AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 147–156

Total vertex irregularity strength of wheel related graphs*

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Abstract

For a simple graph G with vertex set $V(G)$ and edge set $E(G)$, a labeling $\phi : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, k\}$ is called a *vertex irregular total k-labeling* of G if for any two different vertices x and y, their weights $wt(x)$

* The work was supported by the Higher Education Commission Pakistan.

and $wt(y)$ are distinct. The weight $wt(x)$ of a vertex x in G is the sum of its label and the labels of all edges incident with the given vertex x. The *total vertex irregularity strength* of G, denoted by $\mathrm{tvs}(G)$, is the smallest positive integer k for which G has a vertex irregular total k-labeling. In this paper, we study the total vertex irregularity strength of flower, helm, generalized friendship and web graphs.

1 Introduction

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$. For a graph G we define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be a *vertex irregular total k-labeling* of the graph G if for every two different vertices x and y of G, one has $wt(x) \neq wt(y)$ where the weight of a vertex x in the labeling ϕ is

$$wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy),$$

where $N(x)$ is the set of neighbors of x. In [3] Bača, Jendroľ, Miller and Ryan defined a new graph invariant, called the *total vertex irregularity strength* of G, $\mathrm{tvs}(G)$, which is the minimum k for which the graph G has a vertex irregular total k-labeling.

The original motivation for the definition of the total vertex irregularity strength came from irregular assignments and the irregularity strength of graphs introduced in [5] by Chartrand et al., and studied by numerous authors [4, 6, 7, 8, 9].

An *irregular assignment* is a k-labeling of the edges

$$f : E \to \{1, 2, \dots, k\}$$

such that the vertex weights

$$w(x) = \sum_{y \in N(x)} f(xy)$$

are different for all vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$.

The lower bound on $s(G)$ is given by the inequality

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}.$$

The first upper bounds including the vertex degrees in the denominator were given in [7]. The best upper bound known so far can be found in [10]. Namely, the authors have proved that

$$s(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil.$$

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

It is easy to see that the irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $\mathrm{tvs}(G)$ is defined for every graph G.

If an edge labeling $f : E \to \{1, 2, \ldots, s(G)\}$ provides the irregularity strength $s(G)$, then we extend this labeling to a total labeling ϕ as follows:

$$\begin{array}{ll} \phi(xy) = f(xy) & \text{for every } xy \in E(G), \\ \phi(x) = 1 & \text{for every } x \in V(G). \end{array}$$

Thus, the total labeling ϕ is a vertex irregular total labeling, and for graphs with no component of order at most 2, $\mathrm{tvs}(G) \leq s(G)$.

In [3] several bounds and exact values of $\mathrm{tvs}(G)$ were determined for different types of graphs (in particular for stars, cliques and prisms). Among other results, the authors proved the following theorem.

Theorem 1 *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \;\leq\; \mathrm{tvs}(G) \;\leq\; p + \Delta - 2\delta + 1. \tag{1}$$

In the case of r-regular graphs we therefore obtain

$$\left\lceil \frac{p+r}{r+1} \right\rceil \;\leq\; \mathrm{tvs}(G) \;\leq\; p - r + 1. \tag{2}$$

For graphs with no component of order ≤ 2, Bača et al. in [3] strengthened these upper bounds, proving that

$$\mathrm{tvs}(G) \leq p - 1 - \left\lceil \frac{p-2}{\Delta+1} \right\rceil. \tag{3}$$

These results were then improved by Przybylo in [11] for sparse graphs and for graphs with large minimum degree. In the latter case the bounds

$$\mathrm{tvs}(G) < 32\,\frac{p}{\delta} + 8 \tag{4}$$

in general and

$$\mathrm{tvs}(G) < 8\,\frac{p}{r} + 3 \tag{5}$$

for r-regular (p, q)-graphs were proved to hold.

In [2] Anholcer, Kalkowski and Przybylo established a new upper bound of the form

$$\text{tvs}(G) \leq 3\frac{p}{\delta} + 1. \tag{6}$$

Wijaya and Slamin [12] found the exact values of the total vertex irregularity strength of wheels, fans, suns and friendship graphs. Wijaya, Slamin, Surahmat and Jendroľ [13] determined the exact value for complete bipartite graphs. Ahmad and Bača [1] found the exact value of the total vertex irregularity strength for Jahangir graphs $J_{n,2}$ for $n \geq 4$ and for 4-regular circulant graphs $C_n(1,2)$ for $n \geq 5$ namely, $\text{tvs}(J_{n,2}) = \lceil \frac{n+1}{2} \rceil$ and $\text{tvs}(C_n(1,2)) = \lceil \frac{n+4}{5} \rceil$.

In this paper, we determine exact values of the total vertex irregularity strength for wheel related graphs, namely, helm graph, generalised friendship graph, flower graph and web graph. We define such graphs as follows.

The helm graph H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. Thus the vertex set of H_n is $V = \{u, v_i, u_i : 1 \leq i \leq n\}$ and the edge set of H_n is $E = \{uv_i, v_iv_{i+1}, v_iu_i : 1 \leq i \leq n\}$, with indices taken modulo n.

The generalised friendship graph $f_{m,n}$ is a collection of m cycles of order n meeting in a common vertex. Thus the vertex set of $f_{m,n}$ is $V = \{v_j^i : 1 \leq i \leq m \wedge 1 \leq j \leq n-1\} \cup \{v_0^i = v_0 : 1 \leq i \leq m\}$ and the edge set of $f_{m,n}$ is $E = \{v_j^i v_{j+1}^i : 1 \leq i \leq m \wedge 0 \leq j \leq n-1\}$, with indices taken modulo n.

The flower graph F_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. Thus the vertex set of F_n is $V = \{v, v_i, u_i : 1 \leq i \leq n\}$ and the edge set of F_n is $E = \{vv_i, vu_i, v_iv_{i+1}, v_iu_i : 1 \leq i \leq n\}$, with indices taken modulo n.

The web graph Wb_n is the graph obtained from a helm by joining the pendant vertices to form an n-cycle. Thus the vertex set of Wb_n is $V = \{v, v_j^i : 1 \leq i \leq 2 \wedge 1 \leq j \leq n\}$, and the edge set of Wb_n is $E = \{vv_j^1, v_j^1v_j^2, v_j^1v_{j+1}^1, v_j^2v_{j+1}^2 : 1 \leq j \leq n\}$, with indices taken modulo n.

2 Main Result

In this section, we study total vertex irregularity strength of helm graphs, generalized friendship graphs, flower graphs and web graphs.

We start by determining the total vertex irregularity strength of the helm H_n for $n \geq 3$.

Theorem 2 *For $n \geq 4$, the vertex irregularity strength of H_n is*

$$\text{tvs}(H_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Recall that the vertex set and edge set of the helm H_n are

$$\begin{aligned} V(H_n) &= \{u_i; v_i : 1 \le i \le n\} \cup \{u\} \\ E(H_n) &= \{v_i v_{i+1}; u_i v_i; u v_i : 1 \le i \le n\} \end{aligned}$$

Let us consider the vertices of degree 1. There are n such vertices, and if we want to use only the labels $1, 2, \ldots, s$, the lowest and highest weights that we can obtain are respectively 2 and $2s$, which implies that $2s - 2 + 1 \ge n$ and thus

$$\text{tvs}(H_n) \ge \left\lceil \frac{n+1}{2} \right\rceil.$$

To show that $\text{tvs}(H_n) \le \lceil \frac{n+1}{2} \rceil$, we define a labeling $\phi : V(H_n) \cup E(H_n) \to \{1, 2, \ldots, \lceil \frac{n+1}{2} \rceil\}$ as follows:

$$\begin{aligned} \phi(v_i v_{i+1}) = \phi(u) = \phi(u v_i) &= \lceil \tfrac{n+1}{2} \rceil, \qquad \text{for } 1 \le i \le n \\ \phi(v_i) = \phi(u_i) &= \begin{cases} 1, & \text{for } i = 1 \\ 2, & \text{for } 2 \le i \le 4 \\ \lceil \frac{i}{2} \rceil, & \text{for } 5 \le i \le n \end{cases} \\ \phi(v_i u_i) &= \begin{cases} 1, & \text{for } i = 1, 2 \\ 2, & \text{for } i = 3 \\ \lceil \frac{i+1}{2} \rceil, & \text{for } 4 \le i \le n \end{cases} \end{aligned}$$

This labeling gives the weight of the vertices, u, u_i and v_i for $1 \le i \le n$, as follows:

$$\begin{aligned} wt(u) &= \left\lceil \frac{n+1}{2} \right\rceil (n+1); \\ wt(u_i) &= i + 1; \\ wt(v_i) &= 3 \left\lceil \frac{n+1}{2} \right\rceil + 1 + i. \end{aligned}$$

It is easy to check that the weight of the vertices are distinct, and thus ϕ is the vertex irregular total labeling of the helm graph H_n. □

Let $f_{m,n}$ be a generalized friendship graph, where m is the number of cycles, and n is the order of cycles. For $n = 3$, it has been shown in [12] that $\text{tvs}(f_{m,n}) = \lceil \frac{2m+2}{3} \rceil$. In the next theorem we determine the total vertex irregularity strength of $f_{m,n}$ for $n \ge 4$.

Theorem 3 *For $n \ge 4$ and $m > 1$, the total vertex irregularity strength of $f_{m,n}$ is*

$$\text{tvs}(f_{m,n}) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

Proof. Let us consider the vertices of degree 2. There are $m(n-1)$ such vertices, and if we want to use only the labels $1, 2, \ldots, s$, the lowest and highest weights that we can obtain are respectively 3 and $3s$, which implies that $3s - 3 + 1 \geq m(n-1)$ and thus

$$\text{tvs}(f_{m,n}) \geq \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

We now prove the upper bound by providing labeling construction for $f_{m,n}$. The value of j is taken modulo n. The labeling ϕ is defined as:

$$\begin{aligned}
\phi(v_0^i) = \phi(v_0) &= \left\lceil \tfrac{m(n-1)+2}{3} \right\rceil, && \text{for } 1 \leq i \leq m; \\
\phi(v_j^i) &= \left\lceil \tfrac{j+(i-1)(n-1)}{3} \right\rceil, && \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n-1; \\
\phi(v_j^i v_{j+1}^i) &= \left\lceil \tfrac{j+2+(i-1)(n-1)}{3} \right\rceil, && \text{for } 1 \leq i \leq m, \quad 0 \leq j \leq n-1.
\end{aligned}$$

This labeling gives the weight of the vertices as follows:

$$wt(v_j^i) = \begin{cases} j+2+(i-1)(n-1), & \text{for } 1 \leq i \leq m; \\ & \text{and } 1 \leq j \leq n-1; \\ \sum_{i=1}^{m} \left(\left\lceil \frac{2+(i-1)(n-1)}{3} \right\rceil + \left\lceil \frac{n+1+(i-1)(n-1)}{3} \right\rceil \right) & \\ + \left\lceil \frac{m(n-1)+2}{3} \right\rceil, & \text{for } j = 0. \end{cases}$$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that

$$\text{tvs}(f_{m,n}) \leq \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

Combining this with the lower bounds, we conclude that

$$\text{tvs}(f_{m,n}) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil.$$

□

We now determine the total vertex irregularity strength of the flower graph F_n.

Theorem 4 *For $n \geq 4$, the total vertex irregularity strength of the flower graph F_n is*

$$\left\lceil \frac{2n+2}{5} \right\rceil.$$

Proof. Recall that the vertex set and edge set of F are

$$\begin{aligned} V(F) &= \{u_i; v_i : 1 \le i \le n\} \cup \{v\}, \\ E(F) &= \{v_i v_{i+1}; u_i v_i; v v_i; v u_i : 1 \le i \le n\}. \end{aligned}$$

Let us consider the vertices of degree 2 and 4. There are $2n$ such vertices, and if we want to use only the labels $1, 2, \ldots, s$, the lowest and highest weights that we can obtain are respectively 3 and $5s$, which implies that $5s - 3 + 1 \ge 2n$, and thus

$$\text{tvs}(F) \ge \left\lceil \frac{2n+2}{5} \right\rceil.$$

Let $k = \left\lceil \frac{2n+2}{5} \right\rceil$. To show that $\text{tvs}(F) \le k$, we define a labeling $\phi : V(F) \cup E(F) \to \{1, 2, \ldots, k\}$ as follows:

$$\phi(u_i) = \max\{1, i + 2 - 2k\}, \text{ for } 1 \le i \le n$$

$$\phi(u_i v_i) = \min\{i, k\}, \quad \text{for } 1 \le i \le n$$

$$\phi(v_i v_{i+1}) = \phi(v) = k, \quad \text{for } 1 \le i \le n$$

$$\phi(v_i) = \phi(v u_i) = \begin{cases} 1, & \text{for } 1 \le i \le k \\ i + 1 - k, & \text{for } k + 1 \le i \le 2k - 1 \\ k, & \text{for } 2k \le i \le n. \end{cases}$$

If $n - 2k = -1$, then $\phi(v v_i) = 1$, $1 \le i \le n$, otherwise

$$\phi(v v_i) = \max\{n + 1 - 2k, i + 2 + n - 4k\}, \quad \text{for } 1 \le i \le n.$$

The weights of vertices u_i, $1 \le i \le n$, successively attain values $3, 4, \ldots, n+2$, and the weights of vertices v_i, $1 \le i \le n$, successively attain values from $n + 3$ up to $2n + 2$ when $n - 2k \ne -1$, and successively attain values from $n + 4$ up to $2n + 3$ when $n - 2k = -1$. It is easy to see that the weight of the central vertex v receives distinct values from u_i and v_i, $1 \le i \le n$. Thus the labeling ϕ is the desired vertex irregular total k-labeling. □

Let Wb_n be a web graph; we find the total vertex irregularity strength of Wb_n in the following theorem.

Theorem 5 *For $n > 4$, the total vertex irregularity strength of Wb_n is*

$$\left\lceil \frac{2n+3}{5} \right\rceil.$$

Proof. Let us consider the vertices of degrees 3 and 4. There are $2n$ such vertices, and if we want to use only the labels $1, 2, \ldots, s$, the lowest and highest weights that we can obtain are respectively 4 and $5s$, which implies that $5s - 4 + 1 \geq 2n$ and thus

$$\text{tvs}(Wb_n) \geq \left\lceil \frac{2n+3}{5} \right\rceil.$$

Let $\left\lceil \frac{2n+3}{5} \right\rceil = k$. Recall that $V(Wb_n) = \{v, v_j^i \mid 1 \leq i \leq 2 \vee 1 \leq j \leq n\}$ and $E(Wb_n) = \{vv_j^1, v_j^1 v_j^2, v_j^1 v_{j+1}^1, v_j^2 v_{j+1}^2 \mid 1 \leq j \leq n\}$ with indices taken modulo n, are the vertex and edge sets of the graph Wb_n. To show that $\text{tvs}(Wb_n) \leq \left\lceil \frac{2n+3}{5} \right\rceil$, we distinguish three cases, and define a labeling $\phi : V(Wb_n) \cup E(Wb_n) \to \{1, 2, \ldots, k\}$ as follows:

When $n - 2k = -1$:

For $1 \leq j \leq n$,

$\phi(v) = \phi(vv_j^1) = k, \phi(v_j^1) = \phi(v_j^2) = \max\{1, j+1-k\}, \phi(v_j^1 v_j^2) = \min\{j, k\}$, $\phi(v_j^2 v_{j+1}^2) = 1$,

$$\phi(v_j^1 v_{j+1}^1) = \begin{cases} 1, & \text{if } 1 \leq j \leq n-2 \text{ is odd,} \\ k, & \text{otherwise.} \end{cases}$$

Thus the vertex weights of Wb_n are as follows:

$wt(v_j^2) = j + 3$ for $1 \leq j \leq n$, $wt(v_j^1) = n + j + 3$ for $1 \leq j \leq n-1$, $wt(v_n^1) = 5k$, and $wt(v) = (n+1)k$

When $n - 2k = 0$:

$$\begin{aligned} \phi(v) = \phi(vv_j^1) &= k, && \text{for } 1 \leq j \leq n, \\ \phi(v_j^1 v_j^2) &= \min\{j, k\}, && \text{for } 1 \leq j \leq n, \end{aligned}$$

$$\phi(v_j^1) = \begin{cases} \max\{1, j+1-k\}, & \text{for } 1 \leq j \leq n-2 \\ 2, & \text{for } j = n-1 \\ 3, & \text{for } j = n \end{cases}$$

$$\phi(v_j^2) = \begin{cases} \max\{1, j+1-k\}, & \text{for } 1 \leq j \leq n-2 \\ 1, & \text{for } j = n-1 \\ 2, & \text{for } j = n \end{cases}$$

$$\phi(v_j^1 v_{j+1}^1) = \begin{cases} 2, & \text{for } 1 \leq j \leq n-2 \text{ odd} \\ k, & \text{otherwise} \end{cases}$$

$$\phi(v_j^2 v_{j+1}^2) = \begin{cases} k, & \text{for } j = n-1 \\ 1, & \text{otherwise.} \end{cases}$$

Thus the vertex weights of Wb_n are $wt(v_j^2) = j+3$, $wt(v_j^1) = n+j+3$, and $wt(v) = (n+1)k$ for $1 \le j \le n$.

When $n - 2k \neq \{0, -1\}$**:**

$$\begin{aligned} \phi(v_j^1 v_{j+1}^1) = \phi(v) &= k, && \text{for } 1 \le j \le n, \\ \phi(v_j^1 v_j^2) &= \min\{j, k\}, && \text{for } 1 \le j \le n, \\ \phi(v v_j^1) &= \max\{n+2-2k, j+n+3-4k\}, && \text{for } 1 \le j \le n, \end{aligned}$$

$$\phi(v_j^i) = \begin{cases} \max\{1, j+1-k\}, & \text{for } i = 1,2 \text{ and } 1 \le j \le 2k-1 \\ k, & \text{for } i = 1,2 \text{ and } 2k \le j \le n-2 \\ k, & \text{for } i = 1 \text{ and } j = n-1, n \\ \lceil \frac{n+2-2k}{2} \rceil, & \text{for } i = 2 \text{ and } j = n-1 \\ n+2-2k, & \text{for } i = 2 \text{ and } j = n \end{cases}$$

$$\phi(v_j^2 v_{j+1}^2) = \begin{cases} \max\{1, \lceil \frac{j+3-2k}{2} \rceil\}, & \text{for } 1 \le j \le n-2 \\ k, & \text{for } j = n-1 \\ 1, & \text{for } j = n. \end{cases}$$

Thus the vertex weights of Wb_n are as follows:

$$wt(v_j^i) = \begin{cases} j+3, & \text{for } i = 2 \text{ and } 1 \le j \le n-2 \\ n+2, & \text{for } i = 2\ ; j = n-1 \\ n+3, & \text{for } i = 2\ ; j = n \\ n+3+j, & \text{for } i = 1\ ; 1 \le j \le n \end{cases}$$

$wt(v) = k + (2k-1)(n+2-2k) + \sum_{j=2k}^{n} (j+n+3-4k)$.

Thus the labeling ϕ is a vertex irregular total k-labeling. □

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(Received 7 Dec 2010; revised 18 May 2011)