AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **53** (2012), Pages 221–235

Extended Skolem-type difference sets

Tina Helms Heather Jordon

Maggie Murray Stephanie Zeppetello

Department of Mathematics
Illinois State University
Normal, IL 61790–4520
U.S.A.

Abstract

A k-extended Skolem-type 5-tuple difference set of order t is a set of t 5-tuples $\{(d_{i,1}, d_{i,2}, d_{i,3}, d_{i,4}, d_{i,5}) \mid i = 1, 2, \ldots, t\}$ such that $d_{i,1} + d_{i,2} + d_{i,3} + d_{i,4} + d_{i,5} = 0$ for $1 \leq i \leq t$ and $\{|d_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq 5\} = \{1, 2, \ldots, 5t+1\} \setminus \{k\}$. In this paper, we give necessary and sufficient conditions on t and k for the existence of a k-extended Skolem-type 5-tuple difference set of order t. We also consider hooked k-extended Skolem-type 5-tuple difference sets of order t and provide necessary and sufficient conditions for their existence. We then show how these k-extended Skolem-type difference sets can be used to find decompositions of circulant and complete graphs of order n into 5-cycles, d-cycles, where d is a divisor of n, Hamilton cycles, and possibly a 1-factor.

1 Introduction

Let $[1, n]$ denote the set $\{1, 2, \ldots, n\}$. A *Skolem sequence of order* t is a sequence $S = (s_1, s_2, \ldots, s_{2t})$ of $2t$ integers satisfying the conditions

(S1) for every $\ell \in [1, t]$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = \ell$;

(S2) if $s_i = s_j = \ell$ with $i < j$, then $j - i = \ell$.

A Skolem sequence of order t provides a partition of the set $[1, 3t]$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$; for example, if $S = (s_1, s_2, \ldots, s_{2t})$ is a Skolem sequence of order t, then $\{(\ell, t + i, t + j) \mid 1 \leq \ell \leq t, s_i = s_j = \ell \text{ with } i < j\}$ is a partition of the set $[1, 3t]$ into t such triples.

Since a Skolem sequence does not exist for every order t, the natural alternative is a hooked Skolem sequence. A *hooked Skolem sequence of order* t is a sequence $HS = (s_1, s_2, \ldots, s_{2t+1})$ of $2t + 1$ integers satisfying conditions (S1) and (S2) above and

(S3) $s_{2t} = 0$.

A hooked Skolem sequence of order t provides a partition of the set $[1, 3t+1] \setminus \{3t\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. The following theorem provides necessary and sufficient conditions for Skolem [25] and hooked Skolem [18] sequences to exist.

Theorem 1.1 (Skolem [25], O'Keefe [18]) *For a positive integer t, a Skolem sequence of order t exists if and only if $t \equiv 0, 1 \pmod 4$, and a hooked Skolem sequence of order t exists if and only if $t \equiv 2,\ 3 \pmod 4$.*

A *Langford sequence of order t and defect d* is a sequence $L = (s_1, s_2, \ldots, s_{2t})$ of $2t$ integers satisfying the conditions

(L1) for every $\ell \in [d, d+t-1]$ there exists exactly two elements $s_i, s_j \in L$ such that $s_i = s_j = \ell$, and

and (S2) above. In a similar manner to which a Skolem sequence of order t provides a partition of the set $[1, 3t]$ into triples, a Langford sequence of order t and defect d provides a partition of the set $[d, d+3t-1]$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. A *hooked Langford sequence of order t and defect d* is a sequence $L = (s_1, s_2, \ldots, s_{2t+1})$ of $2t+1$ integers satisfying conditions (L1), (S2) and (S3) above. A hooked Langford sequence of order t and defect d provides a partition of the set $[d, d+3t] \setminus \{d+3t-1\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. Clearly, a (hooked) Langford sequence with defect 1 is a (hooked) Skolem sequence. The following theorem gives necessary and sufficient conditions for the existence of Langford and hooked Langford sequences [24].

Theorem 1.2 (Simpson, [24]) *For positive integers d and t, there exists a Langford sequence of order t and defect d if and only if*

(1) $t \geq 2d - 1$, and

(2) d is odd and $t \equiv 0, 1 \pmod 4$, or d is even and $t \equiv 0, 3 \pmod 4$; and

there exists a hooked Langford sequence of order t and defect d if and only if

(1) $t(t - 2d + 1) + 2 \geq 0$, and

(2) d is odd and $t \equiv 2, 3 \pmod 4$, or d is even and $t \equiv 1, 2 \pmod 4$.

For positive integers k and t with $k \leq 2t + 1$, a *k-extended Skolem sequence of order t* is a sequence $ES_k = (s_1, s_2, \ldots, s_{2t+1})$ of $2t+1$ integers satisfying conditions (S1) and (S2) above and

(E1) $s_k = 0$.

Clearly, a k-extended Skolem sequence of order t provides a partition of the set $[1, 3t+1] \setminus \{t+k\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$, and a $(2t)$-extended Skolem sequence is a hooked Skolem sequence while a $(2t+1)$-extended Skolem sequence is a Skolem sequence. Since a k-extended Skolem sequence does not exist for every order t, the natural alternative is a hooked k-extended Skolem sequence. For positive integers k and t with $k < 2t+1$, a *hooked k-extended Skolem sequence of order t* is a sequence $EHS_k = (s_1, s_2, \ldots, s_{2t+2})$ of $2t+2$ integers satisfying conditions (S1), (S2), and (E1) above, and

(E2) $s_{2t+1} = 0$.

A hooked k-extended Skolem sequence provides a partition of the set $[1, 3t+2] \setminus \{t+k, 3t+1\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$. The following theorem provides necessary and sufficient conditions for k-extended and hooked k-extended Skolem sequences to exist. The extended case is handled in [2] while the hooked case is given in [17] as a consequence of a more general result.

Theorem 1.3 (Baker [2], Linek and Shalaby [17]) *For positive integers k and t with $k \leq 2t+1$, a k-extended Skolem sequence of order t exists if and only if k is odd and $t \equiv 0, 1 \pmod 4$, or k is even and $t \equiv 2, 3 \pmod 4$. For positive integers k and t with $k < 2t+1$,a hooked k-extended Skolem sequence of order t exists if and only if k is even and $t \equiv 0, 1 \pmod 4$, or k is odd and $t \equiv 2, 3 \pmod 4$.*

For positive integers k and t with $k \leq t$, a *near-Skolem sequence of order t and defect k* is a sequence $NS_k = (s_1, s_2, \cdots, s_{2t-2})$ of $2t-2$ integers satisfying the conditions

(N1) for every $\ell \in [1, t] \setminus \{k\}$, there exist unique $s_i, s_j \in S$ with $s_i = s_j = \ell$

and (S2) above. A *hooked near-Skolem sequence of order t and defect k*, with $k \leq t$, is a sequence $HNS_k = (s_1, s_2, \cdots, s_{2t-1})$ of $2t-1$ integers satisfying (N1), (S2), and

(N2) $s_{2t-2} = 0$.

Necessary and sufficient conditions for near-Skolem and hooked near-Skolem sequences were given in [23].

Theorem 1.4 (Shalaby [23]) *For integers $t \geq k \geq 1$, a near-Skolem sequence of order t and defect k exists if and only if k is odd and $t \equiv 0, 1 \pmod 4$, or k is even and $t \equiv 2, 3 \pmod 4$; and a hooked near-Skolem sequence of order t and defect k exists if and only if k is even and $t \equiv 0, 1 \pmod 4$, or k is odd and $t \equiv 2, 3 \pmod 4$.*

Clearly, a near-Skolem sequence of order t and defect k provides a partition of the set $[1, 3t-2] \setminus \{k\}$ into $t-1$ triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ and a hooked near-Skolem sequence of order t and defect k provides a partition of the set $[1, 3t-1] \setminus \{k, 3t-2\}$ into $t-1$ triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$.

Skolem sequences and their generalizations have been used widely in the construction of combinatorial designs, and a survey on Skolem sequences can be found in [13]. In the literature, difference triples obtained from Skolem sequences are usually written (a, b, c) with $a + b = c$. However, the equivalent representation, with c replaced by $-c$ so that $a+b+c = 0$, is more convenient for the purpose of extending these ideas to m-tuples with $m > 3$. As such, the following definition was given in [5].

Definition 1.5 An m-tuple $(d_1, d_2, \dots, d_m)$ is of *Skolem-type* if $d_1+d_2+\cdots+d_m = 0$. A set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid i = 1, 2, \dots, t\}$ such that $\{|d_{i,j}| \mid 1 \le i \le t, 1 \le j \le m\} = [1, mt]$ is called a *Skolem-type m-tuple difference set of order t*. A set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid i = 1, 2, \dots, t\}$ such that $\{|d_{i,j}| \mid 1 \le i \le t, 1 \le j \le m\} = [1, mt+1] \setminus \{mt\}$ is called a *hooked Skolem-type m-tuple difference set of order t.*

Necessary and sufficient conditions for the existence of Skolem-type m-tuple difference sets of order t and hooked Skolem-type m-tuple difference sets of order t were found in [5], where the following theorem was given.

Theorem 1.6 (Bryant, Gavlas, Ling [5]) *For positive integers m and t with $m \ge 3$, there exists a Skolem-type m-tuple difference set of order t if and only if $mt \equiv 0, 3 \pmod 4$, and there exists a hooked Skolem-type m-tuple difference set of order t if and only if $mt \equiv 1, 2 \pmod 4$.*

In this paper, we are interested in (hooked) k-extended Skolem-type m-tuple different sets, and thus we make the following definition.

Definition 1.7 For positive integers k and t with $k \le mt+1$, a *k-extended Skolem-type m-tuple difference set of order t* is a set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid i = 1, 2, \dots, t\}$ such that $\{|d_{i,j}| \mid 1 \le i \le t, 1 \le j \le m\} = [1, mt+1] \setminus \{k\}$. For positive integers k and t with $k < mt + 1$, a *hooked k-extended Skolem-type m-tuple difference set of order t* is a set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid i = 1, 2, \dots, t\}$ such that $\{|d_{i,j}| \mid 1 \le i \le t, 1 \le j \le m\} = [1, mt + 2] \setminus \{k, mt + 1\}$.

Clearly, a $(mt + 1)$-extended Skolem-type m-tuple difference set is a Skolem-type m-tuple difference set and a mt-extended Skolem-type m-tuple difference set is a hooked Skolem type m-tuple difference set. In Section 2, we give necessary and sufficient conditions for the existence of k-extended Skolem-type 5-tuple difference sets of order t for all k and t with $k \le 5t+1$. We also provide necessary and sufficient conditions for the existence of hooked k extended Skolem-type 5-tuple difference sets of order t. Our main result is as follows.

Theorem 1.8 *For positive integers k and t with $k \le 5t+1$, there exists a k-extended Skolem-type 5-tuple difference set of order t if and only if k is odd and $t \equiv 0, 1 \pmod 4$ or k is even and $t \equiv 2, 3 \pmod 4$. For positive integers k and t with $k < 5t+1$, there exists a hooked k-extended Skolem-type 5-tuple difference set of order t if and only if k is odd and $t \equiv 2, 3 \pmod 4$ or k is even and $t \equiv 0, 1 \pmod 4$.*

In Section 3, we apply Theorem 1.8 to obtain cyclic 5-cycle systems of the circulant graphs as well as cyclic decompositions of circulant graphs of order n into 5-cycles and d-cycles, where d is a divisor of of n. These results are further applied to find decompositions of complete graphs of order n into 5-cycles, d-cycles where $d \mid n$, Hamilton cycles, and possibly a 1-factor.

2 Extended Skolem-type 5-tuple Difference Sets

Let k and t be positive integers with $k \leq mt+1$, and note that if a k-extended Skolem-type m-tuple difference set of order t exists, then each of the t m-tuples must contain an even number of odd integers. Thus the set $[1, mt+1] \setminus \{k\}$ must contain an even number of odd integers. Similarly, if there exists a k-extended hooked Skolem-type m-tuple difference set of order t, then the set $[1, mt+2] \setminus \{k, mt+1\}$ contains an even number of odd integers. Hence, the following lemma provides necessary conditions for the existence of (hooked) k-extended Skolem-type m-tuple difference sets in the case that $m = 5$.

Lemma 2.1 *Let k and t be positive integers. If there exists a k-extended Skolem-type 5-tuple difference set of order t, then*

(1) k is odd when $t \equiv 0, 1 \pmod 4$, and

(2) k is even when $t \equiv 2, 3 \pmod 4$.

If there exists a k-extended hooked Skolem-type 5-tuple difference set of order t, then

(1) k is even when $t \equiv 0, 1 \pmod 4$, and

(2) k is odd when $t \equiv 2, 3 \pmod 4$.

We now construct k-extended Skolem-type 5-tuple difference sets.

Lemma 2.2 *For positive integers k and t with $k \leq 5t+1$, there exists a k-extended Skolem-type 5-tuple difference set of order t under the following conditions:*

(1) k is odd when $t \equiv 0, 1 \pmod 4$, and

(2) k is even when $t \equiv 2, 3 \pmod 4$.

Proof. Let $t \geq 1$ be a positive integer. Let k be a positive integer such that $1 \leq k \leq 5t+1$, and k is odd when $t \equiv 0, 1 \pmod 4$ or k is even when $t \equiv 2, 3 \pmod 4$. We seek to partition the set $[1, 5t+1] \setminus \{k\}$ into Skolem-type 5-tuples. In each of the following cases, we construct a $t \times 5$ array $X = [x_{ij}]$ such that the entries of X in absolute value are $[1, 5t+1] \setminus \{k\}$, and for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{5} x_{ij} = 0$. Thus, the t rows of X give a k-extended Skolem-type 5-tuple difference set of order t.

Case 1: *Suppose that* $1 \le k \le t$. We begin by considering the case when $t = 1$. Then $k = 1$, and let $X = \begin{bmatrix} 2 & 3 & -4 & 5 & -6 \end{bmatrix}$. Thus we may assume $t \ge 2$. Now suppose $k = 1$ so that $t \equiv 0, 1 \pmod 4$. By Theorem 1.1, there exists a Skolem sequence of order t giving a partition of $[1, 3t]$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \le i \le t$. The set $[3t+2, 5t+1]$ consists of $2t$ integers, and these $2t$ integers can be paired into sets $\{d_i, d_i + 1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_t+1 & b_t+1 & -(c_t+1) & d_t & -(d_t+1) \end{bmatrix},$$

and note that the entries of X, in absolute value, are $[2, 5t+1]$. Thus, we may assume $k \ge 2$. By Theorem 1.4, there exists a hooked near-Skolem sequence of order t and defect $k-1$ giving a partition of $[1, 3t-1] \setminus \{k-1, 3t-2\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \le i \le t-1$. As before, the set $[3t+2, 5t+1]$ consists of $2t$ integers, and these $2t$ integers can be paired into sets $\{d_i, d_i+1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+1 & b_{t-1}+1 & -(c_{t-1}+1) & d_{t-1} & -(d_{t-1}+1) \\ 1 & 3t-1 & -(3t+1) & d_t+1 & -d_t \end{bmatrix},$$

and note that the entries of X in absolute value are $[1, 5t+1] \setminus \{k\}$.

Case 2: *Suppose k is an integer such that* $t + 1 \le k < 3t + 1$. We begin by considering the case when $t = 1$. Then k is odd with $2 \le k < 4$ implying $k = 3$. Let $X = \begin{bmatrix} 1 & 2 & -4 & 6 & -5 \end{bmatrix}$. Thus we may assume $t \ge 2$. By Theorem 1.3, there exists a hooked $(k-t)$-extended Skolem sequence of order $t-1$ giving a partition of $[1, 3t-1] \setminus \{k-1, 3t-2\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \le i \le t-1$. Consider the set $[3t+2, 5t+1]$ consisting of $2t$ integers. These $2t$ integers can be paired into sets $\{d_i, d_i+1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+1 & b_{t-1}+1 & -(c_{t-1}+1) & d_{t-1} & -(d_{t-1}+1) \\ 1 & 3t-1 & -(3t+1) & d_t+1 & -d_t \end{bmatrix}.$$

Case 3: *Suppose k is an integer such that* $3t + 1 \le k \le 5t + 1$. Clearly, if $k = 5t + 1$, then $t \equiv 0, 3 \pmod 4$. Hence, by Theorem 1.6, there exists a Skolem-type 5-tuple difference set of order t and thus there exists a $(5t+1)$-extended Skolem-type difference set of order t. Therefore, we may assume $k < 5t + 1$. We proceed by considering the congruence class of t modulo 4.

SUBCASE 3.1: *Suppose* $t \equiv 0 \pmod 4$ *and* k *is odd, or* $t \equiv 3 \pmod 4$ *and* k *is even.* Now $t \geq 3$, and since $t - 1 \equiv 2, 3 \pmod 4$, by Theorem 1.1, there exists a hooked Skolem sequence of order $t - 1$. Thus, there exists a partition of the set $[1, 3t-2] \setminus \{3t-3\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. Consider the sets $[3t+1, k-1]$ and $[k+1, 5t+1]$ containing $k - 3t - 1$ and $5t - k + 1$ integers respectively. Since the number of integers in each set is even, these $2t$ integers can be paired into sets $\{d_i, d_i + 1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1 + 1 & b_1 + 1 & -(c_1 + 1) & d_1 & -(d_1 + 1) \\ a_2 + 1 & b_2 + 1 & -(c_2 + 1) & d_2 & -(d_2 + 1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1} + 1 & b_{t-1} + 1 & -(c_{t-1} + 1) & d_{t-1} & -(d_{t-1} + 1) \\ 1 & 3t - 2 & -3t & d_t + 1 & -d_t \end{bmatrix}.$$

SUBCASE 3.2: *Suppose* $t \equiv 1 \pmod 4$ *and* k *is odd or* $t \equiv 2 \pmod 4$ *and* k *is even.* We begin by considering the case when $t = 1$. Then k is odd and $4 \leq k < 6$ implies $k = 5$. Let $X = \begin{bmatrix} 1 & 3 & -2 & 4 & -6 \end{bmatrix}$. Thus we may assume $t \geq 2$. Since $t - 1 \equiv 0, 1 \pmod 4$, by Theorem 1.1, there exists a Skolem sequence of order $t - 1$. Thus, there exists a partition of the set $[1, 3t - 3]$ into $t - 1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t - 1$. Consider the sets $[3t - 1, k - 2]$ and $[k + 2, 5t + 1]$ containing $k - 3t$ and $5t - k$ integers respectively. Since the number of integers in each set is even, these $2t$ integers can be paired into sets $\{d_i, d_i + 1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1 + 1 & b_1 + 1 & -(c_1 + 1) & d_1 & -(d_1 + 1) \\ a_2 + 1 & b_2 + 1 & -(c_2 + 1) & d_2 & -(d_2 + 1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1} + 1 & b_{t-1} + 1 & -(c_{t-1} + 1) & d_{t-1} & -(d_{t-1} + 1) \\ 1 & k - 1 & -(k + 1) & d_t + 1 & -d_t \end{bmatrix}.$$

■

We now find hooked k-extended Skolem-type 5-tuple difference sets of order t for all positive integers k with $1 \leq k < 5t + 1$.

Lemma 2.3 *For each positive integer* t*, there exists a hooked* k*-extended Skolem-type* 5*-tuple difference set of order* t *for all* k *with* $1 \leq k < 5t + 1$ *under the following conditions:*

(1) k *is even when* $t \equiv 0, 1 \pmod 4$*, and*

(2) k *is odd when* $t \equiv 2, 3 \pmod 4$.

Proof. Let $t \geq 1$ be a positive integer. Let k be a positive integer such that $1 \leq k < 5t+1$, and k is even when $t \equiv 0, 1 \pmod 4$, or k is odd when $t \equiv 2, 3 \pmod 4$. We seek to partition the set $[1, 5t+2] \setminus \{k, 5t+1\}$ into t Skolem-type 5-tuples. In each of the following cases, we construct a $t \times 5$ array $X = [x_{ij}]$ such that the entries of X, in absolute value, are $[1, 5t+2] \setminus \{k, 5t+1\}$ and for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{5} x_{ij} = 0$. Thus, the t rows of X give a hooked k-extended Skolem-type 5-tuple difference set of order t.

Case 1: *Suppose that* $1 \leq k \leq t$. Suppose first that $k = 1$ so that $t \equiv 2, 3 \pmod 4$. We handle each of these congruence classes separately. Let $t \equiv 2 \pmod 4$. By Theorem 1.1, there exists a Skolem sequence of order $t-1$ giving a partition of $[1, 3t-3]$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. Without loss of generality, we may assume $a_1 = 1$. The set $[3t+2, 5t-3]$ consists of $2t-4$ integers, and since $2t-4 \equiv 0 \pmod 4$, these $2t-4$ integers can be paired into sets $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t-2$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} 2 & b_1+2 & -(c_1+2) & 5t-2 & -(5t-1) \\ a_2+2 & b_2+2 & -(c_2+2) & d_1 & -(d_1+2) \\ a_3+2 & b_3+2 & -(c_3+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-2} & -(d_{t-2}+2) \\ 3 & 3t & -(3t+1) & 5t & -(5t+2) \end{bmatrix},$$

and note that the entries of X in absolute value are $[2, 5t+2] \setminus \{5t+1\}$.

Now let $t \equiv 3 \pmod 4$. By Theorem 1.1, there exists a hooked Skolem sequence of order $t-1$ giving a partition of $[1, 3t-2] \setminus \{3t-3\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. Again, without loss of generality, we may assume $a_1 = 1$. The set $[3t+4, 5t-3]$ consists of $2t-6$ integers, and since $2t-6 \equiv 0 \pmod 4$, these $2t-6$ integers can be paired into sets $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t-3$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} 2 & b_1+2 & -(c_1+2) & 5t-2 & -(5t-1) \\ a_2+2 & b_2+2 & -(c_2+2) & 3t-1 & -(3t+1) \\ a_3+2 & b_3+2 & -(c_3+2) & d_1 & -(d_1+2) \\ a_4+2 & b_4+2 & -(c_4+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-3} & -(d_{t-3}+2) \\ 3 & 3t+2 & -(3t+3) & 5t & -(5t+2) \end{bmatrix},$$

and note that the entries of X in absolute value are $[2, 5t+2] \setminus \{5t+1\}$.

Thus, we may assume $k \geq 2$. By Theorem 1.4, there exists a near-Skolem sequence of order t and defect $k-1$ giving a partition of $[1, 3t-2] \setminus \{k-1\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. Consider the set $[3t, 5t-1]$ consisting of $2t$ integers. These $2t$ integers can be paired into sets $\{d_i, d_i+1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+1 & b_{t-1}+1 & -(c_{t-1}+1) & d_{t-1} & -(d_{t-1}+1) \\ 1 & 5t & -(5t+2) & d_t+1 & -d_t \end{bmatrix}.$$

Case 2: *Suppose that* $t+1 \le k < 3t+1$. Let $t \ge 1$ be a positive integer. We begin by considering the case when $t = 1$. Then k is even and $2 \le k < 4$ implies $k = 2$. Let $X = \begin{bmatrix} 1 & 4 & -3 & 5 & -7 \end{bmatrix}$. Thus we may assume $t \ge 2$. By Theorem 1.3, there exists a $(k-t)$-extended Skolem sequence of order $t-1$ giving a partition of $[1, 3t-2] \setminus \{k-1\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \le i \le t-1$. Consider the set $[3t, 5t-1]$, consisting of $2t$ integers. These $2t$ integers can be paired into sets $\{d_i, d_i+1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+1 & b_{t-1}+1 & -(c_{t-1}+1) & d_{t-1} & -(d_{t-1}+1) \\ 1 & 5t & -(5t+2) & d_t+1 & -d_t \end{bmatrix}.$$

Case 3: *Suppose that* $3t+1 \le k < 5t+1$. We proceed by considering the congruence class of t modulo 4.

Subcase 3.1: *Suppose* $t \equiv 0 \pmod 4$ *and* k *is even.* Now $t \ge 4$, and since $t-1 \equiv 3 \pmod 4$, by Theorem 1.2, there exists a Langford sequence of order $t-1$ and defect 2. Thus, there exists a partition of the set $[2, 3t-2]$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \le i \le t-1$. The sets $[3t+1, k-1]$ and $[k+1, 5t-1]$ contain $k-3t-1$ and $5t-k-1$ integers respectively. Suppose first that $k \equiv 2 \pmod 4$. Then the $2(t-1)$ integers in the set $[3t+1, k-1] \cup [k+1, 5t-1]$ can be paired into sets $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t-1$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+2 & b_1+2 & -(c_1+2) & d_1 & -(d_1+2) \\ a_2+2 & b_2+2 & -(c_2+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-1} & -(d_{t-1}+2) \\ 1 & 3 & -2 & 5t & -(5t+2) \end{bmatrix}.$$

Now consider the case when $k \equiv 0 \pmod 4$. The set $[3t+1, 5t+2] \setminus \{k, 5t+1\}$, containing $2t$ integers, can be paired into sets $\{k-2, k+2\}$ and $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t-1$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+2 & b_1+2 & -(c_1+2) & d_1 & -(d_1+2) \\ a_2+2 & b_2+2 & -(c_2+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-1} & -(d_{t-1}+2) \\ 2 & 3 & -1 & k-2 & -(k+2) \end{bmatrix}.$$

SUBCASE 3.2: *Suppose* $t \equiv 3 \pmod 4$ *so that* k *is odd.* Now $t \geq 3$, and since $t-1 \equiv 2 \pmod 4$, there exists a hooked Langford sequence of order $t-1$ and defect 2. Thus, there exists a partition of the set $[2, 3t-1] \setminus \{3t-2\}$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. The sets $[3t, k-1] \setminus \{3t+1\}$ and $[k+1, 5t+2] \setminus \{5t+1\}$ contain $k-3t-1$ and $5t-k+1$ integers respectively. When $k \equiv 1 \pmod 4$ these $2t$ integers can be paired into sets $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+2 & b_1+2 & -(c_1+2) & d_1 & -(d_1+2) \\ a_2+2 & b_2+2 & -(c_2+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-1} & -(d_{t-1}+2) \\ 1 & 3 & -2 & dt & -(dt+2) \end{bmatrix}.$$

Now consider the case when $k \equiv 3 \pmod 4$. The sets $[3t, k-1] \setminus \{3t+1\}$ and $[k+1, 5t+2] \setminus \{5t+1\}$, containing $2t$ integers can be paired into sets $\{k-2, k+2\}$ and $\{d_i, d_i+2\}$ for each $i = 1, 2, \ldots, t-1$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+2 & b_1+2 & -(c_1+2) & d_1 & -(d_1+2) \\ a_2+2 & b_2+2 & -(c_2+2) & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+2 & b_{t-1}+2 & -(c_{t-1}+2) & d_{t-1} & -(d_{t-1}+2) \\ 2 & 3 & -1 & k-2 & -(k+2) \end{bmatrix}.$$

SUBCASE 3.3: *Suppose* $t \equiv 1 \pmod 4$ *and* k *is even, or* $t \equiv 2 \pmod 4$ *and* k *is odd.* We begin by considering the case when $t = 1$. Then k is even and $4 \leq k < 6$ implies $k = 4$. Let $X = \begin{bmatrix} 1 & 3 & -2 & 5 & -7 \end{bmatrix}$. Thus we may assume $t \geq 2$. Since $t-1 \equiv 0, 1 \pmod 4$, by Theorem 1.1, there exists a Skolem sequence of order $t-1$. Thus, there exists a partition of the set $[1, 3t-3]$ into $t-1$ triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $1 \leq i \leq t-1$. Consider the sets $[3t-1, k-1]$ and $[k+1, 5t-1]$ containing $k-3t+1$ and $5t-k-1$ integers respectively. Since the number of integers in each set is even, these $2t$ integers can be paired into sets $\{d_i, d_i+1\}$ for each $i = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times 5$ matrix

$$X = \begin{bmatrix} a_1+1 & b_1+1 & -(c_1+1) & d_1 & -(d_1+1) \\ a_2+1 & b_2+1 & -(c_2+1) & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t-1}+1 & b_{t-1}+1 & -(c_{t-1}+1) & d_{t-1} & -(d_{t-1}+1) \\ 1 & 5t & -(5t+2) & d_t+1 & -d_t \end{bmatrix}.$$

■

Theorem 1.8 now follows from Lemmas 2.2 and 2.3.

3 Applications to Circulants and Complete Graphs

Let K_n denote the complete graph on n vertices and C_m denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An n-cycle in a graph with n vertices is called a *Hamilton* cycle. A *decom-*

position of a graph G is a set $\{H_1, H_2, \dots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i for some i with $1 \le i \le k$.

An *m-cycle system* of a graph G is a decomposition of G into m-cycles. Several obvious necessary conditions for an m-cycle system of a graph G to exist are immediate: $3 \le m \le |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G. A survey on cycle systems is given in [9], and necessary and sufficient conditions for the existence of an m-cycle system of G in the cases $G = K_n$ and $G = K_n - I$ (the complete graph of order n with a 1-factor I removed) were given in [1, 22]. Such m-cycle systems exist if and only if $n \ge m$, every vertex of G has even degree, and m divides the number of edges in G.

Let ρ denote the permutation $(0, 1, \dots, n-1)$, so $\langle \rho \rangle = \mathbb{Z}_n$, the additive group of integers modulo n. An m-cycle system $\mathcal{C}$ of a graph G with vertex set $\mathbb{Z}_n$ is *cyclic* if, for every m-cycle $C = (v_1, v_2, \dots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \dots, \rho(v_m))$ is also in $\mathcal{C}$. Finding necessary and sufficient conditions for cyclic m-cycle systems of a given graph G is an interesting problem and has attracted much attention (see, for example, [4, 5, 7, 8, 12, 14, 16, 20, 21, 27]). The obvious necessary conditions for a cyclic m-cycle system of a graph G are the same as for an m-cycle system of G; that is, $3 \le m \le |V(G)|$, the degree of the vertices of G must be even, and m must divide the number of edges in G. However, these conditions are not sufficient. For example, it is not difficult to see that there is no cyclic 15-cycle system of K_{15}. Also, if p is an odd prime and $\alpha \ge 2$, then there is no cyclic p^α-cycle system of K_{p^α} [8].

The existence question for cyclic m-cycle systems of K_n has been completely settled in a few small cases, namely $m = 3$ [19], 5 and 7 [21]. For even m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are constructed for $m \equiv 0 \pmod 4$ in [16] and for $m \equiv 2 \pmod 4$ in [20]. Both of these cases are handled simultaneously in [12] as a consequence of a more general result. For odd m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are found using different methods in [4, 7, 14]. In [5], as a consequence of a more general result, cyclic m-cycle systems of K_n for all positive integers m and $n \equiv 1 \pmod{2m}$ with $n \ge m \ge 3$ are given. In [8], it is shown that a cyclic hamiltonian cycle system of K_n exists if and only if $n \ne 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. Thus, as a consequence of a result in [7], cyclic m-cycle systems of K_{2mk+m} exist for all $m \ne 15$ and $m \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. In [26], for m odd, the last remaining cases for cyclic m-cycle systems of K_{2mk+m} are settled, i.e., it is shown that, for $k \ge 1$, cyclic m-cycle systems of K_{2km+m} exist if $m = 15$ or $m \in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. In [27], necessary and sufficient conditions for the existence of cyclic $2q$-cycle and m-cycle systems of the complete graph are given when q is an odd prime power and $3 \le m \le 32$. In [6], cycle systems with a sharply vertex-transitive automorphism group that is not necessarily cyclic are investigated. As a result, it is shown in [6] that no cyclic k-cycle system of K_n exists if $k < n < 2k$ with n odd and $\gcd(k, n)$ a prime power.

For $x \not\equiv 0 \pmod n$, the *modulo n length* of an integer x, denoted $|x|_n$, is defined to be the smallest positive integer y such that $x \equiv y \pmod n$ or $x \equiv -y \pmod n$. Note that for any integer $x \not\equiv 0 \pmod n$, it follows that $|x|_n \in [1, \lfloor \frac{n}{2} \rfloor]$. If L is a set of modulo n lengths, we define the *circulant graph* $\langle L \rangle_n$ to be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{i,j\} \mid |i-j|_n \in L\}$. Observe that $K_n \cong \langle[1, \lfloor \frac{n}{2} \rfloor]\rangle_n$. An edge $\{i,j\}$ in a graph with vertex set $\mathbb{Z}_n$ is called an *edge of length* $|i-j|_n$. Notice that in order for a graph G to admit a cyclic m-cycle system, G must be a circulant graph; thus circulant graphs provide a natural setting in which to construct cyclic m-cycle systems.

For $t \equiv 0,1 \pmod 4$, a Skolem sequence $S = (s_1, s_2, \ldots, s_{2t})$ of order t can be used to construct a cyclic 3-cycle system of $\langle[1,3t]\rangle_n$ for all $n \geq 6t+1$ in the following way. Let $\{(a_i, b_i, c_i) \mid i = 1, 2, \ldots, t\}$ be a set of t triples constructed from a Skolem sequence of order t. Thus, $\{(a_i, b_i, c_i) \mid i = 1, 2, \ldots, t\}$ provides a partition of $[1, 3t]$ and we may assume $a_i + b_i = c_i$ for $1 \leq i \leq t$. Then, a cyclic 3-cycle system of $\langle[1,3t]\rangle_n$ is given by $\{(j, j+a_i, j+a_i+b_i) \mid 0 \leq j \leq n-1, 1 \leq i \leq t\}$. Similarly, for $t \equiv 2,3 \pmod 4$, hooked Skolem sequences can be used to construct cyclic 3-cycle systems of $\langle[1,3t+1] \setminus \{3t\}\rangle_n$ for $n = 6t+1$ (since $|3t|_{6t+1} = |3t+1|_{6t+1}$) and all $n \geq 6t+3$. Note that by using Skolem and hooked Skolem sequences, we obtain cyclic 3-cycle systems of K_{6t+1} for all positive integers t.

In a similar manner to which 3-cycle difference sets are constructed from Skolem sequences, Langford sequences, extended Skolem sequences, and near Skolem sequences can be used to construct cyclic 3-cycle systems of the appropriate circulant graph. Note that in constructing cyclic 3-cycle systems from k-extended Skolem sequences, by choosing k appropriately, we can obtain some interesting decompositions. For example, $(t+1)$-extended Skolem sequences for $t \equiv 0,3 \pmod 4$ and hooked $(t+1)$-extended Skolem sequences for $t \equiv 1,2 \pmod 4$ give cyclic 3-cycle systems of K_{6t+3} for all positive integers t as $\{(i, i+2t+1, i+4t+2) \mid 0 \leq i \leq 2t\}$ is a 3-cycle system of $\langle\{2t+1\}\rangle_{6t+3}$.

As a consequence of Theorem 1.6, the following result was given in [5] regarding cyclic m-cycle systems of specific circulant graphs.

Theorem 3.1 (Bryant, Gavlas, Ling [5]) *For positive integers m and t with $m \geq 3$, there exists a cyclic m-cycle system of $\langle[1,mt]\rangle_n$ for $mt \equiv 0,3 \pmod 4$ and all $n \geq 2mt+1$, and there exists a cyclic m-cycle system of $\langle[1,mt+1] \setminus \{mt\}\rangle_n$ for $mt \equiv 1,2 \pmod 4$ and $n = 2mt+1$ and all $n \geq 2mt+3$.*

In a similar fashion, we obtain the following corollary to Theorem 1.8 regarding cyclic 5-cycle systems of specific circulant graphs.

Corollary 3.2 *For positive integers k and t with $k \leq 5t+1$ with k odd and $t \equiv 0,1 \pmod 4$ or k even and $t \equiv 2,3 \pmod 4$, there exists a cyclic 5-cycle system of $\langle[1,5t+1] \setminus \{k\}\rangle_n$ for all $n \geq 10t+3$. For positive integers k and t with $k < 5t+1$ with k odd and $t \equiv 2,3 \pmod 4$ or k even and $t \equiv 0,1 \pmod 4$, there exists a cyclic 5-cycle system of $\langle[1,5t+2] \setminus \{k, 5t+1\}\rangle_n$ for $n = 10t+3$ and all $n \geq 10t+5$.*

Proof. Suppose first that k and t be positive integers such that $k \leq 5t+1$, $t \equiv 0,1 \pmod 4$ when k is odd, or $t \equiv 2,3 \pmod 4$ when k is even. By Lemma 2.2, there exists a k-extended Skolem-type 5-tuple difference set of order t. Let $\{(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,5}) \mid i = 1, 2, \cdots, t\}$ denote the set of t Skolem-type 5-tuples given by the rows of the $t \times 5$ matrix X in the proof of Lemma 2.2. Let n be a positive integer such that $n \geq 10t+3$. Observe that the required set of nt 5-cycles is given by

$$\{(j, j+x_{i,1}, j+x_{i,1}+x_{i,2}, j+x_{i,1}+x_{i,2}+x_{i,3}, j+x_{i,1}+x_{i,2}+x_{i,3}+x_{i,4}) \mid 0 \leq j \leq n-1, 1 \leq i \leq t\},$$

thereby giving an cyclic 5-cycle system of $\langle [1, 5t+1] \setminus \{k\} \rangle_n$.

In a similar manner, cyclic 5-cycle systems of $\langle [1, 5t+2] \setminus \{k, 5t+1\} \rangle_n$ for $n = 10t+3$ and all $n \geq 10t+5$ can be constructed from the matrix X given in the proof of Lemma 2.3 for all positive integers k and t with $k < 5t+1$ with k odd and $t \equiv 2, 3 \pmod 4$ or k even and $t \equiv 0, 1 \pmod 4$. ■

A subgraph F of a graph G is a *factor* if F contains all the vertices of G. If each component of F is isomorphic to a graph H, then F is called an *H-factor*; while if every vertex of F has the same degree d, then F is called a *d-factor*. Let $n \geq 3$ be a positive integer and suppose $d \geq 3$ is a divisor of n. Clearly, the circulant graph $\langle \{\frac{n}{d}\} \rangle_n$ is a 2-factor consisting of $\frac{n}{d}$ d-cycles, or a C_d-factor. Thus, the following corollary is an immediate consequence of this fact and Corollary 3.2.

Corollary 3.3 *Let t and n be positive integers such that $n \geq 10t+3$. Let $d \geq 3$ be a positive divisor of n with $\frac{n}{d} < 5t+1$.*

- *If $\frac{n}{d}$ is odd and $t \equiv 0, 1 \pmod 4$, or $\frac{n}{d}$ is even and $t \equiv 2, 3 \pmod 4$, then there exists a decomposition of $\langle [1, 5t+1] \rangle_n$ into nt 5-cycles and a C_d-factor.*
- *If $n \neq 10t+4$, $\frac{n}{d}$ is odd and $t \equiv 2, 3 \pmod 4$, or $\frac{n}{d}$ is even and $t \equiv 0, 1 \pmod 4$, then there exists a decomposition of $\langle [1, 5t+2] \setminus \{5t+1\} \rangle_n$ into nt 5-cycles and a C_d-factor.*

We now consider the case of decomposing the complete graph K_n into 5-cycles, d-cycles for some divisor $d \geq 3$ of n, Hamilton cycles, and possibly a 1-factor. The following result was given in [15], as a direct consequence of the results given in [3, 10, 11], and will be helpful with the Hamilton cycles.

Lemma 3.4 (Jordon, [15]) *For each odd integer $n \geq 5$ and each integer x with $1 \leq x \leq \frac{n-1}{2}$, the graphs $\langle [x, \frac{n-1}{2}] \rangle_n$ and $\langle [x, \frac{n-1}{2}] \setminus \{x+1\} \rangle_n$ decompose into Hamilton cycles. For each even integer $n \geq 6$ and*

(1) for each integer x with $1 \leq x \leq \frac{n}{2}-1$, the graph $\langle [x, \frac{n}{2}] \rangle_n$ decomposes into Hamilton cycles and a 1-factor; and

(2) for each integer x with $1 \leq x \leq \frac{n}{2}-3$, the graph $\langle [x, \frac{n-1}{2}] \setminus \{x+1\} \rangle_n$ decomposes into Hamilton cycles and a 1-factor.

As a consequence of Corollary 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.5 *For each odd integer $n \geq 5$, positive integer t with $n \geq 10t+3$, and divisor $d \geq 3$ of n with $\frac{n}{d} < 5t+1$, the complete graph K_n decomposes into a C_d-factor, nt 5-cycles, and $\frac{n-3}{2} - 5t$ Hamilton cycles. For each even integer $n \geq 6$, positive integer t with $n \geq 10t+6$, and divisor $d \geq 3$ of n with $\frac{n}{d} < 5t+1$, the complete graph K_n decomposes into a C_d-factor, nt 5-cycles, $\frac{n-4}{2} - 5t$ Hamilton cycles and a 1-factor.*

References

[1] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B* **81** (2001), 77–99.

[2] C. A. Baker, Extended Skolem sequences. *J. Combin. Des.* **3** (1995), 363–379.

[3] J.-C. Bermond, O. Favaron, and M. Mahéo, Hamiltonian decomposition of Cayley graphs of degree 4. *J. Combin. Theory Ser. B* **46** (1989), 142–153.

[4] A. Blinco, S. El Zanati and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost complete graphs, *Discrete Math.* **284** (2004), 71–81.

[5] D. Bryant, H. Gavlas, and A. C. H. Ling, Difference Sets for Cycle Systems, *Electron. J. Combin.* **10** (2003), #R38.

[6] M. Buratti, Cycle decompositions with a sharply vertex transitive automorphism group, *Le Matematiche* **59** (2004), 91–105.

[7] M. Buratti and A. Del Fra, Existence of cyclic k-cycle systems of the complete graph, *Discrete Math.* **261** (2003), 113–125.

[8] M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, *Discrete Math.* **279** (2004), 107–119.

[9] D. Bryant and C. A. Rodger, Cycle Decompositions, in the *CRC Handbook of Combinatorial Designs 2nd ed.* (eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton FL (2007), 373–382.

[10] M. Dean, On Hamilton cycle decomposition of 6-regular circulant graphs. *Graphs Combin.* **22** (2006), 331–340.

[11] M. Dean, Hamilton cycle decomposition of 6-regular circulants of odd order. *J. Combin. Des.* **15** (2007), 91–97.

[12] S. I. El-Zanati, N. Punnim, C. Vanden Eynden, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[13] N. Francetic, E. Mendelsohn, A Survey of Skolem-type sequences and Rosa's use of them, *Mathematica Slovaca.* **59** (2009), 39–76.

[14] H. Fu and S. Wu, Cyclically decomposing complete graphs into cycles, *Discrete Math.* **282** (2004), 267–273.

[15] H. Jordon, Alspach's problem: the case of Hamilton cycles and 5-cycles, *Electron. J. Combin.* **18(1)** (2011), #P82.

[16] A. Kotzig, On decompositions of the complete graph into $4k$-gons, *Mat.-Fyz. Cas.* **15** (1965), 227–233.

[17] V. Linek and N. Shalaby, The existence of (p, q)-extended Rosa sequences. *Discrete Math.* **308** (2008), 1583–1602.

[18] E. S. O'Keefe, Verification of a conjecture of Th Skolem, *Math. Scand* **9** (1961), 80–82.

[19] R. Peltesohn, Eine Losung der beiden Heffterschen Differenzenprobleme, *Compos. Math.* **6** (1938), 251–257.

[20] A. Rosa, On cyclic decompositions of the complete graph into $(4m + 2)$-gons, *Mat.-Fyz. Cas.* **16** (1966), 349–352.

[21] A. Rosa, On the cyclic decompositions of the complete graph into polygons with an odd number of edges, *Časopis Pěst. Math.* **91** (1966), 53–63.

[22] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[23] Nabil Shalaby, The existence of near-Skolem and hooked near-Skolem sequences. *Discrete Math.* **135** (1994), 303–319.

[24] J. E. Simpson, Langford sequences: perfect and hooked, *Discrete Math.* **44** (1983), 97–104.

[25] Th. Skolem, On certain distributions of integers in pairs with given difference, *Math. Scand.* **5** (1957), 57–68.

[26] A. Vietri, Cyclic k-Cycle Systems of order $2kn + k$; a solution of the last open cases, *J. Combin. Des* **12** (2004), 299–310.

[27] S.-L. Wu and H.-L. Fu, Cyclic m-cycle systems with $m \leq 32$ or $m = 2q$ with q a prime power, *J. Combin. Des.* **14** (2006), 66–81.

(Received 27 Sep 2011)