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All 2-regular graphs with uniform odd components admit ρ-labelings

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Abstract

Let G be a graph of size n with vertex set $V(G)$ and edge set $E(G)$. A ρ-*labeling* of G is a one-to-one function $h\colon V(G) \to \{0, 1, \ldots, 2n\}$ such that $\{\min\{|h(u)-h(v)|, 2n+1-|h(u)-h(v)|\}\colon \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. Such a labeling of G yields a cyclic G-decomposition of K_{2n+1}. It is known that 2-regular bipartite graphs, the vertex-disjoint union of C_3's, and the vertex-disjoint union of C_5's all admit ρ-labelings. We show that for any odd $n \geq 7$, the vertex-disjoint union of any number of C_n's admits a ρ-labeling.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let rG denote the vertex-disjoint union of r copies of G.

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Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_k is known as a *G-design of order k*. A G-decomposition Γ of K_k is *cyclic* if clicking is a permutation of Γ.

The investigation of G-designs is a popular area of research in combinatorial design theory. For example, if G is K_k, then a G-design of order v is a $(v, k, 1)$-BIBD. If G has n edges, then G-designs of order $2n+1$ are of particular interest. In 1963, Ringel [8] conjectured that there is a G-design of order $2n+1$ for every tree G with n edges. In [9], Rosa introduced graph labelings as a means of attacking Ringel's conjecture.

For any graph G, a one-to-one function $h\colon V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. Let G be a graph with n edges and no isolated vertices and let h be a labeling of G. Let $h(V(G)) = \{h(u)\colon u \in V(G)\}$. Define a function $\bar{h}\colon E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$ and let $(\bar{h}(e))^* = \min\{\bar{h}(e), 2n+1-\bar{h}(e)\}$. We will refer to $\bar{h}(e)$ and $(\bar{h}(e))^*$ as the *label* and the *length* of e, respectively. If $F \subseteq E(G)$, then $\bar{h}(F) = \{\bar{h}(e)\colon e \in F\}$ and $(\bar{h}(F))^* = \{(\bar{h}(e))^*\colon e \in F\}$. We say h is a *ρ-labeling* of G if $h(V(G)) \subseteq [0, 2n]$ and $(\bar{h}(E(G)))^* = [1, n]$. If $h(V(G)) \subseteq [0, n]$ and $\bar{h}(E(G)) = [1, n]$, then h is *β-labeling* or a *graceful* labeling of G.

Labelings are critical to the study of cyclic graph decompositions as seen in the following result from [9].

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

While a ρ-labeling is the most basic of Rosa's labelings, β-labelings (i.e., graceful) are by far the most popular. Graphs that admit a graceful labeling are called *graceful*. A conjecture that every tree is graceful is one of the best known conjectures in design theory. Unfortunately, graceful labelings are too restrictive for many classes of graphs. For example, K_4 is the largest complete graph that is graceful and C_n is graceful if and only if $n \equiv 0$ or $3 \pmod 4$. Kotzig [7] has shown that rC_3 is graceful only if $r = 1$ and that rC_5 is never graceful. For a comprehensive survey of graph labelings that lead to cyclic G-designs, we direct the reader to [5]. A dynamic survey on general graph labelings is maintained by Gallian [6].

In this paper, we will focus on ρ-labelings of rC_n, where $n \geq 7$ is odd. A 1997 result of Dinitz and Rodney [3] on disjoint-starters in cyclic Steiner triple systems is equivalent to showing that rC_3 admits a ρ-labeling for all positive integers r. From results in [1], it can be concluded that every 2-regular bipartite graph admits a ρ-labeling. More recently, it was shown in [2] that rC_{4x+1} has a ρ-labeling for $r \leq 10$ and $x \geq 1$. In [4], we showed that rC_5 admits a ρ-labeling. Here, we shall show that rC_n has a ρ-labeling for all integers $r \geq 1$ and all odd integers $n \geq 7$. We note that we arrived at these results by examining small cases and finding labeling patterns that generalize to larger values of n. Our results provide further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a ρ-labeling.

2 Main Results

Let C_n be the graph with vertex set $\{v_i\colon 1 \le i \le n\}$ and edge set $\{\{v_i, v_{i+1}\}\colon 1 \le i \le n-1\} \cup \{v_n, v_1\}$. For a positive integer r, let $G = rC_n$, the vertex-disjoint union of r copies of C_n. For $1 \le j \le r$, let the j^{th} component of G have vertex set $\{v_{i,j}\colon 1 \le i \le n\}$ and edge set $\{e_{i,j} = \{v_{i,j}, v_{i+1,j}\}\colon 1 \le i \le n-1\} \cup \{e_{n,j} = \{v_{n,j}, v_{1,j}\}\}$. For $1 \le i \le n$, let $V_i = \{v_{i,j}\colon 1 \le j \le r\}$, and let $E_i = \{\{e_{i,j}\}\colon 1 \le j \le r\}$. Finally for $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, we define $e_{A,B} = \{\{a,b\} \in E(G)\colon a \in A, b \in B\}$.

We first show that rC_7 admits a ρ-labeling for all positive integers r.

Lemma 2 *Let r be a positive integer and let $G = rC_7$. Then G admits a ρ-labeling.*

Proof. This is already known for $r = 1$. We consider two cases depending on the parity of r.
Case 1: r is even.
Let $r = 2t$, where $t \ge 1$. Let $h\colon V(G) \to \mathbb{N}$ be defined in the following 12 pieces:

$$h(v_{i,j}) = \begin{cases} 2j-2 & \text{if } i = 1, 1 \le j \le 2t, \\ 28t-2j & \text{if } i = 2, 1 \le j \le t, \\ 22t+j-1 & \text{if } i = 2, t < j \le 2t, \\ 2j-1 & \text{if } i = 3, 1 \le j \le t, \\ 2t+2j-2 & \text{if } i = 3, t < j \le 2t, \\ 9t+j & \text{if } i = 4, 1 \le j \le t, \\ 10t-2j & \text{if } i = 4, t < j \le 2t, \\ 2t+2j-1 & \text{if } i = 5, 1 \le j \le 2t, \\ 14t+4j-2 & \text{if } i = 6, 1 \le j \le t, \\ 17t+3j-1 & \text{if } i = 6, t < j \le 2t, \\ 10t+3j-2 & \text{if } i = 7, 1 \le j \le t, \\ 7t+j-1 & \text{if } i = 7, t < j \le 2t. \end{cases}$$

An example of the ρ-labeling of $4C_7$ obtained from h is shown in Figure 1. Note that

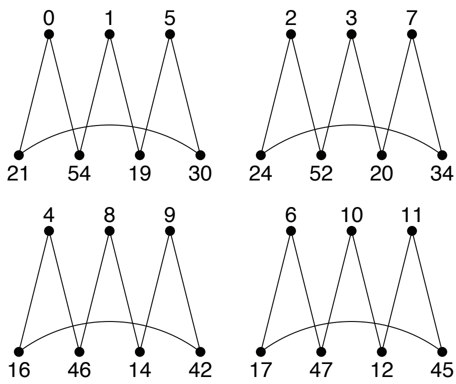

Figure 1: A ρ-labeling of $4C_7$.

h is strictly increasing or strictly decreasing on each of the 12 pieces. Moreover,

$$
\begin{aligned}
&h(V_1) \subseteq [0, 4t-2],\\
&h(V_2) \subseteq [26t, 28t-2] \cup [23t, 24t-1],\\
&h(V_3) \subseteq [1, 2t-1] \cup [4t, 6t-2],\\
&h(V_4) \subseteq [9t+1, 10t] \cup [6t, 8t-2],\\
&h(V_5) \subseteq [2t+1, 6t-1],\\
&h(V_6) \subseteq [14t+2, 18t-2] \cup [20t+2, 23t-1],\\
&h(V_7) \subseteq [10t+1, 13t-2] \cup [8t, 9t-1].
\end{aligned}
$$

We can see from the difference in parities of their elements that $h(V_1) \cap h(V_3) = \emptyset$, $h(V_1) \cap h(V_5) = \emptyset$, and $h(V_3) \cap h(V_5) = \emptyset$. Therefore, h is one-to-one and $h(V(G)) \subseteq [0, 28t] = [0, 2|E(G)|]$.

We now compute the resulting edge labels.

$$
\begin{aligned}
\bar{h}(E_1) &= \{28t - 4j + 2 \colon 1 \le j \le t\} \cup \{22t - j + 1 \colon t < j \le 2t\},\\
\bar{h}(E_2) &= \{28t - 4j + 1 \colon 1 \le j \le t\} \cup \{20t - j + 1 \colon t < j \le 2t\},\\
\bar{h}(E_3) &= \{9t - j + 1 \colon 1 \le j \le t\} \cup \{8t - 4j + 2 \colon t < j \le 2t\},\\
\bar{h}(E_4) &= \{7t - j + 1 \colon 1 \le j \le t\} \cup \{8t - 4j + 1 \colon t < j \le 2t\},\\
\bar{h}(E_5) &= \{12t + 2j - 1 \colon 1 \le j \le t\} \cup \{15t + j \colon t < j \le 2t\},\\
\bar{h}(E_6) &= \{4t + j \colon 1 \le j \le t\} \cup \{10t + 2j \colon t < j \le 2t\},\\
\bar{h}(E_7) &= \{10t + j \colon 1 \le j \le t\} \cup \{7t - j + 1 \colon t < j \le 2t\}.
\end{aligned}
$$

Since the edge labels in $\bar{h}(E_1)$ and $\bar{h}(E_2)$ all exceed $14t$, we have

$$\begin{aligned}(\bar{h}(E_1))^* &= \{2(14t)+1-(28t-4j+2)\colon 1\le j\le t\}\\ &\qquad \cup\{2(14t)+1-(22t-j+1)\colon t<j\le 2t\}\\ &= \{4j-1\colon 1\le j\le t\}\cup\{6t+j\colon t<j\le 2t\},\\ (\bar{h}(E_2))^* &= \{2(14t)+1-(28t-4j+1)\colon 1\le j\le t\}\\ &\qquad \cup\{2(14t)+1-(20t-j+1)\colon t<j\le 2t\}\\ &= \{4j\colon 1\le j\le t\}\cup\{8t+j\colon t<j\le 2t\}.\end{aligned}$$

Additionally, we have $\bar{h}(E_5) > 14t$ only if $j \in [t+1, 2t]$. Thus,

$$(\bar{h}(E_5))^* = \{12t+2j-1\colon 1\le j\le t\}\cup\{13t-j+1\colon t<j\le 2t\}.$$

Otherwise, the label of an edge under $\bar{h}$ is also the length of the edge. Therefore,

$$\begin{aligned}(\bar{h}(E_1))^* &= \{4k-1\colon 1\le k\le t\}\cup[7t+1,8t],\\ (\bar{h}(E_2))^* &= \{4k\colon 1\le k\le t\}\cup[9t+1,10t],\\ (\bar{h}(E_3))^* &= [8t+1,9t]\cup\{4k-2\colon 1\le k\le t\},\\ (\bar{h}(E_4))^* &= [6t+1,7t]\cup\{4k-3\colon 1\le k\le t\},\\ (\bar{h}(E_5))^* &= \{2k+1\colon 6t\le k\le 7t-1\}\cup[11t+1,12t],\\ (\bar{h}(E_6))^* &= [4t+1,5t]\cup\{2k\colon 6t+1\le k\le 7t\},\\ (\bar{h}(E_7))^* &= [10t+1,11t]\cup[5t+1,6t].\end{aligned}$$

Hence, $(\bar{h}(E(G)))^* = [1, 14t]$ and h is a ρ-labeling of G.

Case 2: $r \ge 3$ is odd.

Let $r = 2t-1$. Let $h\colon V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases} 2j-2 & \text{if } i=1, 1\le j\le t,\\ 2j-1 & \text{if } i=1, t<j\le 2t-1,\\ 28t-2j-12 & \text{if } i=2, 1\le j\le t,\\ 22t+j-11 & \text{if } i=2, t<j\le 2t-1,\\ 2j-1 & \text{if } i=3, 1\le j\le t,\\ 2t+2j-2 & \text{if } i=3, t<j\le 2t-1,\\ 22t+j-1 & \text{if } i=4, 1\le j\le t,\\ 10t-2j-2 & \text{if } i=4, t<j\le 2t-1,\\ 2t+2j-2 & \text{if } i=5, 1\le j\le 2t,\\ 2t+2j-1 & \text{if } i=5, t<j\le 2t-1,\\ 14t+4j-9 & \text{if } i=6, 1\le j\le t,\\ 14t+4j-8 & \text{if } i=6, t<j\le 2t-1,\\ 10t+3j-7 & \text{if } i=7, 1\le j\le t,\\ 4t+3j-3 & \text{if } i=7, t<j\le 2t-1.\end{cases}$$

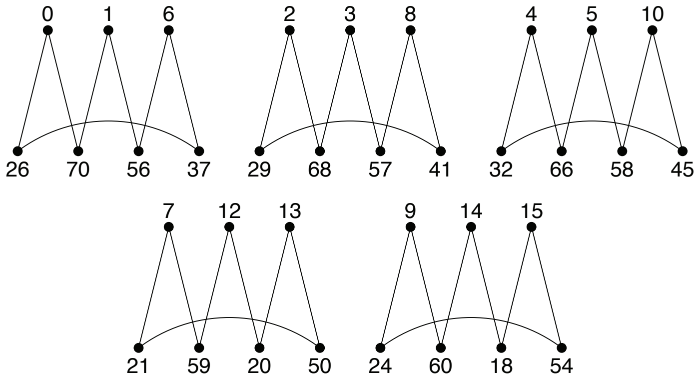

Figure 2: A ρ-labeling of $5C_7$.

An example of the ρ-labeling of $5C_7$ obtained from h is shown in Figure 2. If we proceed as in Case 1, it is easy to verify that the given labeling is a ρ-labeling of rC_7.

Theorem 3 *Let r be a positive integer and let $n \geq 9$ be odd. Let $G = rC_n$. Then G admits a ρ-labeling.*

Proof. This is already known to be true if $r = 1$. We consider four cases.

Case 1a: $n \equiv 3 \pmod 4$, $n \geq 11$, and r is even.
Let $n = 4m+3$ and $r = 2t$, where $m \geq 2$ and $t \geq 1$. Partition $V(G)$ into the following sets:

$$\begin{aligned}
S_1 &= \{v_{i,j}\colon i \text{ odd}, 1 \leq i < 4m+3, 1 \leq j \leq 2t\},\\
S_2 &= \{v_{i,j}\colon i \text{ even}, 1 \leq i \leq 2m-2, 1 \leq j \leq t\},\\
S_3 &= \{v_{i,j}\colon i \text{ even}, 2m-2 < i < 4m+2, 1 \leq j \leq t\},\\
S_4 &= \{v_{i,j}\colon i \text{ even}, 1 \leq i \leq 2m+2, t < j \leq 2t\},\\
S_5 &= \{v_{i,j}\colon i \text{ even}, 2m+2 < i < 4m+2, t < j \leq 2t\},\\
S_6 &= \{v_{i,j}\colon i = 4m+2, 1 \leq j \leq 2t\},\\
S_7 &= \{v_{i,j}\colon i = 4m+3, 1 \leq j \leq 2t\}.
\end{aligned}$$

Let $h\colon V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases}
-t + (ti+j) - 1 & \text{if } v_{i,j} \in S_1,\\
16tm + 14t - (ti+j) + 1 & \text{if } v_{i,j} \in S_2,\\
16tm + 12t - (ti+j) + 1 & \text{if } v_{i,j} \in S_3,\\
8tm + 6t - (ti+j) + 1 & \text{if } v_{i,j} \in S_4,\\
8tm + 4t - (ti+j) + 1 & \text{if } v_{i,j} \in S_5,\\
12tm + 2t + 2j - 1 & \text{if } v_{i,j} \in S_6,\\
8tm + 4t + 3j - 2 & \text{if } v_{i,j} \in S_7.
\end{cases}$$

An example of the ρ-labeling of $4C_{11}$ obtained from h is shown in Figure 3.

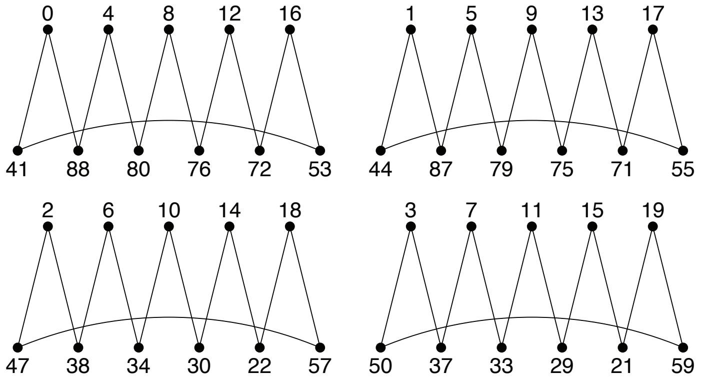

Figure 3: A ρ-labeling of $4C_{11}$.

Let h_k be the restriction of h on S_k. Since t and m are constants, h_6 and h_7 are strictly increasing, and thus one-to-one. For $k \in [1,5]$ and $v_{i,j}, v_{i',j'} \in S_k$, if $h_k(v_{i,j}) = h_k(v_{i',j'})$, then $t(i-i') = j'-j$. Since $i-i'$ is even and $j,j' \in [1,2t]$, if $t(i-i') = j'-j$, then $i=i'$ and $j=j'$. Thus, h_k is one-to-one for $k \in [1,5]$. Moreover,

$$\begin{aligned}
h(S_1) &\subseteq [0, 4tm+2t-1],\\
h(S_2) &\subseteq [14tm+15t+1, 16tm+12t],\\
h(S_3) &\subseteq [12tm+11t+1, 14tm+12t],\\
h(S_4) &\subseteq [6tm+2t+1, 8tm+3t],\\
h(S_5) &\subseteq [4tm+2t+1, 6tm-t],\\
h(S_6) &\subseteq [12tm+2t+1, 12tm+6t-1],\\
h(S_7) &\subseteq [8tm+4t+1, 8tm+10t-2].
\end{aligned}$$

We see that $0 \le h(S_1) < h(S_5) < h(S_4) < h(S_7) < h(S_6) < h(S_3) < h(S_2) \le 16tm+12t$. Therefore, h is one-to-one and $h(V(G)) \subseteq [0, 2|E(G)|]$.

We now compute the resulting edge labels. If $i \in [1, 2m-2]$ and $j \in [1,t]$, then $e_{i,j} \in e_{S_1,S_2}$. Additionally, if i is odd, then

$$\bar{h}(e_{i,j}) = (16tm + 14t - (t(i+1)+j) + 1) - (-t + (ti+j) - 1),$$

and if i is even, then

$$\bar{h}(e_{i,j}) = (16tm + 14t - (ti+j) + 1) - (-t + (t(i+1)+j) - 1).$$

Therefore,

$$\bar{h}(e_{S_1,S_2}) = \{16tm + 14t - 2(ti+j) + 2 \colon 1 \le i \le 2m-2, 1 \le j \le t\}.$$

Similarly, we have

$$\begin{aligned}
\bar{h}(e_{S_1,S_3}) &= \{16tm + 12t - 2(ti+j) + 2\colon 2m-2 < i < 4m+1, 1 \le j \le t\},\\
\bar{h}(e_{S_1,S_4}) &= \{8tm + 6t - 2(ti+j) + 2\colon 1 \le i \le 2m+2, t < j \le 2t\},\\
\bar{h}(e_{S_1,S_5}) &= \{8tm + 4t - 2(ti+j) + 2\colon 2m+2 < i < 4m+1, t < j \le 2t\},\\
\bar{h}(e_{S_1,S_6}) &= \bar{h}(E_{4m+1}) = \{8tm + 2t + j\colon 1 \le j \le 2t\},\\
\bar{h}(e_{S_6,S_7}) &= \bar{h}(E_{4m+2}) = \{4tm - 2t - j + 1\colon 1 \le j \le 2t\},\\
\bar{h}(e_{S_1,S_7}) &= \bar{h}(E_{4m+3}) = \{8tm + 4t + 2j - 1\colon 1 \le j \le 2t\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\bar{h}(e_{S_1,S_2}) &= \{2k\colon k \in [6tm+8t+1, 8tm+6t]\} \subseteq [12tm+16t+2, 16tm+12t],\\
\bar{h}(e_{S_1,S_3}) &= \{2k\colon k \in [4tm+5t+1, 6tm+7t]\} \subseteq [8tm+10t+2, 12tm+14t],\\
\bar{h}(e_{S_1,S_4}) &= \{2k\colon k \in [2tm-t+1, 4tm+t]\} \subseteq [4tm-2t+2, 8tm+2t],\\
\bar{h}(e_{S_1,S_5}) &= \{2k\colon k \in [1, 2tm-2t]\} \subseteq [2, 4tm-4t],\\
\bar{h}(e_{S_1,S_6}) &= [8tm+2t+1, 8tm+4t],\\
\bar{h}(e_{S_6,S_7}) &= [4tm-4t+1, 4tm-2t],\\
\bar{h}(e_{S_1,S_7}) &= \{2k+1\colon k \in [4tm+2t, 4tm+4t-1]\} \subseteq [8tm+4t+1, 8tm+8t-1].
\end{aligned}$$

Since the edge labels in $\bar{h}(e_{S_1,S_2})$ and $\bar{h}(e_{S_1,S_3})$ all exceed $8tm+6t$, we have

$$\begin{aligned}
(\bar{h}(e_{S_1,S_2}))^* &= \{2(8tm+6t)+1-(16tm+14t-2(ti+j)+2)\colon 1 \le i \le 2m-2, 1 \le j \le t\}\\
&= \{-2t + 2(ti+j) - 1\colon 1 \le i \le 2m-2, 1 \le j \le t\}\\
&= \{2k+1\colon k \in [0, 2tm-2t-1]\} \subseteq [1, 4tm-4t-1],\\
(\bar{h}(e_{S_1,S_3}))^* &= \{2(8tm+6t)+1-(16tm+12t-2(ti+j)+2)\colon 2m-2 < i < 4m+1,\\
&\qquad 1 \le j \le t\}\\
&= \{2(ti+j)-1\colon 2m-2 < i < 4m+1, 1 \le j \le t\}\\
&= \{2k+1\colon k \in [2tm-t, 4tm+t-1]\} \subseteq [4tm-2t+1, 8tm+2t-1].
\end{aligned}$$

Additionally, for $e_{i,j} \in e_{S_1,S_7}$ we have $\bar{h}(e_{i,j}) > 8tm+6t$ only if $j \in [t+1, 2t]$. Thus,

$$\begin{aligned}
(\bar{h}(e_{S_1,S_7}))^* &= \{8tm+4t+2j-1\colon 1 \le j \le t\} \cup \{8tm+8t-2j+2\colon t < j \le 2t\}\\
&= [8tm+4t+1, 8tm+6t].
\end{aligned}$$

Otherwise, the label of an edge under $\bar{h}$ is also the length of the edge. Finally, we can see from the difference in parities of their elements that $(\bar{h}(e_{S_1,S_2}))^* \cap (\bar{h}(e_{S_1,S_5}))^* = \emptyset$ and $(\bar{h}(e_{S_1,S_3}))^* \cap (\bar{h}(e_{S_1,S_4}))^* = \emptyset$. Thus,

$$\begin{aligned}
(\bar{h}(e_{S_1,S_2}))^* \cup (\bar{h}(e_{S_1,S_5}))^* &= [1, 4tm-4t],\\
(\bar{h}(e_{S_1,S_3}))^* \cup (\bar{h}(e_{S_1,S_4}))^* &= [4tm-2t+1, 8tm+2t],\\
(\bar{h}(e_{S_1,S_6}))^* &= [8tm+2t+1, 8tm+4t],\\
(\bar{h}(e_{S_6,S_7}))^* &= [4tm-4t+1, 4tm-2t],\\
(\bar{h}(e_{S_1,S_7}))^* &= [8tm+4t+1, 8tm+6t].
\end{aligned}$$

Hence, $(\bar{h}(E(G)))^* = [1, 8tm + 6t]$ and h is a ρ-labeling of G.

Case 1b: $n \equiv 3 \pmod 4$, $n \geq 11$, and r is odd.
Let $n = 4m + 3$ and $r = 2t + 1$ where $m \geq 2$ and $t \geq 1$. Partition $V(G)$ into the following sets:

$$\begin{aligned}
S_1 &= \{v_{i,j} \colon i \text{ odd}, 1 \leq i \leq 4m+3, 1 \leq j \leq 2t+1\},\\
S_2 &= \{v_{i,j} \colon i \text{ even}, 1 \leq i \leq 2m-2, 1 \leq j \leq t+1\},\\
S_3 &= \{v_{i,j} \colon i \text{ even}, 2m \leq i \leq 4m, 1 \leq j \leq t+1\},\\
S_4 &= \{v_{i,j} \colon i \text{ even}, 1 \leq i \leq 2m+2, t+2 \leq j \leq 2t+1\},\\
S_5 &= \{v_{i,j} \colon i \text{ even}, 2m+4 \leq i \leq 4m, t+2 \leq j \leq 2t+1\},\\
S_6 &= \{v_{i,j} \colon i = 4m+2, 1 \leq j \leq 2t+1\},\\
S_7 &= \{v_{i,j} \colon i = 4m+3, 1 \leq j \leq 2t+1\}.
\end{aligned}$$

Let $h \colon V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases}
-t + (ti+j) - 1 + \frac{i-1}{2} & \text{if } v_{i,j} \in S_1,\\
16tmi + 14t + 8m - (ti+j) + 8 - \frac{i}{2} & \text{if } v_{i,j} \in S_2,\\
16tm + 12t + 8m - (ti+j) + 7 - \frac{i}{2} & \text{if } v_{i,j} \in S_3,\\
8tm + 6t + 4m - (ti+j) + 4 - \frac{i}{2} & \text{if } v_{i,j} \in S_4,\\
8tm + 4t + 4m - (ti+j) + 3 - \frac{i}{2} & \text{if } v_{i,j} \in S_5,\\
12tm + 2t + 6m + 2j & \text{if } v_{i,j} \in S_6,\\
8tm + 4t + 4m + 3j & \text{if } v_{i,j} \in S_7.
\end{cases}$$

An example of the ρ-labeling of $5C_{11}$ obtained from h is shown in Figure 4. If we proceed as in Case 1a, it is easy to verify that the given labeling is a ρ-labeling of rC_n.

Case 2a: $n \equiv 1 \pmod 4$, $n \geq 9$, and r even.
Let $n = 4m+1$ and $r = 2t$ where $m \geq 2$ and $t \geq 1$. Partition $V(G)$ into the following sets:

$$\begin{aligned}
S_1 &= \{v_{i,j} \colon i \text{ odd}, 1 \leq i < 4m+1, 1 \leq j \leq 2t\},\\
S_2 &= \{v_{i,j} \colon i \text{ even}, 1 \leq i \leq 2m, 1 \leq j \leq t\},\\
S_3 &= \{v_{i,j} \colon i \text{ even}, 2m < i \leq 4m-2, 1 \leq j \leq t\},\\
S_4 &= \{v_{i,j} \colon i \text{ even}, 1 \leq i \leq 2m-2, t < j \leq 2t\},\\
S_5 &= \{v_{i,j} \colon i \text{ even}, 2m < i \leq 4m-2, t < j \leq 2t\},\\
S_6 &= \{v_{i,j} \colon i = 4m, 1 \leq j \leq t\},\\
S_7 &= \{v_{i,j} \colon i = 4m, t+1 \leq j \leq 2t\},\\
S_8 &= \{v_{i,j} \colon i = 4m+1, 1 \leq j \leq t\},\\
S_9 &= \{v_{i,j} \colon i = 4m+1, t+1 \leq j \leq 2t\}.
\end{aligned}$$

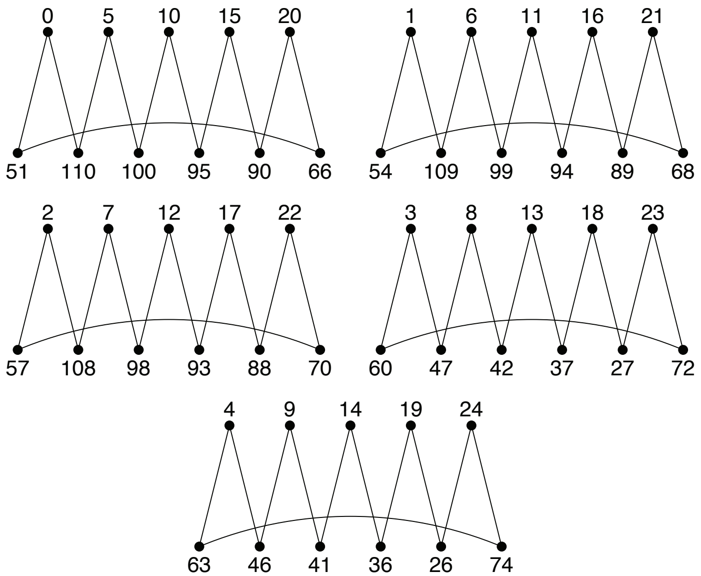

Figure 4: A ρ-labeling of $5C_{11}$.

Let $h\colon V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases} -t + (ti + j) - 1 & \text{if } v_{i,j} \in S_1, \\ 16tm + 6t - (ti + j) + 1 & \text{if } v_{i,j} \in S_2, \\ 16tm + 4t - (ti + j) + 1 & \text{if } v_{i,j} \in S_3, \\ 8tm + 2t - (ti + j) + 1 & \text{if } v_{i,j} \in S_4, \\ 8tm - (ti + j) + 1 & \text{if } v_{i,j} \in S_5, \\ 12tm + 2t - 3j + 3 & \text{if } v_{i,j} \in S_6, \\ 12tm + 6t - j + 1 & \text{if } v_{i,j} \in S_7, \\ 8tm - j + 1 & \text{if } v_{i,j} \in S_8, \\ 8tm + 8t - 3j + 2 & \text{if } v_{i,j} \in S_9. \end{cases}$$

An example of the ρ-labeling of $4C_9$ obtained from h is shown in Figure 5. If we proceed as in Case 1a, it is easy to verify that the given labeling is a ρ-labeling of rC_n.

Case 2b: $n \equiv 1 \pmod 4$, $n \geq 9$, and r odd.

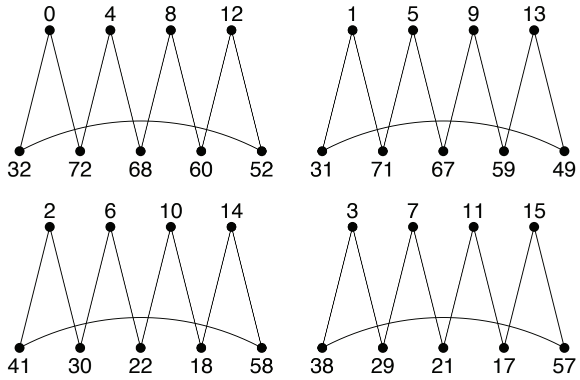

Figure 5: A ρ-labeling of $4C_9$.

Let $n = 4m + 1$ and $r = 2t + 1$. Partition $V(G)$ into the following sets:

$$\begin{aligned}
S_1 &= \{v_{i,j} : i \text{ odd}, 1 \le i < 4m+1, 1 \le j \le 2t+1\},\\
S_2 &= \{v_{i,j} : i \text{ even}, 1 \le i \le 2m, 1 \le j \le t+1\},\\
S_3 &= \{v_{i,j} : i \text{ even}, 2m < i \le 4m-2, 1 \le j \le t+1\},\\
S_4 &= \{v_{i,j} : i \text{ even}, 1 \le i \le 2m-2, t+1 < j \le 2t+1\},\\
S_5 &= \{v_{i,j} : i \text{ even}, 2m \le i \le 4m-2, t+1 < j \le 2t+1\},\\
S_6 &= \{v_{i,j} : i = 4m, 1 \le j \le t+1\},\\
S_7 &= \{v_{i,j} : i = 4m, t+1 < j \le 2t+1\},\\
S_8 &= \{v_{i,j} : i = 4m+1, 1 \le j \le t+1\},\\
S_9 &= \{v_{i,j} : i = 4m+1, t+1 < j \le 2t+1\}.
\end{aligned}$$

Let $h\colon V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases}
-t + (ti+j) - 1 + \frac{i-1}{2} & \text{if } v_{i,j} \in S_1,\\
16tm + 6t + 8m - (ti+j) + 4 - \frac{i}{2} & \text{if } v_{i,j} \in S_2,\\
16tm + 4t + 8m - (ti+j) + 3 - \frac{i}{2} & \text{if } v_{i,j} \in S_3,\\
8tm + 2t + 4m - (ti+j) + 2 - \frac{i}{2} & \text{if } v_{i,j} \in S_4,\\
8tm + 4m - (ti+j) + 1 - \frac{i}{2} & \text{if } v_{i,j} \in S_5,\\
12tm + 2t + 6m - 3j + 4 & \text{if } v_{i,j} \in S_6,\\
12tm + 6t + 6m - j + 4 & \text{if } v_{i,j} \in S_7,\\
8tm + 4m - j + 1 & \text{if } v_{i,j} \in S_8,\\
8tm + 8t + 4m - 3j + 6 & \text{if } v_{i,j} \in S_9.
\end{cases}$$

An example of the ρ-labeling of $5C_9$ obtained from h is shown in Figure 6. If we proceed as in Case 1a, it is easy to verify that the given labeling is a ρ-labeling of rC_n. □

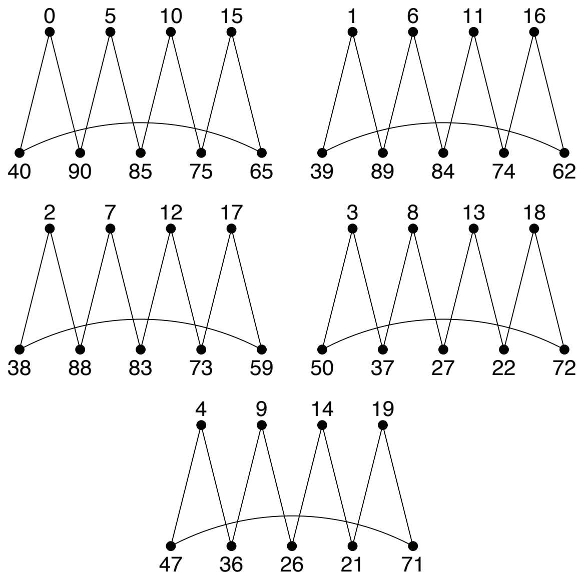

Figure 6: A ρ-labeling of $5C_9$.

In light of Theorem 1 and of the results from [1], [3], and [4], we have the following.

Corollary 4 *Let $r \geq 1$ and $n \geq 3$ be integers and let $G = rC_n$. Then there exists a cyclic G-decomposition of K_{2rn+1}.*

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