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On the total edge irregularity strength of hexagonal grid graphs

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Abstract

An edge irregular total k-labeling of a graph $G = (V, E)$ is a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ such that the total edge-weights $wt(xy) = \phi(x) + \phi(xy) + \phi(y)$ are different for all pairs of distinct edges. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G. In this paper, we determined the exact values of the total edge irregularity strength of hexagonal grid graphs.

1 Introduction

We consider finite undirected graphs $G = (V, E)$ without loops and multiple edges with vertex-set $V(G)$ and edge-set $E(G)$, where $|V(G)| = p$ and $|E(G)| = q$. The degree of a vertex x is the number of edges that have x as an endpoint, and the set of neighbors of x is denoted by $N(x)$.

By a *labeling* we mean any mapping that carries a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V \cup E$ then we call the labeling a *total labeling*. Thus for an edge k-labeling $\sigma : E(G) \to \{1, 2, \dots, k\}$, the associated vertex-weight of a vertex $x \in V(G)$

is

$$w_\sigma(x) = \sum_{y \in N(x)} \sigma(xy)$$

and for a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$, the associated edge-weight is

$$wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y).$$

Chartrand et al. in [10] introduced an edge k-labeling of a graph G such that $w(x) \neq w(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

This parameter has attracted much attention ([3], [4], [8], [11], [12], [13], [15]). Finding the irregularity strength of a graph seems to be hard even for graphs with a simple structure; see a survey article by Lehel [18].

Motivated by these papers and by a book of Wallis [21], Bača et al. in [5] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labelings.

A total k-labeling φ is defined to be an *edge irregular total labeling* of a graph G if for every two different edges xy and $x'y'$ of G one has $wt_\varphi(xy) \neq wt_\varphi(x'y')$. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $\mathrm{tes}(G)$.

Let φ be an edge irregular total k-labeling of $G = (V, E)$. Since $3 \leq wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \leq 3k$ for every edge $xy \in E(G)$, we have $|E(G)| \leq 3k - 2$ which implies $\mathrm{tes}(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil$.

If $x \in V(G)$ is a fixed vertex of maximum degree $\Delta(G)$, then there is a range of $2k - 1$ possible weights $\varphi(x) + 2 \leq wt_\varphi(xy) \leq \varphi(x) + 2k$ for the $\Delta(G)$ edges $xy \in E(G)$ incident with x which implies $\mathrm{tes}(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

So we have

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}. \tag{1}$$

The authors of [5] determined the exact value of the total edge irregularity strength for certain families of graphs, namely paths, cycles, stars, wheels and friendship graphs. They posed the problem of determining the total edge irregularity strength of trees. Recently Ivančo and Jendroľ [14] proved that for any tree T the total edge irregularity strength is equal to its lower bound, i.e.

$$\mathrm{tes}(T) = \max\left\{ \left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil \right\}.$$

Moreover, they posed the following conjecture.

Conjecture 1. [14] *Let $G = (V, E)$ be an arbitrary graph different from K_5 and with maximum degree $\Delta(G)$. Then*

$$\mathrm{tes}(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}.$$

Note that for K_5 the maximum of the lower bounds is 4, while $\mathrm{tes}(K_5) = 5$ (see Theorem 7 in [5]). Conjecture 1 has been verified for complete graphs and complete bipartite graphs by Jendroľ, Miškuf and Soták in [16] and [17], and for the Cartesian product of two paths by Miškuf and Jendroľ in [19]. Brandt et al. in [9] proved Conjecture 1 for large dense graphs, i.e. for graphs G with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$.

Motivated by the papers [1], [2], [11], [19] and [20], we investigate the total edge irregularity strength of the hexagonal grid (honeycomb) graph, denoted by H_n^m. This paper adds further support to Conjecture 1 by demonstrating that the hexagonal grid (honeycomb) graph has total edge irregularity strength equal to $\left\lceil (|E(H_n^m)|+2)/3 \right\rceil$.

In the next section, we determine the exact value of total edge irregularity strength of the hexagonal grid.

2 Total edge irregularity strength of the hexagonal grid

The graphs considered here will be finite. For $n \geq 1$, $m \geq 1$, we denote by H_n^m the hexagonal grid (honeycomb) defined in [6], [7], the planar map labeled as in Figure 1 with m rows and n columns of hexagons. The symbols $V(H_n^m)$, $E(H_n^m)$ and $F(H_n^m)$ will denote the vertex set, the edge set and the face set of H_n^m, respectively. The symbol $|A|$ will denote the cardinality of the set A. The face set $F(H_n^m)$ contains $|F(H_n^m)| - 1$ 6-sided faces and one external infinite face. Also $|V(H_n^m)| = 2mn + 2(m+n)$, $|E(H_n^m)| = |V(H_n^m)| + mn - 1$ and $|F(H_n^m)| = mn + 1$.

In this paper, we deal with the hexagonal grid (honeycomb) graph H_n^m for all n, m. In [5] it is proved that $\mathrm{tes}(G) \geq \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}$. As the maximum degree $\triangle(H_n^m) = 3$, this implies that $\mathrm{tes}(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil$.

To show that $\left\lceil \frac{3nm+2(m+n)+1}{3} \right\rceil$ is an upper bound for the $\mathrm{tes}(H_n^m)$, we describe an edge irregular total $\left\lceil \frac{3nm+2(m+n)+1}{3} \right\rceil$-labeling for H_n^m.

For our convenience, we split the edge set of H_n^m into mutually disjoint subsets A_i and B_i, where

$$\begin{aligned} A_i &= \{a_{i,j}a_{i,j+1} : 1 \leq j \leq 2m\} \text{ for } i = 1 \text{ and } i = n+1; \\ A_i &= \{a_{i,j}a_{i,j+1} : 1 \leq j \leq 2m+1\} \text{ for } 2 \leq i \leq n; \\ B_i &= \{a_{i,j}a_{i+1,j} : 1 \leq j \leq m+1, i \text{ and } j \text{ have same parity}\}, \text{ for } 1 \leq i \leq n. \end{aligned}$$

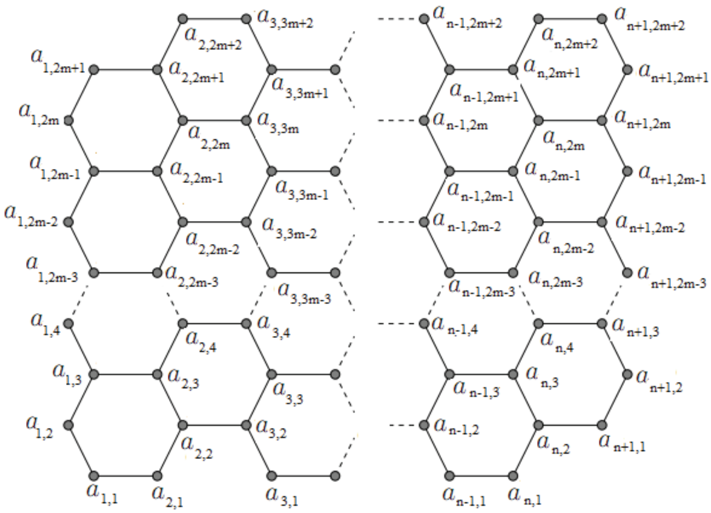

Figure 1: The honeycomb H_n^m

Clearly, $|A_i| = 2m$ for $i = 1, n+1$ and $|A_i| = 2m+1$ for $2 \leq i \leq n$, $|B_i| = m+1$ and $\bigcup\limits_{i=1}^{n+1} A_i \cup \bigcup\limits_{i=1}^{n} B_i = E(H_n^m)$.

Because the graphs H_n^m and H_m^n are isomorphic, it is sufficient to prove the statement for $m \leq n$ or $n \leq m$.

In next theorem, we discuss the total edge irregularity strength of H_n^m, if $n \geq 3, m = 2$. It is easy to verify that $\text{tes}(H_n^m) = \left\lceil \frac{8n+5}{3} \right\rceil$.

Theorem 1. *Let* $n \geq 3, m = 2$*; then* $\text{tes}(H_n^m) = \left\lceil \frac{8n+5}{3} \right\rceil$.

Proof. Let $k = \left\lceil \frac{8n+5}{3} \right\rceil$. For $m = 2$, we define the function ϕ_1 in the following way:

$$\phi_1(a_{i,j}) = \begin{cases} \lfloor \frac{j+1}{2} \rfloor, & \text{if } i = 1,\ 1 \leq j \leq 5; \\ 1 + \lfloor \frac{j+1}{2} \rfloor, & \text{if } i = 2,\ 1 \leq j \leq 6; \\ 3i - 5 + \lfloor \frac{j+1}{2} \rfloor, & \text{if } 3 \leq i \leq n-1,\ 1 \leq j \leq 6; \\ k - 3 + \lfloor \frac{j+1}{2} \rfloor, & \text{if } i = n,\ 1 \leq j \leq 6; \\ k, & \text{if } i = n+1,\ 1 \leq j \leq 5. \end{cases}$$

Observe that under the vertex labeling ϕ_1 the weights of the edges:

(i) from the set A_1 admit the consecutive integers from 2 to 5;

(ii) from the set A_i receive the consecutive integers from $6i-8$ to $6i-4$ for $2 \leq i \leq n-1$;

(iii) from the set A_n receive the consecutive integers from $2k-4$ to $2k$;

(iv) from the set A_{n+1} receive the constant integer $2k$, For every B_i, with i and j having the same parity;

(v) from the set B_1, make an arithmetic progression of positive integers with difference two from 3 to 7;

(vi) from the set B_i, make an arithmetic progression of positive integers with difference two from $6i-5$ to $6i-1$ for $2 \leq i \leq n-2$;

(vii) from the set B_{n-1} make an arithmetic progression of positive integers with difference two from $k+3n-9$ to $k+3n-5$;

(viii) from the set B_n receive the consecutive integers from $2k-2$ to $2k$.

To complete the labeling to a total one we label the edges of the graph H_n^2 as follows:
First we define the edge labeling for the edges $(a_{i,j}a_{i,j+1})$.

$$\phi_1(a_{i,j}a_{i,j+1}) = \begin{cases} 1, & \text{if } i=1,\ 1 \leq j \leq 4, \\ 2i+2, & \text{if } 2 \leq i \leq n-1, 1 \leq j \leq 5, \\ 8n-2k-2, & \text{if } i=n, 1 \leq j \leq 5, \\ 8n-2k+1+j, & \text{if } i=n+1, 1 \leq j \leq 4, \end{cases}$$

Now define the edge labeling for the edges $(a_{i,j}a_{i+1,j})$, with i and j having the same parity.

$$\phi_1(a_{i,j}a_{i+1,j}) = \begin{cases} 5-\frac{j+1}{2}, & \text{if } i=1,\ 1 \leq j \leq 5, \\ 2i+5-\lfloor\frac{j+1}{2}\rfloor, & \text{if } 2 \leq i \leq n-2,\ 1 \leq j \leq 6, \\ 5n-k+1-\lfloor\frac{j+1}{2}\rfloor, & \text{if } i=n-1,\ 1 \leq j \leq 6, \\ 8n-2k+1, & \text{if } i=n,\ 1 \leq j \leq 6. \end{cases}$$

Observe that under the vertex labeling and edge labeling ϕ_1, the total weights of the edges are defined as follows:

(i) from the set A_1 admit the consecutive integers from 3 to 6;

(ii) from the set A_i receive the consecutive integers from $8i-6$ to $8i-2$ for $2 \leq i \leq n-1$;

(iii) from the set A_n receive the consecutive integers from $8n-6$ to $8n-2$;

(iv) from the set A_{n+1} receive the consecutive integers from $8n+2$ to $8n+5$.

For every B_i, with i and j have same parity, the total weight of the edges are defined as follows:

(v) from the set B_1 receive the consecutive integers from 7 to 9;

(vi) from the set B_i receive the consecutive integers from $8i-1$ to $8i+1$ for $2 \leq i \leq n-2$;

(vii) from the set B_{n-1} receive the consecutive integers from $8n-9$ to $8n-7$;

(viii) from the set B_n receive the consecutive integers from $8n-1$ to $8n+1$.

Now, it is a routine matter to verify that all vertex and edge labels are at most k and the edge weights of the edges from the sets A_i and B_i are pairwise distinct. Thus the resulting labeling is the desired edge irregular k-labeling. □

Theorem 2. *Let $m \geq 1$, $n \geq 3$ be positive integers. Then the total edge irregularity strength of the hexagonal grid (honeycomb) graph H_n^m is $\left\lceil \frac{3nm+2(m+n)+1}{3} \right\rceil$.*

Proof. Let m, n be two positive integers and $k = \left\lceil \frac{3nm+2(m+n)+1}{3} \right\rceil$. For $n \geq 3$ and $m = 2$ the assertion follows from Theorem 1. Now for $n \geq 3$ and $m \neq 2$, we define the function ϕ_2 as follows:

$$\phi_2(a_{i,j}) = \begin{cases} \lfloor \frac{j+1}{2} \rfloor, & \text{if } i = 1,2;\ \ 1 \leq j \leq 2m+2 \\ (m+1)(i-2) + \lfloor \frac{j+1}{2} \rfloor, & \text{if } 3 \leq i \leq n-1, 1 \leq j \leq 2m+2 \\ k-m-1+\lfloor \frac{j+1}{2} \rfloor, & \text{if } i = n, 1 \leq j \leq 2m+2 \\ k, & \text{if } i = n+1, 1 \leq j \leq 2m+1 \end{cases}$$

Observe that, under the vertex labeling ϕ_2, the weights of the edges:

(i) from the set A_1 admits the consecutive integers from 2 to $2m+1$;

(ii) from the set A_i receive the consecutive integers from $2(m+1)(i-2)+2$ to $2(m+1)(i-1)$ for $2 \leq i \leq n-1$;

(iii) from the set A_n receive the consecutive integers from $2k-2m$ to $2k$;

(iv) from the set A_{n+1} receive the constant integer $2k$.

For every B_i, with i and j having the same parity, the weight of the edges are defined as follows:

(v) from the set B_1 receive the consecutive even integers from 2 to $2(m+1)$;

(vi) from the set B_i, make an arithmetic progression of positive integers with difference two from $2+(m+1)(2i-3)$ to $(m+1)(2i-1)$ for $2 \le i \le n-2$;

(vii) from the set B_{n-1}, make an arithmetic progression of positive integers with difference two from $k+n(m+1)-4m-2$ to $k+n(m+1)-2m-2$;

(viii) from the set B_n receive the consecutive integers from $2k-m$ to $2k$.

To complete the labeling to a total one, we label the edges of the graph H_n^m as follows:
First we define the edge labeling for the edges $(a_{i,j}a_{i,j+1})$

$$\phi_2(a_{i,j}a_{i,j+1}) = \begin{cases} 1, & \text{if } i=1,\ 1 \le j \le 2m, \\ 3m+2+m(i-2), & \text{if } 2 \le i \le n-1, 1 \le j \le 2m+1, \\ m(3n-1)-2k+2n, & \text{if } i=n, 1 \le j \le 2m+1, \\ n(3m+2)-2k+1+j, & \text{if } i=n+1, 1 \le j \le 2m, \end{cases}$$

Now we define the edge labeling for the edges $(a_{i,j}a_{i+1,j})$ with i and j having the same parity.

$$\phi_2(a_{i,j}a_{i+1,j}) = \begin{cases} 2m+2-\frac{j+1}{2}, & \text{if } i=1,\ 1 \le j \le 2m+1 \\ 4m+3+m(i-2)-\lfloor\frac{j+1}{2}\rfloor, & \text{if } 2 \le i \le n-2, 1 \le j \le 2m+2, \\ n(2m+1)-k+2-\lfloor\frac{j+1}{2}\rfloor, & \text{if } i=n-1, 1 \le j \le 2m+2, \\ n(3m+2)-2k+1, & \text{if } i=n, 1 \le j \le 2m+2, \end{cases}$$

Observe that under the vertex labeling and edge labeling ϕ_2, the total weights of the edges are defined as follows:

(i) from the set A_1 admit the consecutive integers from 3 to $2m+2$;

(ii) from the set A_i receive the consecutive integers from $(3m+2)i-3m$ to $(3m+2)i-m$ for $2 \le i \le n-1$;

(iii) from the set A_n receive the consecutive integers from $3m(n-1)+2n$ to $m(3n-1)+2n$;

(iv) from the set A_{n+1} receive the constant integer $n(3m+2)+2$ to $n(3m+2)+2m+1$.

For every B_i, with i and j having the same parity, the total weights of the edges are defined as follows:

(v) from the set B_1 receive the consecutive integers from $2m+3$ to $3m+3$;

(vi) from the set B_i receive the consecutive integers from $(3m+2)i-m+1$ to $(3m+2)i+1$ for $2 \le i \le n-2$;

(vii) from the set B_{n-1} receive the consecutive integers from $n(3m+2)-4m-1$ to $n(3m+2)-3m-1$;

(viii) from the set B_n receive the consecutive integers from $n(3m+2)-m+1$ to $n(3m+2)+1$.

Now it is not hard to see that all vertex and edge labels are at most k and the edge weights of the edges from the sets A_i and B_i are pairwise distinct. In fact, our total labeling has been chosen in such a way that the edge-weights of the edges from the set $\bigcup_{i=1}^{n+1} A_i \cup \bigcup_{i=1}^{n} B_i$ form a consecutive sequence of integers from 3 to $n(3m+2)+2m+1$. Thus the resulting labeling is the desired edge irregular k-labeling. □

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