for sufficiently large n, there is a dominating set of H_n of size at most

$$\left(1 + \frac{b \ln \ln n}{\ln n}\right) \frac{2^n}{n+1}.$$

This gives a good asymptotic bound.

Corollary 2 *There is a constant b such that for sufficiently large n:*

$$\frac{1}{n+1} \leq \mathcal{D}(L_n, 1) \leq \left(1 + \frac{b \ln \ln n}{\ln n}\right) \frac{1}{n+1}.$$

Finally, in Section 3, we prove both an upper and lower bound for $\mathcal{D}(L_n, r)$ and show:

Theorem 3 *For fixed n, $\mathcal{D}(L_n, r) = \Theta(1/r^{n-1})$ as $r \to \infty$.*

Given a connected, undirected graph $G = (V, E)$, we define $B_r(v)$, called the ball of radius r centered at v to be

$$B_r(v) = \{u \in V(G) : d(u, v) \leq r\}.$$

We call any nonempty subset C of $V(G)$ a *code* and its elements *codewords*. A code C is called *r-identifying* if it has the properties:

1. $B_r(v) \cap C \neq \emptyset$ for all v

2. $B_r(u) \cap C \neq B_r(v) \cap C$, for all $u \neq v$

When $r = 1$ we simply call C an identifying code. When C is understood, we define $I_r(v) = I_r(v, C) = B_r(v) \cap C$. We call $I_r(v)$ the identifying set of v. If $I_r(u) \neq I_r(v)$ for some $u \neq v$, the we say u and v are *distinguishable*. Otherwise, we say they are *indistinguishable*.

We formally define the n-dimensional lattice $L_n = (V, E)$ where

$$V = \mathbb{Z}^n, \qquad E = \left\{ \{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\} : \sum_{i=1}^{n} |x_i - y_i| = 1 \right\}.$$

The density of a code C for a finite graph G is defined as $|C|/|V(G)|$. Let Q_m denote the set of vertices $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ with $|x_i| \leq m$ for all $1 \leq i \leq n$. We define the density D of a code C in L_n similarly to how it is defined in [3] by

$$D = \limsup_{m \to \infty} \frac{|C \cap Q_m|}{|Q_m|}.$$

2 The 4-dimensional case

The king grid, G_K, is defined to be the graph on vertex set $\mathbb{Z} \times \mathbb{Z}$ with edge set $E_K = \{\{u, v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1)\}\}$.

Theorem 4 $\mathcal{D}(L_4, 1) \leq 2/9$.

Proof. The idea of our proof is to take a code for the king grid and copy it to two-dimensional cross-sections of L_4–shifting it "up and to the right" when moving in the x_3 direction and "up and to the left" when moving in the x_4 direction.

Let C be the identifying code of density $2/9$ for the king grid given by Charon, Hudry, and Lobstein in [3]. For the remainder of this proof, let $B_1^G(v)$ denote the ball of radius 1 in the graph G and likewise, let $I_1^G(v)$ denote the identifying set of v in G.

Let $v \in V(L_4)$. Since $(1, 0, 0, 0), (0, 1, 0, 0), (1, 1, 1, 0)$, and $(1, -1, 0, 1)$ are linearly independent, we may uniquely write $v = (x, y, 0, 0) + i(1, 1, 1, 0) + j(1, -1, 0, 1)$. Next, we define $\varphi : V(L_4) \to V(G_K)$ by $\varphi(v) = (x, y)$ and then define

$$C' = \{v \in V(L_4) : \varphi(v) \in C\}.$$

Fixing, i and j, we see that C' consists of isomorphic copies of C and so C' has the same density as C.

It is easy to check that $\varphi(B_1^{L_4}(v)) = B_1^{G_K}((x, y))$. For instance, $\varphi(v + (1, 0, 0, 0)) = (x + 1, y) \in B_1^{G_K}(v)$ and $\varphi(v + (0, 0, 1, 0)) = \varphi((x - 1, y - 1, 0, 0) + (i + 1)(1, 1, 1, 0) + j(1, -1, 0, 1)) = (x - 1, y - 1) \in B_1^{G_K}(x, y)$. This shows two things. First, each vertex has a non-empty identifying set, since $|I_1^{L_4}(v)| = |I_1^{G_K}(\varphi(v))| \geq 1$. Secondly, it shows that if $\varphi(u) \neq \varphi(v)$, then u and v are distinguishable. Hence, we only need to distinguish between vertices where $\varphi(u) = \varphi(v)$.

Without loss of generality, let $u = (x, y, 0, 0)$ and $v = (x, y, 0, 0) + i(1, 1, 1, 0) + j(1, -1, 0, 1)$ and so

$$d(u, v) = |i + j| + |i - j| + |i| + |j|.$$

If i and j are both non-zero, then either $|i+j|$ or $|i-j|$ is at least 1 and so $d(u, v) \geq 3$. If $j = 0$, then $d(u, v) = 3|i| \geq 3$ and likewise if $i = 0$ then $d(u, v) \geq 3|j| \geq 3$. Since $d(u, v) \geq 3$ in all cases, we only need to consider $I_1^{L_n}(v)$. It is nonempty and does not intersect with $B_1^{L_n}(u)$. Thus, u and v are distinguishable, completing the proof. □

3 General Bounds and Construction

We finally wish to produce some general bounds for r-identifying codes on the L_n. We start with a lower bound proof, in the style of Charon, Honkala, Hudry and Lobstein[2]. First, we define $b_k^{(n)} = |B_k(v)|$ for $v \in V(L_n)$.

Theorem 5 *The minimum density of an r-identifying code for L_n is at least*

$$\mathcal{D}(L_n, r) \geq \frac{(n-1)!\lceil \log_2(2n+1) \rceil}{2^{n+1}r^{n-1} + p_{n-2}(r)}$$

where $p_{n-2}(r)$ is a polynomial in r of degree no more than $n - 2$.

Proof. Let $v \in V(L_n)$ and $u_1, u_2, \ldots, u_{2n}$ be its neighbors. If $d(v, x) > r + 1$, then it is easy to see that $d(u_i, x) \geq r + 1$ for all i. Likewise, it is easy to check that if $d(v, x) \leq r - 1$, then $d(u_i, x) \leq r$ for all i. In other words, all vertices outside of $B_{r+1}(v)$ are not in $B_r(s)$ for any $s \in S = \{v, u_1, u_2, \ldots, u_{2n}\}$ and all vertices inside of $B_{r-1}(v)$ are in $B_r(s)$ for all $s \in S$.

Next, let C be an r-identifying code for L_n. For $s, s' \in S$ with $s \neq s'$, we must have $I_r(s) \triangle I_r(s') \subset B_{r+1}(v) \setminus B_{r-1}(v)$. Let $K(s) = I_r(s) \cap (B_{r+1}(v) \setminus B_{r-1}(v))$. We claim for $K(s) \neq K(s')$. Suppose not. Then $I_r(s) = K(s) \cup (C \cap B_{r-1}(v)) = I_r(s')$ and so they are not distinguishable. Hence, $K(s)$ must be distinct for each $s \in S$. Since the minimum number of elements of a set to produce $2n + 1$ distinct subsets is $\lceil \log_2(2n + 1) \rceil$, there must be $\lceil \log_2(2n + 1) \rceil$ codewords in $B_{r+1}(v) \setminus B_{r-1}(v)$. We refer to the methods used by Charon, Honkala, Hudry and Lobstein [2] to show this gives the lower bound:

$$\frac{\lceil \log_2(2n+1) \rceil}{b_{r+1}^{(n)} - b_{r-1}^{(n)}}.$$

It is easy to check that $b_r^{(n)}$ is the number of solutions in integers to

$$|x_1| + |x_2| + \cdots + |x_n| \leq r \tag{1}$$

and so $b^{(n)}_{r+1} - b^{(n)}_{r-1}$ is the number of solutions to

$$|x_1| + |x_2| + \cdots + |x_n| = k$$

where $k = r$ or $k = r + 1$. Since the number of solutions to $x_1 + x_2 + \cdots + x_n = k$ is known to be $\binom{n+k-1}{n-1}$, this gives us an upper bound

$$\begin{aligned} b^{(n)}_{r+1} - b^{(n)}_{r-1} &\leq 2^n \left(\binom{n+r-1}{n-1} + \binom{n+r}{n-1} \right) \\ &\leq 2^n \left(\frac{(r+n-1)^{n-1}}{(n-1)!} + \frac{(r+n)^{n-1}}{(n-1)!} \right) \\ &= \frac{2^{n+1} r^{n-1} + p_{n-2}(r)}{(n-1)!} \end{aligned}$$

which comes from choosing each term to be either positive or negative and then using a standard binomial inequality. Plugging this in gives us the result described in the theorem. □

Theorem 6 *If n is odd, $0 \leq j < n+1$, $r \geq n+2$, and $r \equiv j \pmod{(n+2)}$ then*

$$\mathcal{D}(L_n, r) \leq \frac{(n+2)^{n-1}}{2^n (r-j)^{n-1}}.$$

If n is even, $0 \leq j < (n+2)/2$, $r \geq (n+2)/2$, and $r \equiv j \pmod{(n+2)/2}$ then

$$\mathcal{D}(L_n, r) \leq \frac{(n+2)^{n-1}}{2^n (r-j)^{n-1}}.$$

Proof. Let $2r_0$ be divisible by $n+2$ and let $k = 2r_0/(n+2)$. We wish to find an r-identifying code for $r \geq r_0$. We define a code

$$C = \{(kx_1, kx_2, \ldots, kx_{n-1}, \ell) : x_1 + x_2 + \cdots + x_{n-1} \equiv 1 \pmod{2}\}.$$

Further, let

$$S = \{(kx_1, kx_2, \ldots, kx_{n-1}, \ell) : x_1 + x_2 + \cdots + x_{n-1} \equiv 0 \pmod{2}\}.$$

C will be our code and S will serve as a set of reference points which we will use later.

We first wish to calculate the density $C \cup S$. This is simply a tiling of $\mathbb{Z}^n$ by the region $[0, k-1]^{n-1} \times \{0\}$ which has only a single codeword in it. Hence, the density of $C \cup S$ is $1/k^{n-1} = (n+2)^{n-1}/(2^{n-1} r_0^{n-1})$. Then C is half this density. Then, if $r \geq r_0$ we write $r - r_0 = j \geq 0$. Replacing r_0 with $r - j$ gives us the same value as in the statement of the theorem.

Next, we wish to show that C is an r-identifying code for $r \geq r_0$. Let $e^{(i)}$ represent the vector with a 1 in the ith coordinate and a 0 in all other coordinates. For any vertex u and $1 \leq j \leq n$, let u_j denote the value of the jth coordinate of u.

For $s \in S$, we define the *corners* of s to be the codewords c of the form $c = s \pm ke^{(i)}$ for some $1 \leq i \leq n-1$.

The remainder of the proof consists of three steps:

1. Each vertex $v \in V(G)$ has distance at most $nk/2$ from some $s \in S$ and v has distance at most r from each of the corners of s (in addition, this shows that $I_r(v)$ is nonempty).
2. If $v = (\mathbf{v}, \ell)$, we can uniquely determine ℓ from $I_r(v)$. Furthermore, if $c = (\mathbf{c}, \ell) \in I_r(v)$, we can determine $d(v, c)$.
3. If $v = (v_1, \ldots, v_{n-1}, \ell)$, we can uniquely determine v_i from $I_r(v)$ for each i. Thus, v is distinguishable from all other vertices in the graph.

Step 1: Let $v = (v_1, v_2, \ldots, v_{n-1}, \ell)$. Without loss of generality, we may assume that $(v_1, v_2, \ldots, v_{n-1}) \in [0, k]^{n-1}$. For $i = 1, 2, \ldots, n-2$ define

$$a_i = \begin{cases} 0 & \text{if } v_i \leq k/2 \\ k & \text{if } v_i > k/2 \end{cases}.$$

We then see that $|v_i - a_i| \leq k/2$ in either case. Now consider the vertices $(a_1, a_2, \ldots, a_{n-2}, 0, \ell)$ and $(a_1, a_2, \ldots, a_{n-2}, k, \ell)$. One of these is in S. Let $a_{n-1} = 0$ if the former is in S and $a_{n-1} = k$ if the latter is in S. Then $|v_{n-1} - a_{n-1}| \leq k$. Hence we have

$$\begin{aligned} d(v, (a_1, a_2, \ldots, a_{n-2}, a_{n-1}, \ell)) &= |v_{n-1} - a_{n-1}| + \sum_{i=1}^{n-2} |v_i - a_i| \\ &\leq k + (n-2)k/2 = nk/2. \end{aligned}$$

Let c be a corner of $s = (a_1, a_2, \ldots, a_{n-2}, a_{n-1}, \ell)$. Then

$$d(v, c) \leq d(v, s) + d(s, c) \leq nk/2 + k = (n+2)k/2 = r_0 \leq r.$$

Step 2: Next, we need to determine the last coordinate of v. Write $v = (\mathbf{v}, \ell)$. Suppose that $c = (\mathbf{c}, \ell_0) \in I_r(v)$. We then see that $(\mathbf{c}, \ell) \in I_r(v)$ since $d(v, (\mathbf{c}, \ell)) \leq d(v, c)$. Writing $d(v, (\mathbf{c}, \ell)) = d_1 \leq r$, then we see that $(\mathbf{c}, \ell \pm j) \in I_r(v)$ for $j = 0, 1, \ldots, r - d_1$. Hence, these codewords form a path of length $2(r - d_1) + 1$. Thus, if $\ell_1 = \min\{j : (\mathbf{c}, j) \in I_r(v)\}$ and $\ell_2 = \max\{j : (\mathbf{c}, j) \in I_r(v)\}$, it follows that

$$\ell = \frac{\ell_1 + \ell_2}{2}.$$

Furthermore, this tells us once we know ℓ, we can determine the distance between v and c to be $r - (\ell_2 - \ell)$.

Step 3: Finally, from Step 1 we know that there is some vertex $s \in S$ such that the codewords $s \pm ke^{(i)} \in I_r(v)$ for each i, $1 \leq i \leq n-1$. Thus, for each i we are guaranteed that there are $m \geq 2$ codewords $c^{(0)}, \ldots, c^{(m-1)}$ such that $c^{(j)} = c^{(0)} + 2kje^{(i)}$ and $c^{(j)} \in I_r(v)$ for each j.

Now let

$$D = \sum_{\substack{p=1 \\ p \neq i}}^{n-1} |v_p - c_p^{(0)}|.$$

We then see that $d(v, c^{(j)}) = |v_i - c_i^{(j)}| + D$ which is minimized by minimizing $|v_i - c_i^{(j)}|$. Furthermore, the expression $|v_i - x|$ is unimodal and so the two smallest values of $|v_i - c_i^{(j)}|$ must happen for consecutive integers and they must be amongst our aforementioned m codewords. Let $a = c^{(\ell)}$ and $b = c^{(\ell+1)}$ be these codewords. It is easy to check that $a_i \leq v_i \leq b_i$ by considering evenly spaced point plotted along the graph of $f(x) = |v_i - x|$.

This gives

$$\begin{aligned} d(v, a) &= v_i - a_i + D \\ d(v, b) &= b_i - v_i + D \end{aligned}$$

Since a and b are codewords, the distances listed above are all known quantities from Step 2. Subtracting the second line from the first and solving for v_i gives:

$$v_i = \frac{d(v, a) - d(v, b) + a_i + b_i}{2}.$$

Since these are all known quantities, we can compute v_i, completing step 3. Finally, we get the values described in the theorem by taking r_0 to be the largest integer smaller than r satisfying the condition that $2r_0/(n+2)$ is an integer, completing the proof. □

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4 Conclusions

It is worth noting that the lower bound given in Theorem 5 can only be evaluated as $r \to \infty$ and not as $n \to \infty$ since the polynomial in the denominator is a polynomial in

r, but the coefficients depend on n. However, for fixed n we can make a comparison of the bounds by taking the ratio of the upper bound to the lower bound. This gives:

$$\begin{aligned}
& \frac{(n+1)^{n-1}}{2^n r^{n-1}} \Bigg/ \frac{(n-1)!\lceil \log_2(2n+1)\rceil}{2^{n+1}r^{n-1}+o(r^{n-1})} \\
= \quad & \frac{2^{n+1}r^{n-1}+o(r^{n-1})}{2^n r^{n-1}} \cdot \frac{(n+1)^{n-1}}{(n-1)!\lceil \log_2(2n+1)\rceil} \\
\approx \quad & (2+o(1)) \cdot \frac{(n+1)^{n-1}}{(n-1)^{n-1}} \cdot \frac{e^{n-1}}{\sqrt{2\pi n}\lceil \log_2(2n+1)\rceil} \\
\approx \quad & \frac{2e^{n+1}}{\sqrt{2\pi n}\lceil \log_2(2n+1)\rceil}
\end{aligned}$$

and so our lower bound differs from our upper bound by slightly less than a multiplicative factor of e^n when $r \gg n \gg 0$.

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