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On the independence polynomials of path-like graphs

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Abstract

We investigate the independence polynomials of members of various infinite families of path-like graphs, showing that the coefficient sequences of such polynomials are logarithmically concave.

1 Introduction

In many ways the study of independence polynomials mirrors the older and more well-established study of chromatic polynomials. Both polynomials are known to yield a good deal of information about the graph from which they are derived, and this information is of use even in areas outside of mathematics. For instance, independence polynomials figure prominently in applications in chemistry (see [7], [23], [24]) and physics ([8] and [19]). Therefore structural aspects of these polynomials (degree, behavior of coefficients, location of roots, *etc.*) are very active subjects of research.

Levit and Mandrescu [11] offer a relatively recent and robust overview of the current research on independence polynomials; these authors have also contributed a number of original results to the area (relevant here are [12], [13], [14], [15], and [17]). A number of other scholars develop ideas related to the properties we introduce below: see Chen and Wang [2], Chudnovsky and Seymour [3], Godsil and McKay [4]; Gutman [5], [6], [7]; Hamidoune [9], Keilson and Gerber [10], Makowsky [16], Rosenfeld [18], Stevanović [21], Wang and Zhu [22], and Zhu [25]. There is considerable variety in the results these authors obtain, and in the methods used to obtain those results. Some are quite specific: for instance, Zhu [25], and more recently Wang and Zhu [22], demonstrate the unimodality of the independence polynomials of families of caterpillars. More generally, Hamidoune [9] showed that the independence polynomial of any claw-free graph is logarithmically concave (and therefore unimodal), and Chudnosvky and Seymour [3] generalized this by showing that the roots of the independence polynomial of any claw-free graph are all real.

A review of these articles will convince the reader that developing general techniques for working with independence polynomials is a difficult task and is often only possible on an *ad hoc* basis. For instance, the constructions due to Levit and Mandrescu, to Stevanović, and to Zhu are somewhat constrained, often focusing on a single family of graphs. Moreover, when techniques apply to broad families of graphs, frequently the focus is not on proving properties of the graphs' independence polynomials. For instance, though the techniques of Gutman apply to many graphs, they concern general identities involving independence polynomials and say less about the properties of the polynomials so constructed.

For these reasons, very general statements about independence polynomials are very difficult to make, and even general conjectures are hard to come by. (Perhaps the most outstanding such conjectures are that the independence polynomial of any tree is unimodal, and that the independence polynomial of any very well-covered graph is unimodal; see [11] for discussions of both.) Our goal in this article is more modest, but still far-reaching: here we show that the graphs obtained by certain inductive path-like constructions yield independence polynomials that are logarithmically concave, and therefore unimodal. The constructions we describe here, similar to those developed by the author and Salazar in [1], are relatively flexible, permitting construction of a wide variety of graph families. These families are considerably broader than many of the collections of path-like graphs constructed before.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Recall that an *independent set* S in G is a set of pairwise non-adjacent vertices. The *independence number* of G, $\alpha(G)$, is the cardinality of a largest independent set in G. The *independence polynomial* of G, denoted $I(G; x)$, is defined by

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + \cdots + s_{\alpha(G)} x^{\alpha(G)},$$

where s_k is the number of independent sets with cardinality k.

We say a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ is called *logarithmically concave* (or *log-concave*) if for all i, $1 \le i \le n-1$, $a_i^2 \ge a_{i-1}a_{i+1}$. A polynomial is called *unimodal* if the sequence of its coefficients satisfies $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$ for some j, $0 \le j \le n$. It is trivial to show that if a polynomial is log-concave then it is unimodal. For a fixed natural number n the binomial coefficients $(\binom{n}{i})_{i=0}^{n}$ give what is probably the best-known example of a log-concave sequence.

The following fact from [11] will be useful:

Proposition 1.1 *Suppose that every root of the polynomial* $p(x) = \sum_{i=0}^{n} a_i x^i$ $(a_i \ge 0)$ *is real. Then the sequence* $(a_i/\binom{n}{i})_{i=0}^{n}$ *is log-concave. (As a consequence,* (a_i) *itself is log-concave.)*

We now define the graphs which we will study; they are closely related to the graphs considered in [1]. Given $n \ge 1$ and $t \ge 2$, we define the *K_t-path of length n*

to be the graph $P(t,n) = (V,E)$ in which $V = \{v_1, v_2, \ldots, v_{t+n-1}\}$ and

$$E = \Big\{\{v_i, v_{i+j}\} \mid 1 \le i \le t+n-2,\ 1 \le j \le \min\{t-1, t+n-i-1\}\Big\}.$$

Such a graph consists of n copies of K_t, each glued to the previous one by identifying certain prescribed subgraphs isomorphic to K_{t-1}.

Now suppose that $G = (V,E)$ is a graph and let $U \subseteq V$ be a subset of its vertices. As in [1], let $v \notin V$ and define the *cone of G on U with vertex v*, denoted $G^*(U,v)$ (or $G^*(U)$, if v is understood), by

$$V(G^*(U,v)) = V \cup \{v\} \text{ and } E(G^*(U,v)) = E \cup \Big\{\{u,v\} \mid u \in U\Big\}.$$

Given G and U as above and a graph Γ, we denote by $\Gamma\nabla(G,U)$ the graph obtained by forming a cone of G on U with vertex v for every v in Γ. Furthermore, if $G = (V,E)$ and $U \subseteq V$, $G - U$ denotes the subgraph of G induced by $V \setminus U$.

Theorem 1.2 *Let $G = (V,E)$ and $U \subseteq V$. Let $b(x) = I(G;x)$ and $c(x) = I(G-U;x)$, and suppose that $\deg(b) = \deg(c) + 2$, $c|b$ in $\mathbb{Z}[x]$, and that all roots of b (and thus c) are real. Then for any $t \ge 2$ and $n \ge 1$, $p_{t,n,G,U}(x) = I(P(t,n)\nabla(G,U);x)$ has only real roots and is therefore log-concave.*

Example. As a special case, let $t = 2$ and let $s \ge 2$ be a fixed integer. The graph $P(2,n)\nabla(K_s - e, V(K_s - e))$ is obtained by attaching a copy of the graph $K_s - e$ to each vertex of the path P_{n+1} with $n+1$ vertices. (The graph corresponding to $n = 4$ and $s = 5$ is shown in Figure 1.) Since $b(x) = x^2 + sx + 1$ and $c(x) = 1$ satisfy the hypotheses of Theorem 1.2, the independence polynomials of these graphs are log-concave.

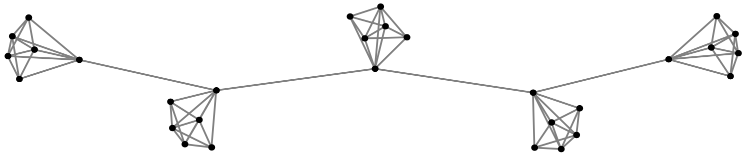

Figure 1: The graph $P(2,4)\nabla(K_5 - e, V(K_5 - e))$

As a consequence of Theorem 1.2, every one of the above independence polynomials will be unimodal, but indeed the author and Salazar have already proven unimodality for these and yet more general graphs in [1]. On the other hand, Theorem 1.2 will apply to a very large number of graphs that are not addressed by the former paper (see Section 4).

The proof for the case $t = 2$ will differ slightly from the proof in cases $t \ge 3$, and so we will consider it separately in the following section. In Section 3 we prove the cases in which $t \geq 3$, and in Section 4 we describe some of the specific classes of graphs to which this article's results apply.

The author wishes to extend his profoundest thanks to this article's referee, who made several helpful suggestions and helped clarify the context in which the present work sits.

2 The case $t = 2$

We now fix the graph $G = (V, E)$ and the vertex subset $U \subseteq V$. Let $b(x)$, $c(x)$, and $p_{t,n}(x) = p_{t,n,G,U}(x)$ as defined in Section 1. For now we will write p_n for $p_{2,n}$; our goal in this section is to prove that every root of p_n is real, for $n \geq 1$, thus proving Theorem 1.2 for the case $t = 2$.

In order to compute p_n efficiently, we apply the lemma below, which can be found in [11].

Lemma 2.1 *Let* $\Gamma = (V, E)$ *be a graph and let* $v \in V$*. Then* $I(\Gamma; x) = I(\Gamma - v; x) + xI(\Gamma - N[v]; x)$*.*

Remark. The proof of Theorem 1.2 in all cases will hinge upon the fact that under that theorem's hypotheses, we may inductively assume that the roots of the two polynomial terms on the righthand side of the recurrence in Lemma 2.1 will give rise to d intervals on which these terms differ in sign, where d is the degree of the lefthand side, $I(\Gamma; x)$. Thus $I(\Gamma; x)$, as the sum of the two terms, will have precisely one real root on each of these intervals, showing that every root of $I(\Gamma; x)$ is in fact real.

The following result now comes easily, where we let $B = \deg(b)$:

Lemma 2.2 *Let* $p_0 = b + xc$*. Then* $p_1 = b^2 + 2xcb$ *and* $p_n = b(p_{n-1} + xcp_{n-2})$ *for* $n \geq 2$*. Thus* $\deg(p_n) = B(n + 1)$ *for all* $n \geq 1$*.*

Proof: The formulas for p_n, $n \geq 1$, are easily proven by induction on n. The base cases $n = 0$ and $n = 1$ follow from direct computation, while the inductive step follows from an application of Lemma 2.1 to one of the terminal vertices of the underlying path $P(2, n)$. The degree formula too is easily proven inductively. □

Let f and g lie in the polynomial ring $\mathbb{Z}[x]$ and let $m \geq 0$ be an integer. We say that f^m *exactly divides* g (denoted $f^m \| g$) if f^m divides g in the ring $\mathbb{Z}[x]$ yet f^{m+1} does not. Clearly $1 = b^0 \| p_0$, $b = b^1 \| p_1$, and an easy induction leads to the following fact:

Lemma 2.3 *Let* $n \geq 0$*. Then* $b^{\lceil n/2 \rceil} \| p_n$*.*

We let $q_n(x)$ denote the polynomial $p_n/b^{\lceil n/2\rceil}$. Lemma 2.3 leads to easy recurrence formulas involving q_n:

Lemma 2.4 *Let $q_0 = b + xc$. Then $q_1 = b + 2xc$ and*

$$q_n = \begin{cases} bq_{n-1} + xcq_{n-2} & \text{if } n \text{ is even,} \\ q_{n-1} + xcq_{n-2} & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \geq 2$.

Proof: The base case is checked by direct computation. Both cases for $n \geq 3$ follow easily as well; we prove the first, and the second is proven analogously.

Let n be even. By Lemma 2.2, $p_n = b(p_{n-1} + xcp_{n-2})$, so Lemma 2.3 implies

$$b^{\frac{n}{2}} q_n = b(b^{\frac{n}{2}} q_{n-1} + xcb^{\frac{n}{2}-1} q_{n-2}) = b^{\frac{n}{2}}(bq_{n-1} + xcq_{n-2}).$$

Canceling $b^{\frac{n}{2}}$ from both sides gives us $q_n = bq_{n-1} + xcq_{n-2}$, as desired. □

The following lemma is analogous to Lemma 2.3:

Lemma 2.5 *Let $n \geq 0$. Then $c^{\lceil \frac{n+1}{2} \rceil} \| q_n$.*

Proof: The base cases of Lemma 2.4 show that $c \| q_0$ and $c \| q_1$, and an easy induction gives us the remaining cases, applying the recurrence formulas from Lemma 2.4. Indeed, if n is even, then $q_n = bq_{n-1} + xcq_{n-2}$, where c exactly divides the first summand $\lceil \frac{n}{2} \rceil + 1 = \frac{n}{2} + 1$ times and the second summand $\lceil \frac{n-1}{2} \rceil + 1 = \frac{n}{2} + 1$ times as well. Since $\lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$ in this case, $c^{\lceil \frac{n+1}{2} \rceil} \| q_n$, as needed. The case in which n is odd is proven analogously. □

Let $r_n(x)$ denote the polynomial $q_n/c^{\lceil \frac{n+1}{2} \rceil}$. The following is proven in exactly the same manner as was Lemma 2.4:

Lemma 2.6 *$r_1 = \frac{b}{c} + x$, $r_2 = \frac{b}{c} + 2x$, and*

$$r_n = \begin{cases} \frac{b}{c} r_{n-1} + xr_{n-2} & \text{if } n \text{ is even,} \\ r_{n-1} + xr_{n-2} & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \geq 3$.

Since the roots $\{\gamma_1, \ldots \gamma_B\}$ of b are assumed real, and since $c|b$ implies that every root of c is a root of b, to show that all of the roots of p_n are real we need only examine the roots of r_n. Regarding these we will prove the following

Proposition 2.7 *Let $n \geq 1$. Then every root of r_n is real and distinct.*

Proof: Note that since $\deg(p_n) = B(n+1)$,

$$\deg(r_n) = B(n+1) - B\left\lceil \frac{n}{2} \right\rceil - (B-2)\left\lceil \frac{n+1}{2} \right\rceil = \begin{cases} n+2 & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

By hypothesis the polynomial $\frac{b}{c}$ is a quadratic polynomial with positive coefficients and real roots, which are necessarily negative. Knowing this it is easy to show that $r_1 = \frac{b}{c} + x$ and $r_2 = \frac{b}{c} + 2x$ likewise have real roots.

Suppose now that we have shown all roots of r_k are real and distinct for $k < n$, some $n \geq 3$. We will also make the following additional inductive hypotheses:

(i) If $k \leq n-2$ is even, then the $k+3$ roots $\{\alpha_1, \ldots, \alpha_{k+2}, 0\}$ of xr_k and the $k+4$ roots $\{\beta_1, \ldots, \beta_{k+4}\}$ of $\frac{b}{c}r_{k+1}$, when listed in order by decreasing modulus, appear as

$$\beta_1, \alpha_1, \ldots, \alpha_{\frac{k}{2}+1}, \beta_{\frac{k}{2}+2}, \beta_{\frac{k}{2}+3}, \alpha_{\frac{k}{2}++2}, \ldots, \alpha_{k+2}, \beta_{k+4}, 0,$$

where $\beta_{\frac{k}{2}+1} = \gamma_-$ and $\beta_{\frac{k}{2}+2} = \gamma_+$ are the roots of $\frac{b}{c}$.

(ii) If $k \leq n-2$ is odd, then the $k+2$ roots $\{\alpha_1, \ldots, \alpha_{k+1}, 0\}$ of xr_k and the $k+3$ roots $\{\beta_1, \ldots, \beta_{k+3}\}$ of r_{k+1}, when listed in order by decreasing modulus, appear as

$$\beta_1, \alpha_1, \ldots, \alpha_{\frac{k+1}{2}}, \beta_{\frac{k+3}{2}}, \beta_{\frac{k+5}{2}}, \alpha_{\frac{k+3}{2}}, \ldots, \alpha_{k+1}, \beta_{k+3}, 0.$$

Moreover, the roots $\gamma_\pm$ of $\frac{b}{c}$ satisfy $\beta_{\frac{k+3}{2}} < \gamma_- < \gamma_+ < \beta_{\frac{k+5}{2}}$.

Case 1. Suppose n is even, so that $r_n = \frac{b}{c}r_{n-1} + xr_{n-2}$. Applying the additional hypothesis (i) above with $k = n-2$, the roots α_i of xr_{n-2} and β_j of $\frac{b}{c}r_{n-1}$ determine the intervals

$$(-\infty, \beta_1), (\beta_1, \alpha_1), \ldots, (\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}+1}), (\beta_{\frac{n}{2}+1}, \beta_{\frac{n}{2}+2}), (\beta_{\frac{n}{2}+2}, \alpha_{\frac{n}{2}+1}), \ldots, (\alpha_n, \beta_{n+2}), (\beta_{n+2}, 0).$$

Since all of the roots are distinct, both $\frac{b}{c}r_{n-1}$ and xr_{n-2} change signs at each root and it is thus not hard to see that these two functions have the same sign on the precisely following intervals:

$$(\beta_1, \alpha_1), \ldots, (\beta_{\frac{n}{2}}, \alpha_{\frac{n}{2}}), (\beta_{\frac{n}{2}+1}, \beta_{\frac{n}{2}+2}), (\alpha_{\frac{n}{2}+1}, \beta_{\frac{n}{2}+3}), \ldots, (\alpha_n, \beta_{n+2}).$$

Thus $r_n = \frac{b}{c}r_{n-1} + xr_{n-2}$ cannot have a root on any one of these intervals. However, for the same reasons r_n *will* have a root on each of the remaining intervals,

$$(-\infty, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_{\frac{n}{2}-1}, \beta_{\frac{n}{2}}), (\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}+1}), (\beta_{\frac{n}{2}+2}, \alpha_{\frac{n}{2}+1}), \ldots, (\beta_{n+1}, \alpha_n), (\beta_{n+2}, 0).$$

There are $n+2$ such intervals, so there must be $n+2$ roots, all distinct from one another, as desired. Moreover, the locations of these roots (along with those of $\frac{b}{c}$) satisfy the inductive hypothesis (ii) above, relative to the roots of xr_{n-1}.

Case 2. Now suppose n is odd, so that $r_n = r_{n-1} + xr_{n-2}$. Applying the additional hypothesis (ii) above with $k = n-2$, the roots α_i of xr_{n-2} and β_j of r_{n-1} determine the intervals

$$(-\infty, \beta_1), (\beta_1, \alpha_1), \ldots, (\alpha_{\frac{n-1}{2}}, \beta_{\frac{n+1}{2}}), (\beta_{\frac{n+1}{2}}, \beta_{\frac{n+3}{2}}), (\beta_{\frac{n+3}{2}}, \alpha_{\frac{n+1}{2}}), \ldots, (\alpha_{n-1}, \beta_{n+1}), (\beta_{n+1}, 0).$$

Since all of the roots are distinct, both r_{n-1} and xr_{n-2} change signs at each root and have the same sign on some intervals and opposite signs on others. The intervals on which the signs differ, and therefore on which roots of r_n lie, are

$$(-\infty, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_{\frac{n-1}{2}}, \beta_{\frac{n+1}{2}}), (\beta_{\frac{n+3}{2}}, \alpha_{\frac{n+1}{2}}), \ldots, (\beta_n, \alpha_{n-1}), (\beta_{n+1}, 0).$$

There are $n+1$ such intervals, so there must be $n+1$ roots, all distinct from one another, as desired. Moreover, the locations of these roots satisfy the inductive hypothesis (i) above, relative to the roots of xr_{n-1}. □

Theorem 1.2 now follows immediately by applying Proposition 1.1, since we now know that every root of p_n is real. That is, every $p_{2,n,G,U}$ (with G and U as above) is log-concave.

3 The case $t \geq 3$

We now consider the more general case, $t \geq 3$. Fixing $G = (V, E)$ and $U \subseteq V$, we let $p_{t,n} = p_{t,n,G,U}$. Let $B = \deg(b)$ as before, so that $\deg(c) = B - 2$. We will prove Theorem 1.2 for $t \geq 3$ by inducting on n, showing that every root of $p_{t,n}$ is real.

We begin with the following recurrence formulas, the first of which is needed to deal with some of our intermediate computations:

Lemma 3.1 *Let $n \geq 1$ and define $p_{1,1} = b + xc$. Then*

(i) $p_{t,1} = bp_{t-1,1} + xcb^{t-1}$ *for $t \geq 2$, and*

(ii) for $t \geq 3$,

$$p_{t,n} = \begin{cases} bp_{t,n-1} + xcb^{t-1}p_{n-1,1} & \text{if } 2 \leq n \leq t, \\ bp_{t,n-1} + xcb^{t-1}p_{t,n-t} & \text{if } t+1 \leq n. \end{cases}$$

In any case, $\deg(p_{t,n}) = B(t+n-1)$.

Proof: To prove (i), apply Lemma 2.1, letting v be any one of the vertices of the single K_t. The formulas in (ii) also follow from Lemma 2.1, applying it to the terminal vertex $v = v_1$ in the K_t-path underlying $P(t, n)$.

The purported degree formula follows easily as well. Consider first $n = 1$. Clearly $\deg(p_{1,1}) = \deg(b + xc) = B$, as needed. Suppose we have shown $\deg(p_{t-1,1}) = B(t-1)$; the first recurrence formula gives $p_{t,1} = bp_{t-1,1} + xcb^{t-1}$, so $\deg(p_{t,1}) = \max\{Bt, Bt-1\} = Bt$, as needed.

Suppose we have proven the degree formula for $n-1$, $1 \leq n-1 \leq t-1$ (so $2 \leq n \leq t$). The second recurrence formula, $p_{t,n} = bp_{t,n-1} + xcb^{t-1}p_{n-1,1}$, gives

$$\begin{aligned} \deg(p_{t,n}) &= \max\{B + B(t+n-1-1), 1+B-2+B(t-1)+B(n-1+1-1)\} \\ &= \max\{B(t+n-1), B(t+n-1)-1\} \\ &= B(t+n-1), \end{aligned}$$

as needed.

Finally, suppose we have proven the formula for $n-1$, $t \leq n-1$ (so $t+1 \leq n$). The third recurrence formula, $p_{t,n} = bp_{t,n-1} + xcb^{t-1}p_{t,n-t}$, gives

$$\begin{aligned} \deg(p_{t,n}) &= \max\{B + B(t+n-1-1), 1+B-2+B(t-1)+B(t+n-t-1)\} \\ &= \max\{B(t+n-1), B(t+n-1)-1\} \\ &= B(t+n-1), \end{aligned}$$

as needed. □

The following is an easy consequence of Lemma 3.1:

Lemma 3.2 *Suppose $b^{\alpha(t,n)} \| p_{t,n}$. Then $\alpha(t,1) = t-1$, and if $t \geq 3$ then*

$$\alpha(t,n) = \begin{cases} t+n-\lfloor \frac{n}{t} \rfloor - 2 & \text{if } n \equiv 0,1 \bmod t, \\ t+n-\lfloor \frac{n}{t} \rfloor - 3 & \text{otherwise.} \end{cases}$$

Proof: As in the proof of the degree formulas, consider first $n = 1$. Note that $\alpha(1,1) = 0$, and if we assume inductively that $\alpha(t-1,1) = (t-1)+1-\lfloor \frac{1}{t-1} \rfloor - 2 = t-2$, then the first recurrence formula of Lemma 3.1 shows that b^{t-1} (and no higher power of b) divides $p_{t,1}$, so $\alpha(t,1) = t-1$, as needed.

Now suppose $t \geq 3$ and the formula for α has been proven for $n-1$, $2 \leq n-1 \leq t-1$ (so $n \leq t$). Then $b^{1+t+(n-1)-\lfloor (n-1)/t \rfloor - 3} = b^{t+n-3}$ divides the first term of the righthand side of the second recurrence formula in Lemma 3.1, and $b^{t-1+(n-1)+1-\lfloor 1/(n-1) \rfloor - 2} = b^{t+n-3}$ divides the second term as well, as needed. (Note that if $n = t$, $\lfloor \frac{n}{t} \rfloor = 1$, so the formula $\alpha(t,n) = t+n-3 = t+n-\lfloor \frac{n}{t} \rfloor - 2$ obtains.)

Now suppose that $n \geq t+1$, $n \equiv 0 \bmod t$. Arguing as in the previous paragraph, $b^{t+n-\lfloor (n-1)/t \rfloor - 3}$ divides the first term in the third recurrence formula from Lemma 3.1, and $b^{t+n-\lfloor (n-t)/t \rfloor - 3}$ divides the second. Since $\lfloor \frac{n-1}{t} \rfloor = \lfloor \frac{n-t}{t} \rfloor = \lfloor \frac{n}{t} \rfloor - 1$, $\alpha(t,n) = t+n-\lfloor \frac{n}{t} \rfloor - 2$, as desired. A similar argument holds for $n \geq t+1$, $n \equiv 1 \bmod t$, where now even though $\lfloor \frac{n-1}{t} \rfloor = \lfloor \frac{n}{t} \rfloor$, our inductive hypothesis guarantees that $b^{t+n-\lfloor (n-1)/t \rfloor - 2} = b^{t+n-\lfloor n/t \rfloor - 2}$ divides the first term in the recurrence formula, and thus all of $p_{t,n}$.

Finally, suppose $n \geq t+1$, $n \not\equiv 0,1 \bmod t$. Now $\lfloor \frac{n-1}{t} \rfloor = \lfloor \frac{n}{t} \rfloor$, so the computations performed in the previous paragraph show that the maximal power of b dividing $p_{t,n}$ is that dividing the first term in the righthand side of the third recurrence formula, namely $b^{t+n-\lfloor (n-1)/t \rfloor - 3} = b^{t+n-\lfloor n/t \rfloor - 3}$. This completes our proof. □

As in the previous section, let $q_{t,n} = p_{t,n}/b^{\alpha(t,n)}$.

Lemma 3.2 implies that for $t \geq 3$,

$$\deg(q_{t,n}) = \begin{cases} B\left(\lfloor \frac{n}{t} \rfloor + 1\right) & \text{if } n \equiv 0, 1 \bmod t, \\ B\left(\lfloor \frac{n}{t} \rfloor + 2\right) & \text{otherwise.} \end{cases}$$

We also have enough information for us to develop recurrence relations defining the polynomials $q_{t,n}$:

Lemma 3.3 *For all $t \geq 1$, $q_{t,1} = b + txc$. Moreover, for $t \geq 3$,*

(i) if $2 \leq n \leq t$, then

$$q_{t,n} = \begin{cases} bq_{t,1} + xcq_{1,1} & \text{if } n = 2, \\ q_{t,n-1} + xcq_{n-1,1} & \text{otherwise,} \end{cases}$$

and

(ii) if $n \geq t + 1$, then

$$q_{t,n} = \begin{cases} bq_{t,n-1} + xcq_{t,n-t} & \text{if } n \equiv 2 \bmod t, \\ q_{t,n-1} + xcq_{t,n-t} & \text{otherwise.} \end{cases}$$

Proof: Note $p_{1,1} = q_{1,1} = b + xc$. From Lemma 3.2 it follows that $p_{t,1} = b^{t-1}q_{t,1}$, and thus the first recurrence formula of Lemma 3.1 gives

$$b^{t-1}q_{t,1} = p_{t,1} = bp_{t-1,1} + xcb^{t-1} = b^{t-1}q_{t-1,1} + xcb^{t-1}.$$

Factoring out b^{t-1} and canceling this term gives the desired formula for the polynomial $q_{t,1}$.

For (i), suppose $n = 2$. Lemma 3.2 gives $p_{t,2} = b^{t-1}q_{t,2}$, and the second recurrence formula of Lemma 3.1 gives

$$\begin{aligned} p_{t,2} &= bp_{t,1} + xcb^{t-1}p_{1,1} \\ &= b \cdot b^{t-1}q_{t,1,s} + xcb^{t-1}q_{1,1} \\ &= b^{t-1}(bq_{t,1} + xcq_{1,1}). \end{aligned}$$

Equating these two formulas for $p_{t,2}$ and canceling the common term b^{t-1} gives the desired formula.

Now suppose $3 \leq n < t$. Again Lemma 3.2 gives $p_{t,n} = b^{t+n-3}q_{t,n}$, while now the second recurrence formula of Lemma 3.1 gives

$$\begin{aligned} p_{t,n} &= bp_{t,n-1} + xcb^{t-1}p_{n-1,1} \\ &= b \cdot b^{t+n-1-3}q_{t,n-1} + xcb^{t-1}b^{n-2}q_{n-1,1} \\ &= b^{t+n-3}(q_{t,n-1} + xcq_{n-1,1}). \end{aligned}$$

Once again equating the two formulas for $p_{t,n}$ and canceling b^{t+n-3} gives the desired formula.

If $n = t$, we have $p_{t,t} = b^{t+n-3}q_{t,t} = b^{2t-3}q_{t,t}$ and

$$\begin{aligned} p_{t,t} &= bp_{t,t-1} + xcb^{t-1}p_{t-1,1} \\ &= b \cdot b^{t+t-1-3}q_{t,t-1} + xcb^{t-1}b^{t-2}q_{t-1,1} \\ &= b^{2t-3}(q_{t,t-1} + xcq_{t-1,1}). \end{aligned}$$

Equating and canceling as before gives the desired formula.

For (ii), consider $n \geq t+1$. We prove the lemma in the case $n \equiv 0 \bmod t$; the remaining cases are proven analogously. Applying Lemma 2.5, we have $p_{t,n} = b^{n+t-\lfloor n/t \rfloor -2}q_{t,n}$ and

$$\begin{aligned} p_{t,n} &= bp_{t,n-1} + xcb^{t-1}p_{t,n-t} \\ &= b \cdot b^{n-1+t-(\lfloor n/t \rfloor -1)-3}q_{t,n-1} + xcb^{t-1}b^{n-t+t-(\lfloor n/t \rfloor -1)-2}q_{t,n-t} \\ &= b^{t+n-\lfloor n/t \rfloor -2}q_{t,n-1} + xcb^{t+n-\lfloor n/t \rfloor -2}q_{t,n-t} \\ &= b^{t+n-\lfloor n/t \rfloor -2}(q_{t,n-1} + xcq_{t,n-t}). \end{aligned}$$

Throughout we have used the fact that $\lfloor \frac{n-1}{t} \rfloor = \lfloor \frac{n-t}{t} \rfloor = \lfloor \frac{n}{t} \rfloor - 1$. Equating and canceling as before, we obtain the desired recurrence involving the polynomials q. □

As in the previous section we now reduce the complexity of our polynomials once more by dividing out as many powers of c as we can from each $q_{t,n}$.

Lemma 3.4 *Let $t \geq 3$ and $n \geq 1$, and suppose $c^{\beta(t,n)} \| q_{t,n}$. Then $\beta(t,n) = \lfloor \frac{n-2}{t} \rfloor + 2$.*

Proof: The formulas in (i) of Lemma 3.3 give $c \| q_{t,1}$, and since $\lfloor \frac{1-2}{t} \rfloor + 2 = -1 + 2 = 1$, the lemma is true in case $n = 1$.

Since the formulas in (ii) of Lemma 3.3 give $q_{t,2} = bq_{t,1} + xcq_{1,1}$, and since $c|b$, the previous paragraph shows that $c^2 \| q_{t,2}$. Assuming the lemma true for $n-1$, $2 \leq n-1 \leq t-1$, consider n (where $3 \leq n \leq t$): $q_{t,n} = q_{t,n-1} + xcq_{n-1,1}$ and $c^2 \| q_{t,n-1}$ together imply that $c^2 \| q_{t,n}$ as well.

Now $q_{t,t+1} = q_{t,t} + xcq_{t,1}$, so $c^2 \| q_{t,t+1}$, as needed; however, $q_{t,t+2} = bq_{t,t} + xcq_{t,2}$, and since $c|b$, $c^3 \| q_{t,t+2}$, also as needed.

At last, assume inductively that the lemma is true of $n-1$, $n-1 \geq t+2$. Consider n. If $n \equiv 2 \bmod t$, $q_{t,n} = bq_{t,n-1} + xcq_{t,n-t}$. By inductive hypothesis c divides into $bq_{t,n-1}$ exactly $1 + \beta(t, n-1) = 1 + \lfloor \frac{n-3}{t} \rfloor + 2 = \lfloor \frac{n-2}{t} \rfloor + 2$ times, since $\lfloor \frac{n-3}{t} \rfloor = \lfloor \frac{n-2}{t} \rfloor - 1$. Meanwhile, c divides into $xcq_{t,n-t}$ exactly $1 + \lfloor \frac{n-t-2}{2} \rfloor + 2 = \lfloor \frac{n-2}{t} \rfloor + 2$ times as well, proving the formula for n.

The argument in the case $n \not\equiv 2 \bmod t$ is entirely analogous, using instead the recurrence $q_{t,n} = q_{t,n-1} + xcq_{t,n-t}$. □

Let us now define $r_{t,n} = q_{t,n}/c^{\beta(t,n)}$. Using our degree formulas for $q_{t,n}$ and Lemma 3.4, we obtain degree formulas for $r_{t,n}$ akin to those for $q_{t,n}$:

$$\deg(r_{t,n}) = \begin{cases} 2\left(\lfloor \frac{n}{t} \rfloor + 1\right) & \text{if } n \equiv 0, 1 \bmod t, \\ 2\left(\lfloor \frac{n}{t} \rfloor + 2\right) & \text{otherwise.} \end{cases}$$

We can also develop recurrence formulas for $r_{t,n}$:

Lemma 3.5 *For all* $t \geq 1$, $r_{t,1} = \frac{b}{c} + tx$. *Moreover, for* $t \geq 3$,

(i) if $2 \leq n \leq t$, *then*

$$r_{t,n} = \begin{cases} \frac{b}{c} r_{t,1} + x r_{1,1} & \text{if } n = 2, \\ r_{t,n-1} + x r_{n-1,1} & \text{otherwise,} \end{cases}$$

and

(ii) if $n \geq t+1$, *then*

$$r_{t,n} = \begin{cases} \frac{b}{c} r_{t,n-1} + x r_{t,n-t} & \text{if } n \equiv 2 \bmod t, \\ r_{t,n-1} + x r_{t,n-t} & \text{otherwise.} \end{cases}$$

We are finally ready to examine carefully the location of the roots of $r_{t,n}$, much as was done in the previous section for $t = 2$. From this point on t will be fixed, so we let $r_n = r_{t,n}$ and let $\gamma_{\pm}$ be the two real roots of $\frac{b}{c}$.

First note that the roots of the quadratic polynomial $r_1 = \frac{b}{c} + tx$ are easily shown to be distinct real numbers. For instance, if $\frac{b}{c} = a_2 x^2 + a_1 x + 1$, then the roots of r_1 are $\frac{-a_1 - t \pm \sqrt{(a_1+t)^2 - 4a_2}}{2a_2}$, where the discriminant $(a_1+t)^2 - 4a_2$ is positive. Moreover, from this it follows that the 3 roots $\{\alpha_1, \alpha_2, 0\}$ of $xr_{1,1}$ and the 4 roots $\{\beta_1, \beta_2 = \gamma_-, \beta_3 = \gamma_+, \beta_4\}$ of $\frac{b}{c} r_1$, when listed in order by decreasing modulus, appear as

$$\beta_1, \alpha_1, \beta_2, \beta_3, \alpha_2, \beta_4, 0.$$

As in the proof at the end of Section 2 we see that $xr_{1,1}$ and $\frac{b}{c} r_1$ have opposite sign on the intervals

$$(-\infty, \beta_1), (\alpha_1, \beta_2), (\beta_3, \alpha_2), (\beta_4, 0),$$

so by the first recurrence formula from Lemma 3.5 r_2 has 4 distinct roots, one on each of these intervals. A nearly identical argument applies to show that the roots of r_n are real and distinct whenever $n \leq t$.

Now let $n > t$ be fixed and suppose we have shown all roots of r_k are real and distinct for all $k < n$. As before we make the following additional inductive hypotheses:

(i) If $k \leq n - t$, $k \equiv 2 \bmod t$, let $s = \deg(r_k) = 2\lfloor \frac{k}{t} \rfloor + 4$. Then the $s+1$ roots $\{\alpha_1, \ldots, \alpha_s, 0\}$ of xr_k and the $2\lfloor \frac{k+t-1}{t} \rfloor + 4 = s+2$ roots $\{\beta_1, \ldots, \beta_{s+2}\}$ of $\frac{b}{c} r_{k+t-1}$, when listed in order by decreasing modulus, appear as

$$\beta_1, \alpha_1, \ \ldots, \beta_{\frac{s}{2}}, \alpha_{\frac{s}{2}}, \beta_{\frac{s+2}{2}}, \beta_{\frac{s+4}{2}}, \alpha_{\frac{s+2}{2}}, \beta_{\frac{s+6}{2}}, \ldots, \alpha_s, \beta_{s+2}, 0,$$

where $\beta_{\frac{s+2}{2}} = \gamma_-$ and $\beta_{\frac{s+4}{2}} = \gamma_+$ are the roots of $\frac{b}{c}$.

(ii) If $k \leq n - t$, $k \not\equiv 2 \bmod t$, let $s = \deg(r_k) = 2\lfloor \frac{k}{t} \rfloor + 2$. Then the $s+1$ roots $\{\alpha_1, \ldots, \alpha_s, 0\}$ of xr_k and the $s+2$ roots $\{\beta_1, \ldots, \beta_{s+2}\}$ of r_{k+t-1}, when listed in order by decreasing modulus, appear as

$$\beta_1, \alpha_1, \ldots, \beta_{\frac{s}{2}}, \alpha_{\frac{s}{2}}, \beta_{\frac{s+2}{2}}, \beta_{\frac{s+4}{2}}, \alpha_{\frac{s+2}{2}}, \beta_{\frac{s+6}{2}}, \ldots, \alpha_s, \beta_{s+2}, 0.$$

Moreover, $\beta_{\frac{s+2}{2}} < \gamma_- < \gamma_+ < \beta_{\frac{s+4}{2}}$.

There are two cases to consider, depending on whether or not $n \equiv 2 \bmod t$. We prove only one of these cases, as the proof in the other case is nearly identical.

Suppose $n \equiv 2 \bmod t$, so that Lemma 3.5 gives $r_n = \frac{b}{c} r_{n-1} + x r_{n-t}$. Inductively all roots of r_{n-1} and r_{n-t} are real and distinct, and by the additional inductive hypothesis (i) (with $k = n - t$) we know the roots of $\frac{b}{c} r_{n-1}$ and $x r_{n-t}$ determine the intervals

$$(-\infty, \beta_1), (\beta_1, \alpha_1), \ldots, (\alpha_{\frac{s}{2}}, \beta_{\frac{s+2}{2}}), (\beta_{\frac{s+2}{2}}, \beta_{\frac{s+4}{2}}), (\beta_{\frac{s+4}{2}}, \alpha_{\frac{s+2}{2}}), \ldots, (\alpha_s, \beta_{s+2}), (\beta_{s+2}, 0),$$

where $s = 2\lfloor \frac{n-t}{t} \rfloor + 4$. Since all of the roots present are distinct, both $\frac{b}{c} r_{n-1}$ and $x r_{n-t}$ change sign at each root, and it is easily seen that these polynomials have opposite signs on precisely the following intervals:

$$(-\infty, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_{\frac{s}{2}}, \beta_{\frac{s+2}{2}}), (\beta_{\frac{s+4}{2}}, \alpha_{\frac{s+2}{2}}), \ldots, (\beta_{s+1}, \alpha_s), (\beta_{s+2}, 0).$$

Therefore r_n will have a single root on each of these $s + 2$ intervals, and all of these roots are distinct. Moreover, the location of these roots, along with the roots $\gamma_\pm$ of $\frac{b}{c}$, satisfy the inductive hypothesis (ii).

We have shown that all roots of $r_{t,n}$ are real, and thus so are all roots of $p_{t,n,G,U}$. At last we have finished our proof of Theorem 1.2: each of the independence polynomials $p_{t,n,G,U}$ is log-concave, following an application of Proposition 1.1.

4 Examples

We now describe one general means of constructing a variety of connected graphs $G = (V, E)$ with $U \subseteq V$ whose independence polynomials satisfy the hypotheses of Theorem 1.2.

Let $G = \cup_{i=1}^k G_i$ be the disjoint union of graphs G_i, with $U = \cup_{i=1}^k U_i$, $U_i \subseteq V(G_i)$, and let $H = \cup_{i=1}^k H_i$, $H_i = G_i - U_i$. (Thus $H = G - U$.)

Suppose the following conditions are met:

(i) For all $i = 1, \ldots, k$, $I(G_i; x)$ has only real roots,

(ii) there is a permutation π of $\{1, \ldots, k\}$ such that for all $i = 1, \ldots, k$ satisfying $H_i \neq \emptyset$, $G_{\pi(i)} = H_i$, and

(iii) if $G = H \cup G'$ then $\deg(I(G'; x)) = 2$.

Let $b(x) = I(G; x)$ and $c(x) = I(H; x)$. Obviously (i) implies that $b(x) = \prod_{i=1}^k I(G_i; x)$ and $c(x) = \prod_{i=1}^k I(H_i; x)$ have only real roots, and (iii) implies that $\deg(b) = \deg(c) + 2$. Moreover, (ii) implies that $c | b$ in $\mathbb{Z}[x]$. If $U_i \neq \emptyset$ for all $i = 1, \ldots, k$, $P(t, n)\nabla(G, U)$ will be connected.

For example, let $\vec{m} = (m_1, m_2, \ldots, m_k)$ and $\vec{u} = (u_1, u_2, \ldots, u_k)$ be k-tuples of natural numbers such that the number 0 appears in $\vec{m} - \vec{u} = (m_1 - u_1, m_2 -$

$u_2, \ldots, m_k - u_k$) exactly twice, and each remaining value appears at most as many times as it does in $\vec{m}$. Letting $G_i = K_{m_i}$ and $U_i \subseteq V(G_i)$ such that $|U_i| = u_i$, the three conditions above are met. To be precise,

$$b(x) = \prod_{i=1}^{k}(m_i x + 1) \quad \text{and} \quad c(x) = \prod_{i=1}^{k}\big((m_i - u_i)x + 1\big).$$

Note that although Theorem 1.2 guarantees the polynomials $p_{t,n,G,U}(x)$ are log-concave and therefore unimodal for all $n \geq 1$ and $t \geq 2$, these polynomials are far from symmetric. Figure 2 shows $P(3,6)\nabla(G,U)$ for $\vec{m} = (5,5,3,2,1)$ and $\vec{u} = (2,3,3,1,1)$.

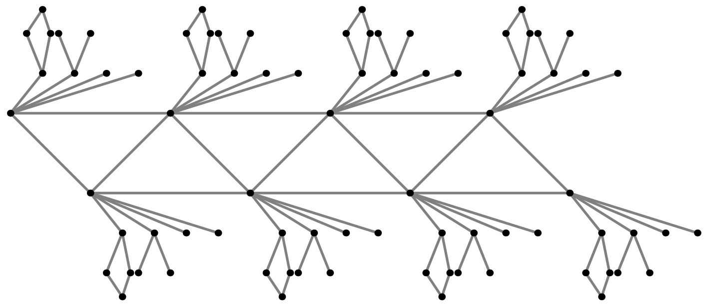

Figure 2: The case $\vec{m} = (5,5,3,2,1)$ and $\vec{u} = (2,3,3,1,1)$

A similar construction involves graphs G_i whose complements $\overline{G}_i$ are triangle-free. If $\overline{G}_i$ has no triangles, then $\alpha(G_i) \leq 2$, so $I(G_i;x) = a_2x^2 + a_1x + 1$, where $a_1 = |V(G_i)|$ and $a_2 = |E(\overline{G}_i)|$. Note that if $a_1 \geq 2\sqrt{a_2}$ the roots of $I(G_i;x)$ are real. Appropriate selection of $U_i \subseteq V(G_i)$ will ensure conditions (ii) and (iii) above are met, guaranteeing that each polynomial $p_{t,n,G,U}$ is log-concave. A very special case of this construction was given in the introduction, in which $k = 1$ and $G_1 = K_s - e$ ($s \geq 2$) so that $a_1 = s - 1$ and $a_2 = 1$. A more general case is shown in Figure 3, in which $k = 3$.

Even more complicated classes of graphs involve graphs G whose components have yet higher degree, so long as care is taken to ensure that the polynomials $I(G_i;x)$ have only real roots.

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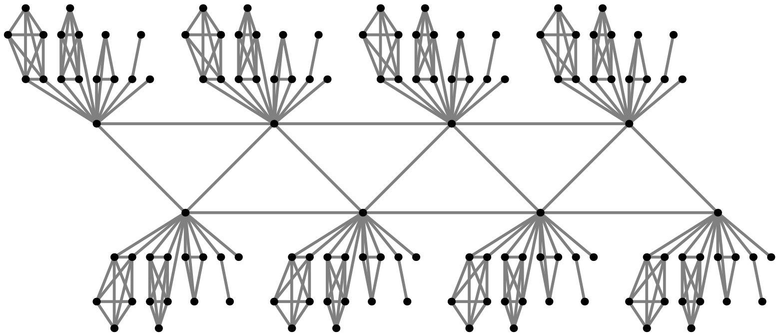

Figure 3: $P(3,6)\nabla(G,U)$ for $G_1 = K_4 - 2e$, $G_2 = K_3 - e$, and $G_3 = K_2 - e$

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