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On uniformly balanced graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A vertex labeling $f : V \to \{0, 1\}$ induces a partial edge labeling $f^* : E \to \{0, 1\}$ defined by $f^*(uv) = f(u)$

if and only if $f(u) = f(v)$. For $i = 0, 1$, let $v_f(i) = |\{v \in V : f(v) = i\}|$, and $e_f(i) = |\{e \in E : f^*(e) = i\}|$. A graph G is uniformly balanced if $|e_f(0) - e_f(1)| \leq 1$ for any vertex labeling f that satisfies $|v_f(0) - v_f(1)| \leq 1$. In this paper, we characterize uniformly balanced graphs, and present several ways to construct uniformly balanced graphs.

In Memory of Professor F.T. Boesch

1 Introduction

In [19], Lee, Liu and Tan introduced a new labeling problem in graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0, 1\}$. For each vertex labeling f of G, we define a partial edge labeling f^* of G in the following manner:

$$f^*(uv) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that the edge uv is not labeled if $f(u) \neq f(v)$. Thus f^* is a partial function from $E(G)$ into $\{0, 1\}$. We call it the induced partial edge labeling. For $i = 0, 1$, define $v_f(i) = |\{v \in V(G) : f(v) = i\}|$, and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. When the context is clear, we will suppress the subscript and simply write $v(0)$, $v(1)$, $e(0)$ and $e(1)$, respectively.

Definition 1. A vertex labeling f of a graph is said to be *friendly* if $|v(0) - v(1)| \leq 1$.

Definition 2. We call a graph *balanced* if it admits a friendly labeling f such that $|e(0) - e(1)| \leq 1$.

Balanced graphs have enjoyed a renewed attention recently. In [4, 5, 8–10, 13, 15, 17, 18, 20–24, 26], many related problems have been posed and studied, among them the balance index set of a graph [4, 9, 10, 13, 15, 17, 18, 20–22, 24, 26].

Definition 3. The *balance index set* of a graph G, denoted $\mathrm{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

Example 1. Let V_1 and V_2 denote the two partite sets of $K_{3,3}$. Due to symmetry, there are only two friendly vertex labelings. If all the vertices in each partite set are assigned the same label, all the edges are unlabeled. For brevity, call a vertex an i-vertex if it is labeled i, and an edge an i-edge if it is labeled i. If one 0-vertex is in V_1, and the other two 0-vertices are in V_2, then V_1 contains two 1-vertices, and V_2 contains only one 1-vertex, and there are two 0-edges and two 1-edges. In both cases, $|e(0) - e(1)| = 0$. Thus $\mathrm{BI}(K_{3,3}) = \{0\}$. □

The balance index set of a complete bipartite graph has been determined.

Theorem 1.1 ([18]) *For $m \leq n$,*

$$BI(K_{m,n}) = \begin{cases} \left\{\left|(n-m)\left(i-\frac{m}{2}\right)\right| : 0 \leq i \leq m\right\} & \text{if } m+n \text{ is even,} \\ \left\{\left|(n-m)\left(i-\frac{m}{2}\right) \pm \frac{m}{2}\right| : 0 \leq i \leq m\right\} & \text{if } m+n \text{ is odd.} \end{cases}$$

Definition 4. A graph G is *uniformly balanced* (which is also referred to as strongly balanced in [24]) if it is balanced for any friendly labeling f. In other words, G is uniformly balanced if $\mathrm{BI}(G) \subseteq \{0, 1\}$.

Corollary 1.2 *The complete bipartite graph $K_{m,n}$ is uniformly balanced if and only if $m = n$, or $(m, n) \in \{(1, 2), (1, 3)\}$.*

Example 2. The cities served by a cable TV company can be represented by vertices, and the connection between the cities by edges, of a simple graph G. The company wants to split into two smaller companies A and B. Each city it currently serves will be covered by either A and B (but not both). A connection between two cities served by the same new company will be owned and maintained by the that company. Otherwise, the connection will be co-owned by the two new companies. The problem of dividing the cities evenly between A and B so that the solely owned connections are also evenly divided between A and B is equivalent to determining whether G is balanced. □

We can view balance index sets as a generalization of balancedness, just as friendly index sets (see, for example, [14]) can be regarded as a generalization of the concept of cordiality introduced by Cahit [1]. The problem of characterizing uniformly cordial graphs had been completely solved by Chartrand, Lee and Zhang [2]. This prompts us to study the characterization of uniformly balanced graphs. To our surprise, the answer is rather simple. We will present the complete solution in the next section. In Section 3, we demonstrate how to generate new uniformly balanced graphs from graphs that are known to be uniformly balanced.

2 Uniformly Balanced Graphs

A graph is uniformly balanced if and only if the maximum value of $|e(1) - e(0)|$ over all friendly labelings is at most 1. To find the maximum value of $|e(1) - e(0)|$, due to symmetry, it suffices to maximize $e(1) - e(0)$. It was shown in [7, 16] that

$$2[e(1) - e(0)] = \sum_{f(v_i)=1} \deg(v_i) - \sum_{f(v_i)=0} \deg(v_i). \tag{1}$$

Consequently, we need to investigate the degree sequence of a graph if we want to study its balance index set.

We first consider graphs with no isolated vertices. To facilitate our discussion, we shall assume that the degree sequence of the graph is $1 \leq d_1 \leq d_2 \leq \cdots \leq d_p$, and define

$$S = \sum_{i=\lfloor p/2 \rfloor+1}^{p-1} d_i - \sum_{i=2}^{\lfloor p/2 \rfloor} d_i.$$

Since $d_{\lfloor p/2\rfloor+j} \geq d_{1+j}$ for $j = 1, 2, \ldots, \lfloor p/2\rfloor - 1$, we find $S \geq 0$. In fact, $S \geq d_{p-1} > 0$ if p is odd.

Lemma 2.1 *If a graph without isolated vertices is uniformly balanced, then* $d_p - d_1 \leq 2$.

Proof. Let the vertices of the graph be $v_1, v_2, \ldots, v_p$ such that $\deg(v_i) = d_i$. Consider the friendly labeling f where

$$f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lfloor p/2\rfloor, \\ 1 & \text{if } \lfloor p/2\rfloor < i \leq p. \end{cases}$$

Thus (1) becomes

$$2\,[e_f(1) - e_f(0)] = d_p - d_1 + S \geq d_p - d_1.$$

If the graph is uniformly balanced, then $|e_f(1) - e_f(0)| \leq 1$, hence $d_p - d_1 \leq 2$. □

Corollary 2.2 *If a graph is uniformly balanced and has no isolated vertices, then its degree sequence contains at most three distinct values.*

Corollary 2.2 provides a necessary condition for a graph to be uniformly balanced: it must be regular, bi-regular, or tri-regular. The case of k-regular graphs had been settled, and can be easily derived from (1).

Theorem 2.3 ([24]) *Let G be a k-regular graph of order p. Then*

$$BI(G) = \begin{cases} \{0\} & \text{if } p \text{ is even}, \\ \{k/2\} & \text{if } p \text{ is odd}. \end{cases}$$

Corollary 2.4 *All 2-regular graphs are uniformly balanced. For $k \neq 2$, a k-regular graph is uniformly balanced if and only if its order is even.*

Corollary 2.5 *Let tK_n denote the disjoint union of t complete graphs of order n. Then tK_n is uniformly balanced if and only if (i) n is even, (ii) $n = 1, 3$, or (iii) $n \geq 5$ is odd and t is even.*

An (r, s)-regular graph is a graph whose degrees are either r or s, where $1 \leq r < s \leq p - 1$. Although the balance index sets of such graphs had been determined in [16, 24], the analysis of bi-regular uniformly balanced graphs can be carried out in the same way we prove Lemma 2.1.

Theorem 2.6 *Any bi-regular uniformly balanced graph of even order has a degree sequence in one of the following forms:*

- $(r, r, r, \ldots, r, r+1, r+1)$;
- $(r, r, r+1, r+1, \ldots, r+1)$;
- $(r, r, r, \ldots, r, r+2)$;

- *$(r, r+2, r+2, \ldots, r+2)$;*

Proof. For a graph to be bi-regular and uniformly balanced, we need $d_p - d_1 \in \{1, 2\}$. It is obvious that the vertex labeling f used in the proof of Lemma 2.1 yields the maximum value in $e(1) - e(0)$, and we know that

$$2[e_f(1) - e_f(0)] = d_p - d_1 + S.$$

If $d_p - d_1 = 1$, then we need $S = 1$, that is, $d_{p-1} - d_2 = 1$, and $d_3 = d_4 = \cdots = d_{p-2}$. There are only two possibilities, as indicated above. Similarly, if $d_p - d_1 = 2$, we need $S = 0$, which requires $d_2 = d_3 = \cdots = d_{p-1}$. The proof is now complete. □

Theorem 2.7 *A bi-regular uniformly balanced graph of odd order must be of the form $P_3 \cup tP_2$ for some integer $t \geq 0$.*

Proof. Since $S > 0$ if p is odd, the only hope for a graph of odd order to be uniformly balanced is $S = 1$ and $d_p - d_1 = 1$. For $S = 1$, we need $d_2 = d_3 = \cdots = d_{p-1} = 1$. Therefore the degree sequence must be of the form $(1, 1, \ldots, 1, 2)$. □

Example 3. All paths of even order are uniformly balanced because they meet the condition stated in Theorem 2.6. □

Example 4. The degree sequence of a connected bi-regular graph on four vertices is $(2, 2, 3, 3)$, $(1, 1, 1, 3)$, or $(1, 1, 2, 2)$. They are realized by the graphs $K_4 - e$, $K_{1,3}$, and P_4, respectively. All of them satisfy the conditions in Theorem 2.6, hence they are all uniformly balanced. □

Example 5. A connected $(p, p+1)$-graph with minimum degree 2 has a degree sequence $(2, 2, \ldots, 2, 4)$ or $(2, 2, \ldots, 2, 3, 3)$. It takes the form of an one-point union of two cycles, a long dumbbell graph, or a generalized theta graph, including cycles with a long chord (see [25]). Theorem 2.6 asserts that all these $(p, p+1)$-graphs of even order are uniformly balanced. □

Example 6. The cylinder graph $C_m \times P_n$, where $m, n \geq 3$, is a $(3, 4)$-regular graph. It cannot be uniformly balanced. □

The case of tri-regular uniformly balanced graphs is even simpler than the bi-regular case.

Theorem 2.8 *Any tri-regular uniformly balanced graph must be of even order, and has a degree sequence of the form $(r, r+1, r+1, \ldots, r+1, r+2)$.*

Proof. For a graph to be tri-regular and uniformly balanced, its degree sequence has to be of the form

$$r \leq d_1 \leq d_2 \leq \cdots \leq d_p = r + 2.$$

The vertex labeling f used in the proof of Lemma 2.1 yields the maximum value in $e(0) - e(1)$, and we know

$$2[e_f(1) - e_f(0)] = d_p - d_1 + S = 2 + S,$$

where $S > 0$ if p is odd. Hence we need p to be even, *and*, at the same time, $S = 0$. The latter requires $d_{\lfloor p/2 \rfloor + j} = d_{1+j}$ for each $j = 1, 2, \ldots, \lfloor p/2 \rfloor - 1$. Since the graph is tri-regular, we also need $d_j = d_1 + 1$ for $2 \leq j \leq p - 1$. □

Example 7. The graph Amal(C_m, P_n) is the one-point union of a cycle with a path. Since its degree sequence is $(1, 2, 2, \ldots, 2, 3)$, according to Theorem 2.8, it is uniformly balanced if $m + n$ is odd. □

Example 8. The wheel W_4 has a degree sequence $(3, 3, 3, 3, 4)$. Subdividing any edge produces a graph with degree sequence $(2, 3, 3, 3, 3, 4)$, which is the signature of a uniformly balanced graph. □

We can now state our main theorem.

Theorem 2.9 (Main Theorem — Part 1) *Let G be a uniformly balanced graph with no isolated vertices.*

(a) If the order of G is even, then its degree sequence must be one of the following forms:

- $(r, r, r, \ldots, r)$;
- $(r, r, r, \ldots, r, r + 1, r + 1)$;
- $(r, r, r + 1, r + 1, \ldots, r + 1)$;
- $(r, r, r, \ldots, r, r + 2)$;
- $(r, r + 2, r + 2, \ldots, r + 2)$;
- $(r, r + 1, r + 1, \ldots, r + 1, r + 2)$.

(b) If the order of G is odd, then either G is 2-regular, or $G = P_3 \cup tP_2$.

Notice that while K_4 is uniformly balanced, $K_4 \cup K_1$ is not. To see why, label one of the vertices on K_4 and the isolated vertex 0, and the other three vertices 1. Then $e(1) - e(0) = 3$. It is clear that the addition of isolated vertices allows a shuffling of the 0- and 1-vertices that may make a graph no longer uniformly balanced. Nevertheless, most uniformly balanced graph have no isolated vertices.

Lemma 2.10 *Let G be a uniformly balanced graph. If $\Delta(G) \geq 3$, where $\Delta(G)$ denotes the maximum degree of G, then G has no isolated vertices.*

Proof. Suppose G has an isolated vertex u. Let v be a vertex with $\deg(v) \geq 3$. Label u with 1, and v with 0, and the remaining vertices of G alternately with 1 and 0 in any order. The resulting vertex labeling f is balanced because G is uniformly balanced. Therefore $e_f(1) - e_f(0) \in \{0, \pm 1\}$. Form a new vertex labeling g by switching the labels of u and v. In other words, $g(u) = 0$, $g(v) = 1$, and $g(x) = f(x)$ if $x \notin \{u, v\}$. Obviously g is still friendly. However, because of (1), we find

$$e_g(1) - e_g(0) = e_f(1) - e_f(0) + \deg(v) \geq 2.$$

Then g is not balanced, which contradicts the assumption that G is uniformly balanced. Hence G cannot contain any isolated vertex. □

Theorem 2.11 (Main Theorem — Part 2) *If G is a uniformly balanced graph with isolated vertices, then G is of the form tK_1, $P_2 \cup tK_1$, $tP_2 \cup 2K_1$, or $H \cup K_1$, where $H \in \{C, tP_2, P_n, P_3 \cup tP_2, P_4 \cup tP_2\}$ for any 2-regular graph C.*

Proof. For any $t \geq 1$, it is obvious that tK_1 is uniformly balanced, so we may assume $G = H \cup tK_1$, where H has no isolated vertices. Lemma 2.10 implies that $\Delta(H) \leq 2$. It is clear that $P_2 \cup tK_1$ is uniformly balanced for any integer $t \geq 1$, so we may assume the order of H is at least 3.

Suppose $t \geq 3$, pick any three isolated vertices u_1, u_2, u_3 and label them 1. Pick any three vertices v_1, v_2, v_3 from H and label them 0. Label the remaining vertices of G alternately with 1 and 0 in any order. The resulting vertex labeling f is balanced because G is uniformly balanced. Switching the labels of u_i with v_i for each $i = 1, 2, 3$ produces a new vertex labeling g with

$$e_g(1) - e_g(0) = e_f(1) - e_f(0) + \sum_{i=1}^{3} \deg(v_i) \geq 2.$$

This shows that g cannot be balanced. Therefore if G is uniformly balanced and the order of H is at least 3, then we need $t \leq 2$.

In fact, the same argument shows that if H has a vertex of degree 2, then we need $t \leq 1$. Thus, if $t = 2$, we also need $\Delta(H) = 1$. This means $H = tP_2$.

For $t = 1$, it is possible to have $\Delta(H) = 2$. It is easy to show that $H \cup K_1$ is uniformly balanced if and only if H is uniformly balanced. Inspecting the graphs in Theorem 2.9 with maximum degree 2 yields the possibilities listed above. □

3 Construction of Uniformly Balanced Graphs

Given a uniformly balanced graph, we can generate other uniformly balanced graphs. We present here four such constructions.

3.1 Construction via Edge Addition or Removal

Theorem 3.1 *Let G be a uniformly balanced graph with $BI(G) = \{0\}$. Then*

1. *$G + e$ is uniformly balanced for any $e \notin E(G)$, and*
2. *$G - e$ is uniformly balanced for any $e \in E(G)$.*

Proof. Consider any friendly labeling of G. Note that, adding any new edge e to G (or deleting any existing edge e from G, respectively) does not affect friendliness. If the two vertices incident to e are labeled differently, the edge e will be unlabeled; hence $e(0) - e(1)$ remains unchanged. If the two vertices are labeled the same, then $e(0) - e(1)$ will be changed by ± 1. Recall that $|e(0) - e(1)| = 0$ in G. Therefore, in $G + e$ (or $G - e$, respectively), we have $|e(0) - e(1)| \leq 1$, which completes the proof immediately. □

Example 9. According to Theorem 2.3, C_n is uniformly balanced, with $\mathrm{BI}(C_n) = \{0\}$ if n is even, in which case $P_n = C_n - e$ is also uniformly balanced. □

Example 10. We learn from Theorem 2.3 that $\mathrm{BI}(C_8) = \mathrm{BI}(C_4 \cup C_4) = \{0\}$. It follows that the path $P_8 = C_8 - e$ and the dumbbell graph $\mathrm{DB}(4,4) = (C_4 \cup C_4) + e$ are uniformly balanced. In addition, $C_8 + e$, a cycle with a chord, is always uniformly balanced regardless of which two vertices the chord joins. □

3.2 Construction via Edge Rotation

Given a graph G, choose three vertices a, b, and c so that their induced subgraph contains only the edge ab. Construct a new graph $\mathrm{RG}(a,b,c)$ by removing the edge ab, and adding a new edge ac.

Example 11. Figure 1 illustrates the construction of $\mathrm{RG}(a,b,c)$ from the graph G on the left. In the figure, we use open and solid circles to represent vertices of different labels. Take note that $|e(0) - e(1)| = 0$ in both graphs. □

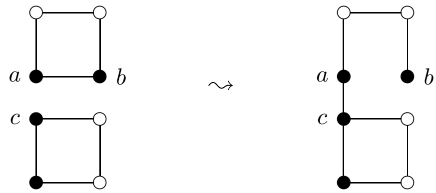

Figure 1: The construction of $\mathrm{RG}(a,b,c)$.

Lemma 3.2 *A friendly vertex labeling f on a graph G induces a friendly vertex labeling h on $RG(a,b,c)$. The difference between $e_f(0) - e_f(1)$ and $e_h(0) - e_h(1)$ is at most one.*

Proof. For a fixed friendly vertex labeling, we consider all 8 possible cases of labeling the three vertices a, b, and c. The changes in the value of $e(0) - e(1)$ are tabulated below.

a	b	c	change in $e(0)-e(1)$
0	0	0	0
0	0	1	-1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	-1
1	1	0	1
1	1	1	0

The lemma follows immediately. □

Theorem 3.3 *If G is uniformly balanced with $BI(G)=\{0\}$, then $RG(a,b,c)$ is also uniformly balanced.*

Example 12. The graph on the left of Figure 1 is $C_4 \cup C_4$, which is known to be uniformly balanced with $\mathrm{BI}(C_4 \cup C_4)=\{0\}$, according to Theorem 2.3. Hence the graph on the right of Figure 1 is also uniformly balanced. □

3.3 Construction via Edge Flipping

Given a graph G, choose three vertices a, b, and c such that their induced subgraph contains only two edges ab and ac. Construct a new graph $\mathrm{FG}(a,b,c)$ by removing the edges ab and ac, and adding a new edge bc. Consequently, if G is a (p,q)-graph, then $\mathrm{FG}(a,b,c)$ is a $(p,q-1)$-graph.

Example 13. Figure 2 illustrates the construction of $\mathrm{FG}(a,b,c)$ from the graph G on the left. Observe that $|e(0)-e(1)|=0$ in the first graph but $|e(0)-e(1)|=1$ in the second graph. □

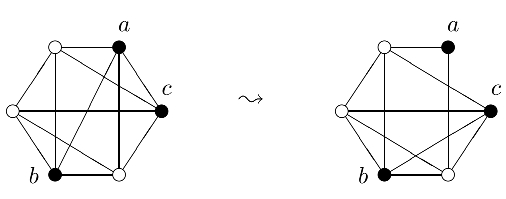

Figure 2: The construction of $\mathrm{FG}(a,b,c)$.

Lemma 3.4 *A friendly vertex labeling f on a graph G induces a friendly vertex labeling h on $FG(a,b,c)$. The difference between $e_f(0)-e_f(1)$ and $e_h(0)-e_h(1)$ is at most one.*

Proof. For a fixed friendly vertex labeling, we consider the all 8 possible cases of labeling the three vertices a, b, and c. The changes in the value of $e(0) - e(1)$ are tabulated below.

a	b	c	change in $e(0) - e(1)$
0	0	0	-1
0	0	1	-1
0	1	0	-1
0	1	1	-1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

The lemma follows immediately. □

Theorem 3.5 *If G is uniformly balanced with $BI(G) = \{0\}$, then $FG(a, b, c)$ is also uniformly balanced.*

Example 14. The graph G on the left of Figure 2 is uniformly balanced. According to Theorem 2.3, $\mathrm{BI}(G) = \{0\}$. Hence the graph on the right is also uniformly balanced. □

Corollary 3.6 *For $n \geq 1$, the graph $C_{2n+1} \cup K_1$ is uniformly balanced.*

Proof. Theorem 2.3 implies that $\mathrm{BI}(C_{2n+2}) = \{0\}$. Let b, a and c be any consecutive vertices on C_{2n+2}, then $\mathrm{FG}(a, b, c) = C_{2n+1} \cup K_1$ is also uniformly balanced. □

3.4 Construction via Edge Swapping

Given a graph G, choose four vertices a, b, c, and d such that $ab, cd \in E(G)$, but $ac, bd \notin E(G)$. Construct the graph $\mathrm{SG}(a, b, c, d)$ by removing the two edges ab and cd, and adding two new edges ac and bd.

Example 15. Figure 3 illustrates the construction of $\mathrm{SG}(a, b, c, d)$ from the graph G on the left. Note that $|e(0) - e(1)| = 1$ in both graphs. □

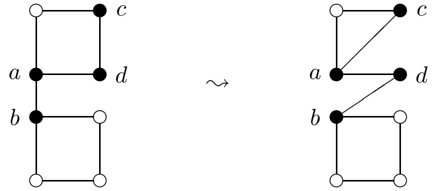

Figure 3: The construction of $\mathrm{SG}(a, b, c, d)$.

Lemma 3.7 *The balance index sets of G and $SG(a, b, c, d)$ are the same.*

Proof. We need to examine 16 cases of how the vertices a, b, c, and d are labeled. It is easy to verify that the value of $e(0) - e(1)$ remains unchanged in every case. □

Theorem 3.8 *If G is uniformly balanced, then $SG(a, b, c, d)$ is also uniformly balanced.*

Example 16. The graph on the left of Figure 3 is the dumbbell graph DB(4, 4), which is uniformly balanced. Thus the graph on the right of Figure 3 is also uniformly balanced. □

Example 17. The graph on the right of Figure 4 is uniformly balanced, because the graph on the left is uniformly balanced (see Example 16). □

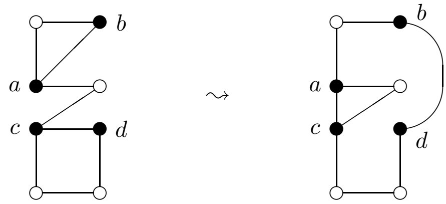

Figure 4: The construction of a new uniformly balanced graph.

Example 18. The graph G on the right of Figure 5 is uniformly balanced, because the graph on the left is uniformly balanced, according to Example 16. □

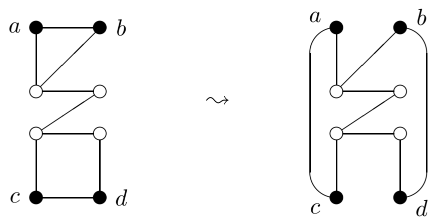

Figure 5: The construction of another uniformly balanced graph.

4 Closing Remarks and Open Questions

It is known that the balanced index set of a graph have a close connection to the graph's degree sequence (see, for example, [9, 24]). Hence we expect the characterization of uniformly balanced graph depends on the degree sequence. Such an expectation is affirmed by our findings.

Edge-balanced graphs, a dual form of balanced graphs, were introduced by Kong and Lee in [6]. Recently, several authors have started studying the associated edge-balance index sets [3, 11, 12]. It would be an interesting project to find a characterization of uniformly edge-balanced graphs.

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