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Weak edge Roman domination in graphs

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Abstract

Let $G = (V, E)$ be a graph and let f be a function $f : E \to \{0, 1, 2\}$. An edge x with $f(x) = 0$ is said to be *undefended* with respect to f if it is not incident to an edge with positive weight. The function f is a *weak edge Roman dominating function* (WERDF) if each edge x with $f(x) = 0$ is incident to an edge y with $f(y) > 0$ such that the function $f' : E \to \{0, 1, 2\}$, defined by $f'(x) = 1$, $f'(y) = f(y) - 1$ and $f'(z) = f(z)$ if $z \in E - \{x, y\}$, has no undefended edge. The weight of f' is $w(f') = \sum_{x \in E} f(x)$. The *weak edge Roman domination number*, denoted by $\gamma'_{WR}(G)$, is the minimum weight of a WERDF in G. We show that for every graph G, $\gamma'(G) \leq \gamma'_{WR}(G) \leq 2\gamma'(G)$, where $\gamma'(G)$ is the edge domination number of G. In this paper first we characterize trees T for which $\gamma'_{WR}(T) = \gamma'(T)$. Then we also characterize trees and unicyclic graphs for which $\gamma'_{WR}(G) = 2\gamma'(G)$.

1 Introduction

Cockayne et al. [1] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real valued function $f : V \to R$ the weight of f is $w(f) = \sum_{v \in V} f(v)$ and for

$S \subseteq V$, $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G; that is $\gamma_R(G) = \min\{w(f) \mid f \text{ is an } RDF \text{ in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-*function.* Roman domination in graphs has been studied in [1,3–5,7–13]. This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire!" [14]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered unsecured if no legions are stationed there (i.e., $f(v) = 0$) and secured otherwise (i.e., if $f(v) \in \{1, 2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from an adjacent location (an adjacent vertex u).

Let $G = (V, E)$ be a graph and let f be a function $f : V \to \{0, 1, 2\}$. Let V_0, V_1 and V_2 be the set of vertices assigned to the values $0, 1, 2$ respectively, under f. Note that there is a 1–1 correspondence between the function $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus $f = (V_0, V_1, V_2)$.

A vertex $u \in V_0$ is *undefended* with respect to f, or simply undefended, if the function f is clear from the context, if it is not adjacent to a vertex in V_1 or V_2. Henning et al. [3] defined a function f to be a *weak Roman dominating function* (WRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$, defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. They defined the weight $w(f)$ to be $|V_1| + 2|V_2|$. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G, that is, a WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. Using notation introduced earlier, we define a location to be *undefended* if the location and every location adjacent to it are unsecured (i.e., have no legion stationed there). Since an undefended location is vulnerable to an attack, we require that every unsecure location be adjacent to a secure location in such a way that the movement of a legion from the secure location to the unsecure location does not create an undefended location. Hence every unsecure location can be defended without creating an undefended location.

The concept of edge domination was introduced by Mitchell and Hedetniemi [6]. A subset X of E is called an *edge dominating set* of G if every edge not in X is incident to some edge in X. The *edge domination number* $\gamma'(G)$ is the minimum cadinality taken over all edge dominating sets of G.

We introduced the concept of edge Roman dominating function (ERDF) in [12].

Let $G = (V, E)$ be a graph. A function $f' : E \to \{0, 1, 2\}$ satisfying the condition that every edge x for which $f'(x) = 0$ is incident to at least one edge y for which $f'(y) = 2$ is called an *edge Roman dominating function* of the graph. The *edge Roman domination number*, denoted by $\gamma'_R(G)$, equals the minimum weight of an ERDF of G.

Let E_0, E_1 and E_2 be the sets of edges assigned to the values 0, 1, 2 respectively, under f. Note that there is a 1–1 correspondence between the function f and the ordered partitions (E_0, E_1, E_2) of E. Thus $f = (E_0, E_1, E_2)$. In this paper we introduce the concept of weak edge Roman dominating function. Let $G = (V, E)$

be a graph and let f be a function $f : E \to \{0, 1, 2\}$. An edge x with $f(x) = 0$ is said to be undefended with respect to f if it is not incident to an edge with positive weight. The function f is a *weak edge Roman dominating function* (WERDF) if each edge x with $f(x) = 0$ is incident to an edge y with $f(y) > 0$ such that the function $f' : E \to \{0, 1, 2\}$, defined by $f'(x) = 1$, $f'(y) = f(y) - 1$ and $f'(z) = f(z)$ if $z \in E - \{x, y\}$, has no undefended edge. The weight of f is $w(f) = \sum_{x \in E} f(x)$. The *weak edge Roman domination number*, denoted by $\gamma'_{WR}(G)$, is the minimum weight of a WERDF in G.

Notice that in a WERDF, every edge in E is dominated by an edge in $E_1 \cup E_2$, while in an ERDF, every edge in E_0 is dominated by at least one edge in E_2 (this is more expensive). Furthermore, in a WERDF, every edge in E_0 can be defended without creating an undefended edge.

In this paper, we characterize graphs G for which $\gamma'_{WR}(G) = \gamma'(G)$ and $\gamma'_{WR}(G) = 2\gamma'(G)$. We also characterize trees with $\gamma'_{WR}(G) = \gamma'(G)$. Furthermore we characterize trees and unicyclic graphs for which $\gamma'_{WR}(G) = 2\gamma'(G)$.

2 Notation

For notation and graph theoretic terminology we in general follow [2]. Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let e be an edge in E. The *open neighborhood of* e is $N(e) = \{x \in E \mid x \text{ is incident with } e\}$ and the *closed neighborhood of* e is $N[e] = \{e\} \cup N(e)$. For a set $S \subseteq E$, its *open neighborhood* is $N(S) = \bigcup_{e \in S} N(e)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. An edge x is called a *private neighbor of* e *with respect to* S, or simply $S{-}pn$ of e, if $N[x] \cap S = \{e\}$. The set $pn(e, S) = N[e] - N[S - \{e\}]$ of all the $S{-}pn$ of e is called the *private neighbor set of* e *with respect to* S. The *external private neighbor set of* e *with respect to* S is defined by $epn(e, S) = pn(e, S) - \{e\}$. Hence the set $epn(e, S)$ consists of all the $S - pn$ of e that belong to $E - S$.

A *pendant vertex* of G is a vertex of degree one, while a *support vertex* of G is a vertex adjacent to a leaf. A *pendant edge* of G is an edge which is incident to a pendant vertex.

A set S of edges is called *independent* if no two edges in S are adjacent. A set S of edges is called a 2-*edge packing* if it is independent and for every pair of edges $u, v \in S$, $N[u] \cap N[v] = \emptyset$.

A star $K_{1,n}$ has one vertex v of degree n and n vertices of degree one. A *unicyclic graph* G is a graph with exactly one cycle.

3 Properties

We begin with an inequality chain relating the edge domination number, the weak edge Roman domination number and the edge Roman domination number.

Property 3.1 *For any graph G, $\gamma'(G) \leq \gamma'_{WR}(G) \leq \gamma'_R(G) \leq 2\gamma'(G)$.*

Note that if $G = P_6$, then $\gamma'(G) = 2$, $\gamma'_{WR}(G) = 3$ and $\gamma'_R(G) = 4$. Hence there exist connected graphs G with $\gamma'(G) < \gamma'_{WR}(G) < \gamma'_R(G)$.

Property 3.2 *If G is a graph that contains a path P of order 8, every internal edge of which has degree two in G, then $f(E(P)) \geq 3$ for any WERDF f of G.*

Property 3.3 *If G is any graph and e is any edge of G, then the graph H obtained from G by attaching a path of length 7 to e satisfies $\gamma'_{WR}(H) = \gamma'_{WR}(G) + 3$.*

We have omitted the proofs of Properties 3.1, 3.2 and 3.3 as they are analogous to the proofs of the results of the vertex version.

4 Paths and Cycles

The weak Roman domination number of a path P_n and cycle C_n on n vertices is established in [3] as follows.

Proposition 4.1 [3] *For $n \geq 1$, $\gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

Proposition 4.2 [3] *For $n \geq 4$, $\gamma_r(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

We determine the weak edge Roman domination number of paths and cycles in the following propositions.

Proposition 4.3 *For $n \geq 2$, $\gamma'_{WR}(P_n) = \left\lceil \frac{3(n-1)}{7} \right\rceil$.*

Proof Let $P_n = v_1, v_2, \ldots, v_n$ be a path of size $n-1$. Define a function f' on $E(P_n)$ by $f'(v_iv_{i+1}) = f(v_i)$, $1 \leq i \leq n-1$, where f is a γ_r-function of P_n. Then f' is a WERDF and by Proposition 4.1, $\gamma'_{WR}(P_n) = \left\lceil \frac{3(n-1)}{7} \right\rceil$. □

Proposition 4.4 *For $n \geq 4$, $\gamma'_{WR}(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

Proof Let f be a γ_r-function of cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Define a function f' on $E(C_n)$ by $f'(v_iv_{i+1}) = f(v_i)$, $1 \leq i \leq n-1$ and $f'(v_nv_1) = f'(v_n)$. Then f' is a WERDF and by Proposition 4.2, clearly $\gamma'_{WR}(C_n) = \left\lceil \frac{3n}{7} \right\rceil$. □

5 Graphs G with $\gamma'_{WR}(G) = \gamma'(G)$

Our aim in this section is to characterize graphs G for which $\gamma'_{WR}(G) = \gamma'(G)$ and trees T for which $\gamma'_{WR}(T) = \gamma'(T)$. For convenience we denote the external private neighbours of an edge $e = uv$, which are incident at u and v by $epn(e_u, S)$ and $epn(e_v, S)$ respectively.

Theorem 5.1 *For any graph G, $\gamma'_{WR}(G) = \gamma'(G)$ if and only if there exists a $\gamma'(G)$-set S such that*

(i) $pn(e, S)$ induces a K_3 or a star for every $e \in S$.

(ii) For every edge $e \in E(G) - S$ that is not a private neighbor of any edge of S, there exists an edge $x \in S$ such that $pn(x, S) \cup \{e\}$ induces a star.

Proof Suppose $G = (V, E)$ and $\gamma'_{WR}(G) = \gamma'(G)$. Let $f = (E_0, E_1, E_2)$ be a γ'_{WR}-function. Then $\gamma'(G) \leq |E_1| + |E_2| \leq |E_1| + 2|E_2| = w(f) = \gamma'_{WR}(G)$. Hence by Theorem 3.1 we must have equality throughout the above inequality chain. In particular it follows that $E_2 = \emptyset$. Then $S = E_1$, is a $\gamma'(G)$-set.

First we claim that if $|epn(e_u, S)| > 1$, then $|epn(e_v, S)| = 0$ where $e = uv \in S$. Suppose not. Then $|epn(e_v, S)| \geq 1$. Then the movement of the legion from e to some edge of $epn(e_u, S)$ leaves edges of $epn(e_v, S)$ undefended which is a contradiction. Hence our claim. Now if $|epn(e_u, S)| = 1$, then clearly $|epn(e_v, S)| \leq 1$. If $|epn(e_v, S)| = 0$ then $pn(e, S)$ induces a star. If $|epn(e_v, S)| = 1$, let $epn(e_u, S) = \{x_1\}$, $epn(e_v, S) = \{x_2\}$. Now x_1 and x_2 are incident. For otherwise the movement of a legion from e to x_1, leaves x_2 undefended which is a contradiction. Hence in this case $pn(e, S)$ induces a K_3. This establishes *(i)*.

Suppose $e \in E - S$ is not a private neighbor of any edge of S. Then there exists an edge $x \in S \cap N(e)$ such that the movement of the legion from x to e will not create an undefended edge. This is possible only when $epn(x, S) \subseteq N(e)$. Hence $epn(x, S) \cup \{e\}$ induces a star.

Conversely suppose there exists a $\gamma'(G)$-set S, satisfying conditions *(i)* and *(ii)*. Let H be the set of all support edges of T. Then $S \cup W$ is a γ'-set of T. Let $f' = (E_0, E_1, E_2)$ be a function defined by $E_1 = \{e/e \in G[W]\} \cup H$, $E_2 = \emptyset$ and $E_0 = E - E_1$. Then f' is a WERDF and so $\gamma'_{WR}(G) \leq |S| = \gamma'(G)$. Consequently $\gamma'_{WR}(G) = \gamma'(G)$. □

Next we proceed to characterize trees for which $\gamma'_{WR}(T) = \gamma'(T)$. For this purpose, we prove the following lemmas.

Lemma 5.2 *Let T be a tree with $\gamma'_{WR}(T) = \gamma'(T)$. Then no two support vertices in T are adjacent.*

Proof Suppose not. Then there exist two support vertices x_1 and x_2 in T such that x_1 and x_2 are adjacent. Let $h = x_1x_2$. By Theorem 5.1, there exists a γ'-set S such that *(i)* $pn(e, S)$ induces a K_3 or a star for every $e \in S$ and

(ii) For every edge $e \in E(G) - S$ that is not a private neighbor of any edge of S, there exists an edge $x \in S$ such that $pn(x, S) \cup \{e\}$ induces a star.

If $h \in S$, then we claim that the pendant edges incident either at x_1 or at x_2 are private neighbors of h with respect to S. Suppose the pendent edges incident at both x_1 and x_2 are not private neighbors of h, then the pendent edges incident at x_i, $i = 1, 2$ are incident to some member e_i of S such that $pn(e_i, S) = \emptyset$. Then $S' = S - \{h\}$ is an edge dominating set, which is a contradiction to the minimality of S. Without loss of generality, let the pendant edges incident at x_1 be private neighbors of h with respect to S. Then the pendant edges incident at x_2 are incident to some member e' of S such that $pn(e', S) = \emptyset$. Then $S' = S - \{e'\}$ is an edge dominating set, contradicting the minimality of S. If $h \notin S$, then there exist edges $e_1, e_2 \in S$ incident at x_1 and x_2 respectively and the pendant edges incident at x_i are private neighbors of e_i with respect to S, $i = 1, 2$ and further $pn(e_i, S) \cup \{h\}$ induces a star, $i = 1, 2$. Then $S' = (S - \{e_1, e_2\}) \cup \{h\}$ is an edge dominating set contradicting the minimality of S. Hence no two support vertices in T are adjacent. □

Lemma 5.3 *Let T be a tree with $\gamma'_{WR}(T) = \gamma'(T)$. For every vertex $u \in V(T)$, there exists a leaf w such that $d(u, w) \leq 3$.*

Proof Suppose not. Then there exists a $u \in V(T)$ with $\deg(u) > 1$ such that there exists no leaf w in T such that $d(u, w) \leq 3$. Let e_1 and e_2 be two edges incident at u. Since $\gamma'_{WR}(T) = \gamma'(T)$ there exists a γ'-set S satisfying the conditions of Theorem 5.1.

Case (i) $e_1, e_2 \in S$.

In this case both e_1 and e_2 have private neighbors with respect to S. For otherwise, $S^{'} = S - \{e_2\}$ is an edge dominating set, which is a contradiction to the minimality of S. Let $\{x_1, x_2, \ldots, x_m\} = epn(e_1, S)$ and $\{y_1, y_2, \ldots, y_k\} = epn(e_2, S)$. To each x_i, $1 \leq i \leq m$, there exist edges u_i, v_i such that the edges x_i, u_i, v_i form paths (in that order) with $u_i \in epn(v_i, S)$, $v_i \in S$ and $pn(v_i, S) \cup \{u_i\}$ induces a star.

Similarly to each y_i, $1 \leq i \leq k$, there exist edges w_i, t_i such that the edges y_i, w_i, t_i form paths (in that order) with $w_i \in epn(t_i, S)$ and $pn(t_i, S) \cup \{w_i\}$ induces a star. Hence $S' = (S - \{v_1, v_2, \ldots, v_m, t_1, t_2, \ldots, t_k, e_1\}) \cup \{u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_k\}$ is an edge dominating set which is a contradiction to the minimality of S.

Case (ii) $e_1 \in S$, $e_2 \notin S$.

Subcase (a) $e_1 \in S$ and $e_2 \notin epn(e_1, S)$.

If $epn(e_1, S) \neq \emptyset$, let $epn(e_1, S) = \{x_1, x_2, \ldots, x_m\}$. Then by Theorem 5.1, there exist edges u_i, v_i such that x_i, u_i, v_i form paths (in that order) and $u_i \in epn(v_i, S)$, $v_i \in S$ and $pn(v_i, S) \cup \{u_i\}$ induces a star.

Since $e_2 \notin pn(e_1, S)$, there exist edges y_1, y_2, y_3 such that e_2, y_1, y_2, y_3 form a path (in that order) with $y_1, y_3 \in S$, $y_2 \notin S$ and $pn(y_1, S) \cup \{e_2\}$ and $pn(y_3, S) \cup \{y_2\}$ induces a star respectively. Hence $S' = (S - \{v_1, v_2, \ldots, v_m, y_1, y_3, e_1\}) \cup \{u_1, u_2, \ldots, u_m, y_2, e_2\}$ is an edge dominating set contradicting the minimality of S.

If $epn(e_1, S) = \emptyset$, then there exist edges x_1, x_2, x_3, x_4 and y_1, y_2, y_3 such that e_1, x_1, x_2, x_3, x_4 and e_2, y_1, y_2, y_3 form paths (in that order) with $x_2, x_4, y_1, y_3 \in S$ and $x_1, x_3, y_2 \notin S$ and $pn(x_2, S) \cup \{x_1\}$, $pn(x_4, S) \cup \{x_3\}$, $pn(y_1, S) \cup \{e_2\}$, $pn(y_3, S) \cup \{y_2\}$ respectively induces a star. Hence $S' = (S - \{e_1, x_2, x_4, y_1, y_3\}) \cup \{x_1, x_3, e_2, y_2\}$ is an edge dominating set, which is a contradiction.

Subcase (b) $e_1 \in S$ and $e_2 \in epn(e_1, S)$

Then by Theorem 5.1, there exist edges x, y, z, w such that e_1, x, y, z, w form a path (in that order) with $y, w \in S$, $x, z \notin S$ and $pn(y, S) \cup \{x\}$ and $pn\{w, S\} \cup \{z\}$ induces star respectively.

Let $pn(y, S) = \{y_1, y_2, \ldots, y_k\}$. To each y_i, $1 \leq i \leq k$, there exist edges f_i, g_i such that y_i, f_i, g_i form a path (in that order) with $g_i \in S$ and $pn(g_i, S) \cup \{f_i\}$ induces a star. Then $S' = (S - \{w, y, g_1, g_2, \ldots, g_k\}) \cup \{f_1, f_2, \ldots, f_k\}$ is an edge dominating set, which is a contradiction.

Case (iii) $e_1, e_2 \notin S$.

In this case we assume that no edge of S is incident at u. For, otherwise we are in *Case (ii)*. Then by Theorem 5.1 there exist edges e_i, x_i, y_i, z_i, $i = 1, 2$ such that $x_i, z_i \in S$, $y_i \notin S$ and $pn(x_i, S) \cup \{e_i\}$ and $pn(z_i, S) \cup \{y_i\}$ induces a star. Let $pn(x_i, S) = \{f_{i1}, f_{i2}, \ldots, f_{ik}\}$. To each f_{ij}, there exist edges g_{ij}, h_{ij}, $i = 1, 2$, $1 \leq j \leq k$ such that $e_i, f_{ij}, g_{ij}, h_{ij}$ form paths (in that order) with $h_{ij} \in S$ and $f_{ij}, g_{ij} \notin S$ and $pn(h_{ij}, S) \cup \{g_{ij}\}$ induces a star. Hence $S' = (S - \{z_1, z_2, x_1, x_2, h_{ij}\}) \cup \{y_1, y_2, e_1, g_{ij}\}$, $i = 1, 2$, $j = 1, 2, \ldots, k$ form an edge dominating set, which is a contradiction. □

We define a family $\mathcal{T}$ of trees as follows.

A tree $T \in \mathcal{T}$ if the following holds. Let $\mathcal{S}$ be the set of all supports of T. Let $W = \{w/w \notin N[\mathcal{S}]$ and there exists a leaf z such that $d(z, w) = 3\}$.

(i) $\mathcal{S}$ is independent

(ii) $V = W \cup N[\mathcal{S}]$

(iii) If $W \neq \emptyset$, each component of $G[W]$ is a K_2

(iv) If $u \in N(\mathcal{S})$ with $\deg(u) > 1$, then $N(u) \subset \mathcal{S} \cup W$

(v) If $u \in \mathcal{S}$, then at most one non-pendant member of $N(u)$ is not adjacent to a member of $\mathcal{S} - \{u\}$

An example of a tree in the family $\mathcal{T}$ is shown in Figure 1.

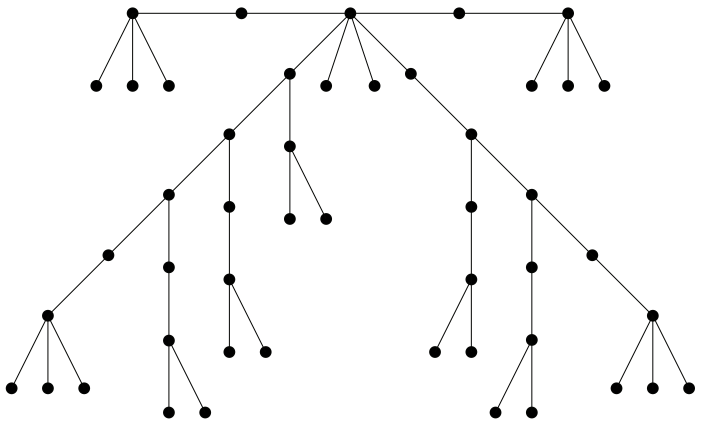

Figure 1: A tree $T \in \mathcal{T}$

Theorem 5.4 *Let T be a tree. Then $\gamma'_{WR}(T) = \gamma'(T)$ if and only if $T \in \mathcal{T}$.*

Proof Suppose $\gamma'_{WR}(T) = \gamma'(T)$. Let $f = (E_0, E_1, E_2)$ be a γ'_{WR}-function of T. Since $\gamma'_{WR}(T) = \gamma'(T)$, $2|E_2| + |E_1| = |E_2| + |E_1|$. Therefore $|E_2| = 0$. Hence clearly E_1 is a γ'-set. Let $\mathcal{S}$ be the set of all supports of T and $W = \{w/w \notin N[\mathcal{S}]$ and there exists a leaf z such that $d(z, w) = 3\}$. By Lemma 5.2, $\mathcal{S}$ is independent. By Lemma 5.3, $V = W \cup N[\mathcal{S}]$.

We now claim that each component of $G[W]$ is a K_2. Suppose not. Then either there exist vertices $u, v, w \in W$ such that u, v, w form a path P_3 or there exists an isolated vertex u in $G[W]$. Now in the first case $uv, uw \in E_1$. For otherwise the edges incident at u, v, w will be undefended. Then $E_1 - \{uv\}$ is an edge dominating set, which is a contradiction to the minimality of E_1. In the second case if u is an isolated vertex in $G[W]$, then one of the edges say e incident at u is in E_1. For, otherwise the edges incident at u will be undefended. Then $E_1 - \{e\}$ is an edge dominating set, which is a contradiction to the minimality of E_1.

Now if $u \in N(\mathcal{S})$ with $\deg(u) > 1$, then we claim that $N(u) \subseteq \mathcal{S} \cup W$. Suppose not. Let $\{z_1, z_2, \ldots, z_k\} \subset N(u)$ such that each z_i is neither in $\mathcal{S}$ nor in W. Hence in order to safeguard the edge uz_i, $1 \leq i \leq k$, at least one edge uz_j is in E_1. Now $E_1 - \{uz_j\}$ is an edge dominating set, which is a contradiction to the minimality of E_1.

Let $u \in \mathcal{S}$. We claim that at most one non-pendant member of $N(u)$ is not adjacent to a member of $\mathcal{S} - \{u\}$. Suppose not. Then there exist at least two non-pendant members say u_1 and u_2 of $N(u)$ such that u_1 and u_2 are not adjacent to any member of $\mathcal{S} - \{u\}$. Since u is a support, at least one edge incident at u is in E_1. Without loss of generality let $uu_1 \in E_1$. Now by *(iv)* u_2 is adjacent to a member w_1 of W and w_1 is adjacent to a member w_2 of W. In order to safeguard the edges u_2w_1, at least one edge incident either at u_2 or at w_1 (other than w_1w_2) is in E_1. Then $E_1 - \{u_2w_1\}$ is an edge dominating set, a contradiction. Hence $T \in \mathcal{T}$.

Conversely suppose $T \in \mathcal{T}$. Let $S = \mathbb{S} \cup G[W]$ where $\mathbb{S}$ is the set of all support edges in T. Then S is a γ' set of T. Let $f' = (E_0, E_1, E_2)$ be a function defined by $E_2 = \emptyset$, $E_1 = S$ and $E_0 = E - E_1$. Then f' is a WERDF as the movement of the legion from any member of E_1 will not create any undefended edge. Now $\gamma'_{WR}(T) = 2|E_2| + |E_1| = |S| = \gamma'(T)$. Hence the claim. □

6 Graphs with $\gamma'_{WR}(G) = 2\gamma'(G)$

In this section we characterize graphs G for which $\gamma'_{WR}(G) = 2\gamma'(G)$. Further we characterize trees and unicyclic graphs for which $\gamma'_{WR}(G) = 2\gamma'(G)$.

Theorem 6.1 *Let G be a graph. Then $\gamma'_{WR}(G) = 2\gamma'(G)$ if and only if every γ'-set S of G satisfies the following conditions*

(i) S is a 2-edge packing

(ii) For every edge $e = u_1u_2 \in S$, corresponding to each u_i, $i = 1, 2$ either there exists an edge x_i incident at u_i with $N(x_i) \subset N[e]$ or there exist at least two edges x_i, y_i, $i = 1, 2$ incident at u_i with $N(x_i) \subset N[e] \cup N(e')$ and $N(y_i) \subset N[e] \cup N(e')$ where $e' \in S$.

(iii) If there exists exactly one edge x_i incident at u_i such that $N(x_i) \subset N[e]$, $i = 1, 2$ and if no two edges w_i, y_i incident at u_i, $i = 1, 2$ are such that $N(w_i) \subset N[e] \cup N(e')$ and $N(y_i) \subset N[e] \cup N(e')$ where $e' \in S$ then x_1 and x_2 are not incident.

Proof Suppose G satisfies the given condition. Let S be any γ'-set. We define a function $f' = (E_0, E_1, E_2)$ by $E_2 = S$, $E_1 = \emptyset$ and $E_0 = E - S$. Then clearly f' is a WERDF and $\gamma'_{WR}(G) = 2|E_2| + |E_1| = 2|S| = 2\gamma'(G)$.

Conversely suppose $\gamma'_{WR}(G) = 2\gamma'(G)$. Let $f' = (E_0, E_1, E_2)$ be a γ'_{WR}-function of G. Then $|E_1| = 0$ and E_2 is a γ'- set. Let S be any γ'-set. We claim that S is a 2-edge packing. Let $e_1, e_2 \in S$. Suppose e_1 and e_2 are incident. Let $f' = (E_0, E_1, E_2)$ be a function defined by $E_2 = S - \{e_1\}$, $E_1 = \{e_1\}$, $E_0 = E - (E_1 \cup E_2)$. The movement of a legion from e_1 to any edge in $epn(e_1, S)$ cannot create an undefended edge. Hence clearly f' is a WERDF. Then $\gamma'_{WR}(G) \leq w(f') = 2|E_2| + |E_1| = 2(|S| - 1) + 1 = 2\gamma'(G) - 1$, a contradiction. Suppose there exists an edge $e \notin S$ such that e_1, e, e_2 form a path (in that order). Let $g = (E_0, E_1, E_2)$ be defined by $E_2 = S - \{e_1, e_2\}$, $E_1 = \{e_1, e_2, e\}$ and $E_0 = E - (E_1 \cup E_2)$. Clearly the movement of a legion from e_i to any member of $epn(e_i, S)$, $i = 1, 2$ will not create an undefended edge. Also the movement of the legion from e to any member of $N(e)$ will not create an undefended edge. Hence f' is a WERDF. Therefore $\gamma'_{WR}(G) \leq 2|E_2| + |E_1| = 2(|S| - 2) + 3 = 2\gamma'(G) - 1$, a contradiction. Hence S is a 2-edge packing.

To prove condition *(ii)* let $e = u_1u_2 \in S$. Consider u_1. Let $Z = \{z_1, z_2, \ldots, z_k\}$ be the collection of edges incident at u_1. If there exists an edge z in Z such that

$N(z) \subset N[e]$, we are through. Otherwise, either no edge z in Z is such that $N(z) \subset N[e] \cup N(e')$ where $e' \in S$ or there exist exactly one edge say z_j in Z such that $N(z_j) \subset N[e] \cup N(e')$ for some $e' \in S$. In the first case, corresponding to each z_i in Z, there exist edges y_i, e_i such that z_i, y_i, e_i form a path (in that order) with $z_i, y_i \notin S$, $e_i \in S$ and $y_i \in epn(e_i, S)$. In the second case, corresponding to each z_i $(i \neq j)$, there exist edges y_i, e_i such that z_i, y_i, e_i form a path (in that order) with $z_i, y_i \notin S$, $e_i \in S$ and $y_i \in epn(e_i, S)$ and corresponding to z_j, there exists an edge y_j such that z_j, y_j, e' form a path (in that order) with $z_j, y_j \notin S$ and $y_j \in epn(e', S)$. In conclusion corresponding to each z_i incident at u_1, there exist edges y_i, e_i such that z_i, y_i, e_i form a path (in that order) with $z_i, y_i \notin S$ and $e_i \in S$ and $y_i \in epn(e_i, S)$. Let $f' = (E_0, E_1, E_2)$ be a function defined by $E_2 = S - \{e, e_1, e_2, \ldots, e_k\}$, $E_1 = \{e, e_1, e_2, \ldots, e_k, y_1, y_2, \ldots, y_k\}$ and $E_0 = E - (E_1 \cup E_2)$. Then the movement of a legion from e_i to $epn(e_i, S) - N(y_i)$ will not create any undefended edge. Further the movement of a legion from y_i to $N(y_i)$ and the movement of a legion from e to $epn(e, S) - Z$ will not create any undefended edge. Hence f' is a WERDF. But then $\gamma'_{WR}(G) \leq w(f') = 2|E_2| + |E_1| = 2(|S| - k - 1) + 2k + 1 = 2\gamma'(G) - 1$, a contradiction. Similarly we deal with a vertex u_2 and get a contradiction. Hence condition *(ii)* holds.

Finally we prove condition *(iii)*. Now if there exists exactly one edge x_i incident at u_i such that $N(x_i) \subset N[e]$ and no two edges x_i, y_i incident at u_i, $i = 1, 2$ are such that $N(x_i) \subset N[e] \cup N(e')$ and $N(y_i) \subset N[e] \cup N(e')$ where $e' \in S$ then we claim that x_1 and x_2 are not incident. Suppose not. Let $z_1, z_2, \ldots, z_k$ be the edges in $N(e) - \{x_1\}$ incident at u_1 and let $v_1, v_2, \ldots, v_m$ be the edges in $N(e) - \{x_2\}$ incident at u_2. Then there exist edges r_i, e_i, $1 \leq i \leq k$ such that z_i, r_i, e_i form a path (in that order) with $z_i, r_i \notin S$, $e_i \in S$ and $r_i \in epn(e_i, S)$. Also there exist edges g_j, h_j, $1 \leq j \leq m$ such that v_j, g_j, h_j form a path (in that order) with $v_j, g_j \notin S$, $h_j \in S$ and $g_j \in epn(h_j, S)$. Let $f' = (E_0, E_1, E_2)$ be a function defined by $E_2 = S - \{e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_m, e\}$, $E_1 = \{e_1, e_2, \ldots, e_k, r_1, r_2, \ldots, r_k, h_1, h_2, \ldots, h_m, g_1, g_2, \ldots, g_m, e\}$, $E_0 = E - (E_1 \cup E_2)$. Then as discussed earlier, we see that f' is a WERDF and $\gamma'_{WR}(G) \leq w(f') = 2|E_2| + |E_1| = 2(|S| - (m+k-1)) - 2(m+k) + 1 = 2\gamma'(G) - 1$, which is a contradiction. Hence condition *(iii)* holds. □

In order to characterize trees and unicyclic graphs with $\gamma'_{WR}(G) = 2\gamma'(G)$, we define a family $\mathcal{G}$ of graphs as follows. For convenience, we call an edge $e = uv$ a *strong support edge* if both u and v are support vertices. A graph $G \in \mathcal{G}$ if the following conditions hold.

(i) Every edge of G is either a strong support edge or incident to exactly one strong support edge.

(ii) No two strong support edges are incident.

An example of a tree in the family $\mathcal{G}$ is shown in Figure 2.

Theorem 6.2 *Let T be a tree. Then $\gamma'_{WR}(T) = 2\gamma'(T)$ if and only if $T \in \mathcal{G}$.*

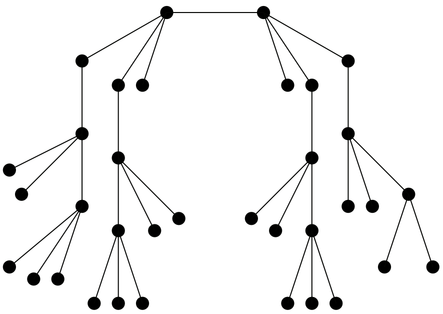

Figure 2: A tree $T \in \mathcal{G}$

Proof Let $\mathbb{S}$ be the set of all strong support edges of T. If $T \in \mathcal{G}$, define $f' = (E_0, E_1, E_2)$ by $E_2 = \mathbb{S}$, $E_1 = \emptyset$ and $E_0 = E - E_2$. Then f' is a WERDF as the movement of the legion from any member of $\mathbb{S}$ cannot leave any undefended edge. Now $\gamma'_{WR}(T) = 2|E_2| + |E_1| = 2|\mathbb{S}| = 2\gamma'(T)$.

Conversely let $\gamma'_{WR}(T) = 2\gamma'(T)$. Let S be any γ'-set of T. Then by Theorem 6.1, S is a 2-edge packing and every member $e = u_1u_2$ of S satisfies condition *(ii)* of Theorem 6.1. Hence there exists an edge x_i incident at u_i such that $N(x_i) \subset N[e]$. i.e., x_i is a pendant edge. Hence each $e \in S$ is a strong support edge. Next we claim that every strong support edge is in S. Suppose $e' = w_1w_2$ be a strong support edge of T not in S. Then there exists two pendent edges w_1v_1 and w_2v_2. Since w_1v_1 is either in S or incident to an edge of S there exists an edge e_1 of S incident at w_1 and distinct from e'. Similarly, there exists an edge e_2 of S incident at w_2 and distinct from e'. Now since S is a 2-edge packing, this is impossible. So $e' \in S$. Hence S is the set of all strong support edges of T. Hence $T \in \mathcal{G}$.

Since S is a γ'-set and S is a 2-edge packing, every $e' \in E - S$ is incident to exactly one strong support edge and no two strong support edges are incident. Hence $T \in \mathcal{G}$. □

Theorem 6.3 *Let G be a unicylic graph with cycle C_n, $n \geq 6$. Then $\gamma'_{WR}(G) = 2\gamma'(G)$ if and only if $G \in \mathcal{G}$.*

Proof If $G \in \mathcal{G}$ then clearly $\gamma'_{WR}(G) = 2\gamma'(G)$.

Conversely suppose $\gamma'_{WR}(G) = 2\gamma'(G)$. Then by Theorem 6.1, there exists a γ'-set S satisfying the conditions of Theorem 6.1. Let $e = u_1u_2 \in S$. Corresponding to each u_i, if there exists a x_i incident at u_i such that $N(x_i) \subset N[e]$ then clearly e is a strong support edge of G. As in the proof of Theorem 6.2 we can prove that S is the set of all strong support edges of G. Suppose at u_1, if there exists no edge x incident at u_1 such that $N(x) \subset N[e]$ and there exist two edges x, y incident at u_1 with $N(x) \subset N[e] \cup N(e')$ and $N(y) \subset N[e] \cup N(e')$ where $e' \in S$ then let $w_x \in N(x) \cap N(e')$ and $w_y \in N(y) \cap N(e')$. Now the edges x, y, w_y, e', w_x form a C_5 (in that order) if w_x and w_y are not incident and the edges x, y, w_y, w_x form a C_4 (in that order) if w_x and w_y are incident which is a contradiction since $n \geq 6$. Hence $G \in \mathcal{G}$. $\square$

Theorem 6.4 *Let G be a unicyclic graph with cycle C_n, $n = 4, 5$. Then $\gamma'_{WR}(G) = 2\gamma'(G)$ if and only if either $G \in \mathcal{G}$ or two non adjacent vertices x, y of C_n are of degree two and each component of $G - \{e_1, e_2\}$ is in $\mathcal{G}$ where e_1 and e_2 are non incident edges which are incident at x and y respectively.*

Proof Let G be unicyclic graph with cycle C_n, $n = 4, 5$ satisfying the given conditions. Let $\mathbb{S}$ be the set of all strong support edges of $G - \{e_1, e_2\}$. We define a function $g' = (E_0, E_1, E_2)$ as follows. $E_2 = \mathbb{S}$, $E_1 = \emptyset$ and $E_0 = (E(G - \{e_1, e_2\}) - E_2) \cup \{e_1, e_2\}$. Then g' is a WERDF as the movement of the legion from one edge to other edge cannot leave any undefended edge. Then $\gamma'_{WR}(T) = 2|E_2| + |E_1| = |\mathbb{S}| = \gamma'(G)$.

Conversely let $\gamma'_{WR}(G) = 2\gamma'(G)$. Let S be any γ'-set. Then S is a 2-edge packing satisfying the conditions of Theorem 6.1.

Case (i) $n = 5$

Clearly exactly one edge of $C_5 = (u_1, u_2, u_3, u_4, u_5, u_1)$ is in S. Without loss of generality let $u_1u_2 \in S$. Then there exist edges x_1 and x_2 incident at u_1 and u_2 respectively such that $N(x_1) \subset N[u_1u_2]$ and $N(x_2) \subset N[u_1u_2]$ which implies that x_1 and x_2 are pendant edges. Hence u_1 and u_2 are support vertices. Now some edge $e \notin C_5$ incident at u_4 is in S. If u_4 is a support then $G \in \mathcal{G}$. If u_4 is a non support, we claim that $\deg(u_3) = \deg(u_5) = 2$. Suppose not. Without loss of generality let $\deg(u_3) > 2$. Since $u_3u_4 \notin S$, some edge x incident at u_3 is in $epn(e', S)$ where $e' \in S$. Let $f' = (E_0, E_1, E_2)$ be defined by $E_2 = S - \{e, e', u_1u_2\}$, $E_1 = \{e, e', x, u_1u_2, u_5u_4\}$, $E_0 = E - (E_1 \cup E_2)$. Then clearly f' is a WERDF such that $\gamma'_{WR}(G) \leq 2|E_2| + |E_1| = 2(|S| - 3) + 5 = 2\gamma'(G) - 1$, a contradiction. Hence if u_4 is not a support, $\deg(u_3) = \deg(u_5) = 2$. Let G_1 and G_2 be the components of $G - \{u_2u_3, u_1u_5\}$. Since S is a 2-edge packing satisfying the conditions of Theorem 6.1, we see that G_1 and G_2 are in $\mathcal{G}$.

Case (ii) $n = 4$

Let $C_4 = (u_1, u_2, u_3, u_4, u_1)$. Clearly no edge of C_4 is in S. Hence there exist edges say e_1, e_2 in S incident at two non adjacent vertices in C_n. Without loss of generality let e_1 and e_2 be incident at u_1 and u_3. If u_1 and u_3 are supports, then clearly $G \in \mathcal{G}$. If at least one of u_1 or u_3 is not a support, then $\deg(u_2) = \deg(u_4) = 2$. Let G_1 and G_2 be the components of $G - \{u_1u_2, u_3u_4\}$. Since S is a 2-edge packing satisfying the conditions of Theorem 6.1, we see that G_1 and G_2 are in $\mathcal{G}$. $\square$

Theorem 6.5 *Let G be a unicyclic graph with cycle $C_3 = (u_1, u_2, u_3, u_1)$. Then $\gamma'_{WR}(G) = 2\gamma'(G)$ if and only if either $G \in \mathcal{G}$ or G satisfies the following conditions.*

(i) Exactly one vertex, say u_1, is a support.

(ii) Either u_2 or u_3 is of degree two.

(iii) $G - \{e\}$ is in $\mathcal{G}$, where e is the edge joining u_1 and the vertex of degree two in C_3.

Proof Let G be a unicyclic graph with cycle $C_3 = (u_1, u_2, u_3, u_1)$. Suppose $G \notin \mathcal{G}$. Then G satisfies the given conditions. Let $\mathbb{S}$ be the set of all strong support edges of $G - \{e\}$. We now define a function $f' = (E_0, E_1, E_2)$ as follows. $E_2 = \mathbb{S}$, $E_1 = \emptyset$ and $E_0 = (E(G - \{e\}) - E_2) \cup \{e\}$. Then f' is a WERDF since the movement of the legion from any member of $\mathbb{S}$ cannot leave any undefended edge. Then $\gamma'_{WR}(T) = 2|E_2| + |E_1| = 2|\mathbb{S}| = 2\gamma'(G)$. Hence we obtain $\gamma'_{WR}(G) = 2\gamma'(G)$.

Conversely, let $\gamma'_{WR}(G) = 2\gamma'(G)$. Let S be any γ'-set of G. By Theorem 6.1, S is a 2-edge packing. Furthermore, S satisfies conditions *(ii)* and *(iii)* of Theorem 6.1. Clearly exactly one edge of C_3 is in S. Without loss of generality, let $u_1u_2 \in S$. Then there exist edges x_1 and x_2 incident at u_1 and u_2 such that $N(x_1) \subset N[u_1u_2]$ and $N(x_2) \subset N[u_1u_2]$. If x_1 and x_2 are the only edges incident at u_1 and u_2 respectively, with $N(x_i) \subset N[e]$, $i = 1, 2$, then by condition *(iii)* of Theorem 6.1, x_1 and x_2 are not incident. Hence at least one of x_1 or x_2 is a pendant edge. If both x_1 and x_2 are pendant edges, then clearly $G \in \mathcal{G}$. Otherwise, without loss of generality, let x_1 be a pendant edge and x_2 be a non-pendant edge. Then u_1 is a support. Since $N(x_2) \subset N[u_1u_2]$, we have $x_2 = u_2u_3$ and $\deg(u_3) = 2$. Now clearly $G - \{u_1u_3\}$ is in $\mathcal{G}$. □

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