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Critical concept for 2-rainbow domination in graphs

Nader Jafari Rad

Department of Mathematics
Shahrood University of Technology
Shahrood
Iran
n.jafarirad@shahroodut.ac.ir

Abstract

For a graph G, let $f : V(G) \rightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$, then f is called a *k-rainbow dominating function* (or simply kRDF) of G. The *weight*, $w(f)$, of a kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, and is denoted by $\gamma_{kr}(G)$. The concept of criticality with respect to various operations on graphs has been studied for several domination parameters. In this paper we study the concept of criticality for 2-rainbow domination in graphs. We characterize 2-rainbow domination vertex (edge) super critical graphs and we will give several characterizations for 2-rainbow domination vertex (edge) critical graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N[v]$. The *degree* of a vertex v is $|N(v)|$, and the maximum degree of vertices of G is denoted by $\Delta(G)$. For a vertex set $S \subseteq V(G)$, we let $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. A set of vertices S in G is a *dominating set* if $N[S] = V(G)$. The *domination number* of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G. For notation and graph theory terminology in general we follow [10].

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, $\gamma_R(G)$, is the minimum weight of an RDF on G; see [3, 7, 8].

For a graph G, let $f : V(G) \rightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$, then f is called a *k-rainbow dominating function* (or simply kRDF) of G. The *weight*, $w(f)$, of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, and is denoted by $\gamma_{kr}(G)$. Clearly $\gamma_{1r}(G) = \gamma(G)$. For references on rainbow domination in graphs, see for example [1, 17, 18].

For many graph parameters, criticality is a fundamental question. The concept of criticality with respect to various operations on graphs has been studied for several domination parameters. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. This concept has been considered for several domination parameters such as domination, total domination, global domination, secure domination and Roman domination, by several authors. This concept is now well studied in domination theory. For references on the criticality concept on various domination parameters see, for example [2, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15]. In this paper we consider this concept for 2-rainbow domination.

We call a graph G:

- 2*-rainbow domination vertex-critical*, or just γ_{2r}*-vertex critical*, if $\gamma_{2r}(G - v) < \gamma_{2r}(G)$ for any vertex $v \in V(G)$;
- 2*-rainbow domination vertex super critical*, or just γ_{2r}*-vertex super critical*, if $\gamma_{2r}(G - v) > \gamma_{2r}(G)$ for any vertex $v \in V(G)$;
- 2*-rainbow domination edge critical*, or just γ_{2r}*-edge critical*, if $\gamma_{2r}(G + e) < \gamma_{2r}(G)$ for any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G;
- 2*-rainbow domination edge super critical*, or just γ_{2r}*-edge super critical*, if $\gamma_{2r}(G - e) > \gamma_{2r}(G)$ for any edge $e \in E(G)$.

In Section 2 we study γ_{2r}-vertex critical graphs and γ_{2r}-vertex super critical graphs. We first show that there is no γ_{2r}-vertex super critical graph, and then study γ_{2r}-vertex critical graphs. We characterize all γ_{2r}-vertex critical trees as well as unicyclic graphs. In Section 3 we characterize γ_{2r}-edge super critical graphs. In Section 4 we study γ_{2r}-edge critical graphs, and prove that no tree is γ_{2r}-edge critical.

We recall that a leaf in a graph G, where G is not isomorphic to K_2, is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. We also refer to a γ_{2r}-function in a graph G as a 2RDF with minimum weight. For a γ_{2r}-function f on a graph G and a vertex $v \in V(G)$ we denote by $f|_{G-v}$ the restriction of f on $V(G) \setminus \{v\}$.

For a 2RDF f on a graph G, and a vertex $v \in V(G)$ with $f(v) \neq \emptyset$, we call the open private neighbor set of v upon f by $pn(v, f) = \{u \in N[v] : f|_{G-v}$ is not a 2RDF for $G - v\}$, and the closed private neighbor set by $pn[v, f] = pn(v, f) \cup \{v\}$.

2 Vertex removal

We begin with the following lemma.

Lemma 1 *For any vertex v in a graph G, $\gamma_{2r}(G) - 1 \leq \gamma_{2r}(G - v) \leq \gamma_{2r}(G) + \Delta(G) - 1$.*

Proof. Let $v \in V(G)$, and let f be a γ_{2r}-function for $G - v$. We define f_1 on $V(G)$ by $f_1(v) = \{1\}$ and $f_1(x) = f(x)$ if $x \neq v$. Then f_1 is a 2RDF for G, and so the lower bound follows. For the upper bound let g be a γ_{2r}-function for G. If $g(v) = \emptyset$, then $g|_{G-v}$ is a 2RDF for $G - v$. So we suppose that $g(v) \neq \emptyset$. Let $A = \{x \in N(v) : g(x) = \emptyset\}$. Now we define g_1 on $G - v$ by $g_1(x) = g(x)$ if $x \notin A$, and $g_1(x) = \{1\}$ if $x \in A$. Then g_1 is a 2RDF for $G - v$, and so the upper bound follows. ■

We show that there is no γ_{2r}-vertex supper critical graph.

Lemma 2 *Let $v \in V(G)$ such that $\gamma_{2r}(G - v) > \gamma_{2r}(G)$. Then for any γ_{2r}-function f for G, $f(v) \neq \emptyset$, and $pn(v, f) \neq \emptyset$.*

Proof. Let $v \in V(G)$ such that $\gamma_{2r}(G - v) > \gamma_{2r}(G)$, and let f be a γ_{2r}-function for G. If $f(v) = \emptyset$ then $f|_{G-v}$ is a 2RDF for $G - v$, and so $\gamma_{2r}(G - v) \leq \gamma_{2r}(G)$, a contradiction. So $f(v) \neq \emptyset$. If $pn(v, f) = \emptyset$, then $f|_{G-v}$ is a 2RDF for $G - v$, and so $\gamma_{2r}(G - v) \leq \gamma_{2r}(G)$, a contradiction. ■

If G is a γ_{2r}-vertex super critical graph of order n, then by Lemma 2, $\gamma_{2r}(G) = n$ and $\Delta(G) \geq 2$. Now if x is a vertex of maximum degree, then f defined on G by $f(x) = \{1, 2\}$, $f(u) = \emptyset$ if $u \in N(x)$, and $f(u) = \{1\}$ if $u \in V(G) - N[x]$ is a 2RDF with weight $n - 1$ which is a contradiction. Thus we have the following.

Theorem 3 *There is no γ_{2r}-vertex super critical graph.*

Next we study γ_{2r}-vertex critical graphs. Many results in the rest of this section are similar to the results in [7, 8] for Roman domination vertex critical graphs. By Lemma 1 we have the following.

Lemma 4 *For any vertex v in a γ_{2r}-vertex critical graph G, $\gamma_{2r}(G-v) = \gamma_{2r}(G)-1$.*

In the following we give a characterization of γ_{2r}-vertex critical graphs.

Theorem 5 *A graph G is γ_{2r}-vertex critical if and only if for every $v \in V(G)$ there is a γ_{2r}-function f for G such that $|f(v)| = 1$ and $pn[v, f] = \{v\}$.*

Proof. Let G be a γ_{2r}-vertex critical graph and let $v \in V(G)$. By Lemma 4, $\gamma_{2r}(G - v) = \gamma_{2r}(G) - 1$. Let f be a γ_{2r}-function for $G - v$. We define g on $V(G)$ by $g(v) = \{1\}$ and $g(x) = f(x)$ if $x \neq v$. Then g is a 2RDF for G and $pn[v, g] = \{v\}$. Since $w(g) = \gamma_{2r}(G)$, g is a γ_{2r}-function for G. The converse is obvious. ■

Lemma 6 *([1]) (1)* $\gamma_{2r}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.
(2) For $n \geq 3$,

$$\gamma_{2r}(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 0 \pmod 4, \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Since removing any vertex of a cycle C_n produces a path P_{n-1}, by a simple calculation we obtain the following.

Proposition 7 *A cycle* C_n *is* γ_{2r}*-vertex critical if and only* $n \equiv 2 \pmod 4$.

From [7] we know that a cycle C_n is Roman domination vertex critical if and only if $n \not\equiv 0 \pmod 3$. Proposition 7 now implies that a γ_{2r}-vertex critical graph is not Roman domination vertex critical in general, and a Roman domination vertex critical graph is not γ_{2r}-vertex critical in general.

Lemma 8 *Any support vertex in a* γ_{2r}*-vertex critical graph is adjacent to exactly one leaf.*

Proof. Let G be a γ_{2r}-vertex critical graph, and x be a support vertex. Suppose to the contrary that y, z be two leaves adjacent to x. By Theorem 5, there is a γ_{2r}-function f for G such that $|f(x)| = 1$ and $pn[x, f] = \{x\}$. Then $f(y) \neq \emptyset$ and $f(z) \neq \emptyset$. We define g on $V(G)$ by $g(x) = \{1, 2\}$, $g(y) = g(z) = \emptyset$, and $g(v) = f(v)$ if $v \notin \{x, y, z\}$. Then g is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. ■

Now we characterize γ_{2r}-vertex critical trees.

Theorem 9 *A tree* T *is* γ_{2r}*-vertex critical if and only if* $T = K_2$.

Proof. It is obvious that K_2 is γ_{2r}-vertex critical. Let T be a γ_{2r}-vertex critical tree of order n. Assume that $n \geq 3$. Let x be a support vertex of T, and z be the unique leaf adjacent to x. Let $y \neq z$ be adjacent to x. By Theorem 5 there is a γ_{2r}-function f for G such that $|f(y)| = 1$ and $pn[y, f] = \{y\}$. Then $f(x) \neq \emptyset$ and $f(z) \neq \emptyset$. Without loss of generality assume that $f(y) = \{1\}$. We define g on $V(G)$ by $g(x) = \emptyset$, $g(z) = \{2\}$, and $g(v) = f(v)$ if $v \notin \{x, y, z\}$. Then g is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. ■

Recall that the *corona*, $\mathrm{cor}(H)$, of a connected graph H is the graph obtained from H by adding a pendant edge to each vertex of H.

Proposition 10 $\gamma_{2r}(\mathrm{cor}(C_n)) = \gamma_{2r}(\mathrm{cor}(P_n)) = n + \lceil \frac{n}{3} \rceil$.

Proof. Let $G = \mathrm{cor}(C_n)$ and f be a γ_{2r}-function for G. Suppose that x, y, z are three support vertices with $\{x, z\} \subseteq N(y)$ and $f(x) = f(y) = f(z) = \emptyset$. Let x_1, y_1, z_1 be the leaves adjacent to x, y, z, respectively. Then $f(y_1) = \{1, 2\}$. We define g on $V(G)$ by $g(y) = g(y_1) = \{1\}$, and $g(v) = f(v)$ if $v \notin \{y, y_1\}$. So we may assume that for any path $x - y - z$ on C_n, $\{f(x), f(y), f(z)\} \neq \{\emptyset\}$. This implies that $\sum_{v \in V(C_n)} |f(v)| \geq \lceil \frac{n}{3} \rceil$. But if x is a leaf of G with $f(x) = \emptyset$, and $x' \in N(x)$, then $f(x') = \{1, 2\}$. We deduce that $w(f) \geq \lceil \frac{n}{3} \rceil + n$. On the other hand we define f on $V(G)$ by the following:

If $n \equiv 0 \pmod 3$, then $f(v_{3i+1}) = f(v_{3i+2}) = \emptyset$ for $0 \leq i \leq \frac{n}{3} - 1$, $f(v_{3i}) = \{2\}$ for $1 \leq i \leq \frac{n}{3}$, and $f(x) = \{1\}$ for any leaf x.

If $n \equiv 1 \pmod 3$, then $f(v_{3i+1}) = f(v_{3i+2}) = \emptyset$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$, $f(v_{3i}) = 2$ for $1 \leq i \leq \lfloor \frac{n}{3} \rfloor$, $f(v_n) = \{2\}$, and $f(x) = \{1\}$ for any leaf x.

If $n \equiv 2 \pmod 3$, then $f(v_{3i+1}) = f(v_{3i+2}) = \emptyset$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$, $f(v_{3i}) = 2$ for $1 \leq i \leq \lfloor \frac{n}{3} \rfloor$, $f(v_n) = \{2\}$, $f(v_{n-1}) = \emptyset$, and $f(x) = \{1\}$ for any leaf x.

Then f is a 2RDF for G with $w(f) = \lceil \frac{n}{3} \rceil + n$. This completes the proof for $\text{cor}(C_n)$.

Next we consider $\text{cor}(P_n)$. Any γ_{2r}-function for $\text{cor}(P_n)$ is also a 2RDF for $\text{cor}(C_n)$. So $\gamma_{2r}(\text{cor}(P_n)) \geq n + \lceil \frac{n}{3} \rceil$. Now consider the 2RDF f defined above for $\text{cor}(C_n)$. There are two adjacent support vertices x, y in $\text{cor}(C_n)$ such that $f(x) = f(y) = \emptyset$. Then f is a 2RDF for $\text{cor}(C_n) - xy \cong \text{cor}(P_n)$. Thus $\gamma_{2r}(\text{cor}(P_n)) \leq n + \lceil \frac{n}{3} \rceil$.

Hence $\gamma_{2r}(\text{cor}(C_n)) = \gamma_{2r}(\text{cor}(P_n)) = n + \lceil \frac{n}{3} \rceil$. ■

Proposition 11 *For $n \geq 3$, $\text{cor}(C_n)$ is γ_{2r}-vertex critical if and only if $n \equiv 1 \pmod 3$.*

Proof. Let $x \in V(\text{cor}(C_n))$. First suppose that x is a leaf. Let y be the support vertex adjacent to x. Consider the 2RDF f for $\text{cor}(C_n)$ described in the proof of Proposition 10. Without loss of generality assume that $f(y) \neq \emptyset$. Then $pn[x, f] = \{x\}$. By Theorem 5, $\gamma_{2r}(\text{cor}(C_n) - x) < \gamma_{2r}(\text{cor}(C_n))$. Next assume that x is a support vertex. Removing x results in a graph $\text{cor}(P_{n-1})$ and an isolated vertex. So $\gamma_{2r}(\text{cor}(C_n) - x) = \gamma_{2r}(\text{cor}(P_{n-1})) + 1$. Now a simple calculation implies that $\lceil \frac{n}{3} \rceil + n - 1 = \lceil \frac{n-1}{3} \rceil + n - 1 + 1$ if and only if $n \equiv 1 \pmod 3$. ■

Now we are ready to characterize γ_{2r}-vertex critical unicyclic graphs. Let $\mathcal{E}$ be the class of graphs such that $G \in \mathcal{E}$ if and only if $G = C_n$, where $n \equiv 2 \pmod 4$, or $G = \text{cor}(C_n)$, where $n \equiv 1 \pmod 3$.

Theorem 12 *A unicyclic graph G is γ_{2r}-vertex critical if and only if $G \in \mathcal{E}$.*

Proof. By Propositions 7 and 11, any graph in $\mathcal{E}$ is γ_{2r}-vertex critical. Let G be a unicyclic γ_{2r}-vertex critical graph. Let C be the unique cycle of G. If $G = C$, then by Proposition 7, $G \in \mathcal{E}$. So we assume that $G \neq C$. Let x be a leaf such that the minimum distance from x to the cycle C is maximized. Let $y \in V(C)$ such that $d(x, C) = d(x, y) = k$, and let P be the shortest path from x to y. If $k \geq 2$, then we let y_1 be a vertex on P at distance 2 from x. Let y_2 be the support vertex adjacent to x. By Lemma 8, $\deg(y_2) = 2$. By Theorem 5 there is a γ_{2r}-function f for G such that $|f(y_1)| = 1$ and $pn[y_1, f] = \{y_1\}$. Then $|f(x)| + |f(y_2)| \geq 2$. Without loss of generality suppose that $f(y_1) = \{1\}$. We define g on $V(G)$ by $g(x) = \{2\}$,

$g(y_2) = \emptyset$, and $g(v) = f(v)$ if $v \notin \{x, y_2\}$. Then g is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. We deduce that $k = 1$. If $G \neq \text{cor}(C)$, then we may choose a support vertex y on C such that y has a neighbor x' with $\deg_G(x') = 2$. By Theorem 5, there is a γ_{2r}-function f for G such that $|f(y)| = 1$ and $pn[y, f] = \{y\}$. Let y_1 be the leaf adjacent to y. Then $f(y_1) \neq \emptyset$. Also $f(x') \neq \emptyset$, since $\deg(x') = 2$. Let $1 \in f(x')$. We define g on $V(G)$ by $g(y) = \emptyset$, $g(y_1) = 2$, and $g(v) = f(v)$ if $v \notin \{y, y_1\}$. Then g is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. So $G = \text{cor}(C)$. By Proposition 11, $G \in \mathcal{E}$. ■

3 Edge removal

Lemma 13 *For any edge e in a graph G, $\gamma_{2r}(G) \leq \gamma_{2r}(G - e) \leq \gamma_{2r}(G) + 1$.*

Proof. Let $e \in E(G)$. The lower bound is obvious, since any γ_{2r}-function for $G - e$ is also a 2RDF for G. Let $e = xy$ and let f be a γ_{2r}-function for G. If $\emptyset \notin \{f(x), f(y)\}$, or $f(x) = f(y) = \emptyset$, then f is a 2RDF for $G - e$, and so $\gamma_{2r}(G - e) \leq \gamma_{2r}(G)$. So we assume that $f(x) = \emptyset$ and $f(y) \neq \emptyset$. Then g defined on $G - e$ by $g(x) = \{1\}$ and $g(v) = f(v)$ if $v \neq x$, is a 2RDF for $G - e$. So the upper bound follows. ■

By Lemma 13, for any edge e in a γ_{2r}-edge super critical graph G, $\gamma_{2r}(G - e) = \gamma_{2r}(G) + 1$. The following proposition provides some examples of γ_{2r}-edge super critical graphs. The proof is straightforward using Lemma 6, and so is omitted.

Proposition 14 *(1) The path P_n is γ_{2r}-edge super critical if and only if n is an odd number greater than* 1.
(2) The cycle C_n is γ_{2r}-edge super critical if and only if $n \equiv 0 \pmod 4$.

In the following we characterize all γ_{2r}-edge super critical graphs. Let $\mathcal{F}$ be the class of all graphs G of order at least 3 such that $G \in \mathcal{F}$ if and only if G is a star, or G is a bipartite graph such that for any γ_{2r}-function f the following hold:
(1) for any $v \in V(G)$, $|f(v)| \leq 1$;
(2) if $f(v) = \emptyset$ for some vertex v, then $\deg(v) = 2$;
(3) for any two adjacent vertices x, y, $f(x) \neq f(y)$, and $\emptyset \in \{f(x), f(y)\}$.

Theorem 15 *A connected graph G of order n is γ_{2r}-edge super critical if and only if $G \in \mathcal{F}$.*

Proof. First, any star of order at least three is γ_{2r}-edge super critical. Let G be a bipartite graph such that for any γ_{2r}-function f:
(1) for any $v \in V(G)$, $|f(v)| \leq 1$;
(2) if $f(v) = \emptyset$ for some vertex v, then $\deg(v) = 2$;
(3) for any two adjacent vertices x, y, $f(x) \neq f(y)$, and $\emptyset \in \{f(x), f(y)\}$.

We show that G is γ_{2r}-edge super critical. Let $e = xy \in E(G)$. Suppose to the contrary that $\gamma_{2r}(G - e) = \gamma_{2r}(G)$. Let f be a γ_{2r}-function for $G - e$. Then f is a γ_{2r}-function for G. By (3) assume that $f(x) = \emptyset$. By (2), $\deg(x) = 2$. Then $\deg_{G-e}(x) = 1$. Let $z \in N_{G-e}(x)$. It follows that $f(z) = \{1, 2\}$. This contradicts (1). So $\gamma_{2r}(G - e) = \gamma_{2r}(G) + 1$. Hence G is γ_{2r}-edge super critical.

Conversely, let G be γ_{2r}-edge super critical. If G is a star, then $G \in \mathcal{F}$. So we assume that G is not a star. Let f be a γ_{2r}-function for G. If there is a vertex x in G with $f(x) = \{1, 2\}$, then there are two vertices y, z different from x such that $y \in N(x)$ and $z \in N(y)$, since G is not a star. Then f is a 2RDF for $G - yz$, a contradiction. We deduce that for any vertex x, $|f(x)| \leq 1$. This proves (1). If there is a vertex v in G such that $f(v) = \emptyset$ and $\deg(v) \geq 3$, then there are y_1, y_2, y_3 in $N(v)$ such that $f(y_1) = \{1\}$ and $f(y_2) = \{2\}$. Then f is a 2RDF for $G - vy_3$, a contradiction. This proves (2). If a, b are two adjacent vertices such that $\emptyset \notin \{f(a), f(b)\}$ or $f(a) = f(b)$, then f is a 2RDF for $G - ab$, a contradiction. This proves (3). Now $\{u : f(u) \in \{\{1\}, \{2\}\}\}$ and $\{u : f(u) = \emptyset\}$ form a bipartition for G, and so G is bipartite. ■

4 Edge addition

Lemma 16 *For any edge $e \in E(\overline{G})$, $\gamma_{2r}(G) - 1 \leq \gamma_{2r}(G + e) \leq \gamma_{2r}(G)$.*

Proof. Let $e \in E(\overline{G})$. The upper bound is obvious, since any γ_{2r}-function for G is also a 2RDF for $G + e$. Let $e = xy$, and let f be a γ_{2r}-function for $G + e$. If $\emptyset \notin \{f(x), f(y)\}$ or $f(x) = f(y) = \emptyset$, then f is a 2RDF for G, and so $\gamma_{2r}(G + e) \geq \gamma_{2r}(G)$. So we may assume that $f(x) = \emptyset$ and $f(y) \neq \emptyset$. Then g defined on $V(G)$ by $g(x) = \{1\}$ and $g(v) = f(v)$ if $v \neq x$, is a 2RDF for G, and so $\gamma_{2r}(G) \leq \gamma_{2r}(G+e)+1$. ■

If G is a γ_{2r}-edge critical graph, then by Lemma 16, for any edge $e \in E(\overline{G})$, $\gamma_{2r}(G+e) = \gamma_{2r}(G)-1$. It is clear that no complete graph is γ_{2r}-edge critical. We will see later that no path is γ_{2r}-edge critical as well. Let $n \geq 3$ and let $G = K_n + \overline{K_m}$. It is a routine matter to see that G is γ_{2r}-edge critical if and only if $m = 1$. Furthermore, if G is a γ_{2r}-edge critical graph, then it is easily seen that $G + K_1$ is γ_{2r}-edge critical.

In the following we give a characterization of γ_{2r}-edge critical graphs.

Theorem 17 *A graph G is γ_{2r}-edge critical if and only if for any two non-adjacent vertices x, y, there is a γ_{2r}-function f for G such that one of the following holds:*
(1) $\{|f(x)|, |f(y)|\} = \{1, 2\}$, and if $|f(w)| = 1$ for $w \in \{x, y\}$ then $pn[w, f] = \{w\}$.
(2) $\{|f(x)|, |f(y)|\} = \{1\}$, and there is $w \in \{x, y\}$ such that $pn[w, f] = \{w\}$ and $\bigcup_{v \in N(w)} f(v) = \{1, 2\} \setminus f(w')$, where $\{w'\} = \{x, y\} \setminus \{w\}$.

Proof. Let G be a γ_{2r}-edge critical graph and x, y be two non-adjacent vertices of G. By definition, $\gamma_{2r}(G + e) = \gamma_{2r}(G) - 1$. Let f be a γ_{2r}-function for $G + e$. If $\emptyset \notin \{f(x), f(y)\}$ or $f(x) = f(y) = \emptyset$, then f is a 2RDF for G, a contradiction. So we assume that $f(x) = \emptyset$ and $f(y) \neq \emptyset$. Furthermore, $|\bigcup_{v \in N_G(x)}| \leq 1$. We consider the following cases.

Case 1. $|f(y)| = 2$. Then g defined on $V(G)$ by $g(x) = \{1\}$, and $g(v) = f(v)$ if $v \neq x$, is a 2RDf for G, and $pn[x, g] = \{x\}$. Since $w(g) = \gamma_{2r}(G)$, g is a γ_{2r}-function for G.

Case 2. $|f(y)| = 1$. Suppose that $f(y) = \{1\}$. There is a vertex $z \in N(x)$ such that $2 \in f(z)$ and clearly $f(z) = \{2\}$. Then g defined on $V(G)$ by $g(x) = \{1\}$ and $g(v) = f(v)$ if $v \neq x$, is a 2RDf for G, and $pn[x, g] = \{x\}$. Since $w(g) = \gamma_{2r}(G)$, g is a γ_{2r}-function for G. Finally we observe that $\bigcup_{v \in N(x)} f(v) = \{2\}$.

The converse is obvious. ■

Lemma 18 *Any support vertex in a γ_{2r}-edge critical graph is adjacent to exactly one leaf.*

Proof. Let G be a γ_{2r}-edge critical graph, and let x be a support vertex adjacent to at least two leaves y, z. By Theorem 17 there is a γ_{2r}-function f for G such that one of the following holds:
(1) $\{|f(y)|, |f(z)|\} = \{1, 2\}$, and if $|f(w)| = 1$ for $w \in \{y, z\}$ then $pn[w, f] = \{w\}$;
(2) $\{|f(y)|, |f(z)|\} = \{1\}$, and there is $w \in \{y, z\}$ such that $pn[w, f] = \{w\}$ and $\bigcup_{v \in N(w)} f(v) = \{1, 2\} \setminus f(w')$, where $\{w'\} = \{y, z\} \setminus \{w\}$.

In both cases we define g on $V(G)$ by $g(x) = \{1, 2\}$, $g(y) = g(z) = \emptyset$, and $g(v) = f(v)$ if $n \notin \{x, y, z\}$. Then g is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. ■

Proposition 19 *There is no γ_{2r}-edge critical tree.*

Proof. Assume to the contrary that T is a γ_{2r}-edge critical tree. Let y, z be two leaves such that $d(y, z) = \mathrm{diam(T)}$, and let P be the path between y and z. It is easy to see that P_2, P_3 and P_4 are not γ_{2r}-edge critical. So $\mathrm{diam(T)} \geq 4$. Let $u \in N(y)$, $v \in N(z)$ and $w \in N(u) \setminus \{y\}$. By Lemma 18, $\deg(u) = \deg(v) = 2$. By Theorem 17, there is a γ_{2r}-function f for T such that one of the following holds:
(1) $\{|f(y)|, |f(z)|\} = \{1, 2\}$, and if $|f(a)| = 1$ for $a \in \{y, z\}$ then $pn[a, f] = \{a\}$;
(2) $\{|f(y)|, |f(z)|\} = \{1\}$, and there is $a \in \{y, z\}$ such that $pn[a, f] = \{a\}$ and $\bigcup_{v \in N(a)} f(v) = \{1, 2\} \setminus f(w')$, where $\{w'\} = \{y, z\} \setminus \{a\}$. We consider the following cases.

- **Case 1.** $\mathrm{diam(T)} = 4$. Assume that (1) holds. Without loss of generality we may assume that $f(y) = \{1\}$, and so $f(u) \neq \emptyset$. If $f(v) \neq \emptyset$, then we replace $f(z)$ by $\{1\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction. So $f(v) = \emptyset$. Also we observe that $|f(u)| = 1$. If $f(w) \neq \emptyset$, then $|f(w)| = 1$ and we replace $f(u)$ by $\emptyset$ and $f(y)$ by $\{1, 2\} \setminus f(w)$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(T)$, a contradiction. So $f(w) = \emptyset$. Then we define f_1 on $V(T)$ by $f_1(w) = \{1\}$, $f_1(y) = f_1(z) = \{2\}$, $f_1(u) = f_1(v) = \emptyset$, and $f_1(x) = f(x)$ if $x \notin \{y, u, w, v, z\}$. Then f_1 is a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction.

So we assume that (2) holds. Let $f(y) = \{1\}$, $pn[y, f] = \{y\}$, $|f(u)| = 1$, and $f(u) \neq f(z)$. We show that $f(w) = \emptyset$. Suppose to the contrary that $f(w) \neq \emptyset$. Without loss of generality assume that $2 \in f(w)$. Then we replace $f(u)$ by $\emptyset$, and $f(y)$ by $\{1\}$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. Thus $f(w) = \emptyset$, and so $f(v) \neq \emptyset$. Then $|f(y) \cup f(u) \cup f(w) \cup f(v) \cup f(z)| \geq 4$. Now we define g on $V(T)$ by $g(y) = \{1\} = g(z)$, $g(w) = \{2\}$, $g(u) = g(v) = \emptyset$, and $g(x) = f(x)$ if $x \notin \{y, u, w, v, z\}$. Then g is a 2RDF for T of weight less than $\gamma_{2r}(T)$, a contradiction.

- **Case 2.** diam(T) = 5. Let $v_1 \in N(v) \setminus \{z\}$. Assume that (1) holds, and without loss of generality assume that $f(z) = \{1, 2\}$, and $f(y) = \{1\}$. As in Case 1, we observe that $f(v) = f(w) = \emptyset$ and $|f(u)| = 1$. If $1 \in f(v_1)$, then we replace $f(z)$ by $\{2\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction. So $1 \notin f(v_1)$ and similarly $2 \notin f(v_1)$, and thus $f(v_1) = \emptyset$. If w is adjacent to a support vertex w_1 different from u, then we let w_2 be the unique leaf adjacent to w_1 by Lemma 18. Then $|f(w_1)| + |f(w_2)| \geq 2$. We define g_1 on $V(T)$ by $g_1(w) = \{2\}$, $g_1(u) = g_1(w_1) = \emptyset$, $g_1(w_2) = \{1\}$, and $g_1(x) = f(x)$ if $x \notin \{u, w, w_1, w_2\}$. Then g_1 is a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. So u is the only support vertex adjacent to w. Suppose that $f(u) = \{1\}$. But $f(w) = \emptyset$. Thus there is exactly one leaf w_3 adjacent to w, and we observe that $f(w_3) = \{2\}$.

  Next we consider v_1. We know that $f(v_1) = \emptyset$. Assume that there is a vertex $v_2 \in N(v_1)$ such that $f(v_2) = \{1, 2\}$. If v_2 is a leaf, then we replace $f(v_2)$ by $\{1\}$, $f(v_1)$ by $\{2\}$, and $f(z)$ by $\{1\}$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(T)$, a contradiction. So v_2 is a support vertex with a unique leaf v_3 by Lemma 18. Then we replace $f(v_3)$ by $\{1\}$, $f(v_1)$ by $\{2\}$, $f(v_2)$ by $\emptyset$ and $f(z)$ by $\{1\}$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(T)$, a contradiction. We deduce that for any neighbor x of v_1, $|f(x)| \leq 1$. But $f(v_1) = f(w) = f(v) = \emptyset$. So there are two vertices $v_4, v_5 \in N(v_1)$ such that $f(v_4) = \{1\}$ and $f(v_5) = \{2\}$. Then we replace $f(v_1)$ by $\{1, 2\}$, $f(v_4)$ and $f(v_5)$ by $\emptyset$, and $f(z)$ by $\{1\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction.

  It remains to assume that (2) holds. Assume that $f(y) = \{1\}$, and $pn[y, f] = \{y\}$, and $f(u) \neq f(z)$. As before, we obtain $f(w) = \emptyset$. Without loss of generality assume that $f(z) = \{2\}$ and $f(u) = \{1\}$. We show that $\deg(w) \leq 3$. If $\deg(w) \geq 4$, then we replace $f(w)$ by $\{1, 2\}$, $f(x)$ by $\emptyset$ if $x \in N(w) \setminus \{v_1\}$, and $f(x)$ by $\{1\}$ if x is a leaf at distance 2 from w and $x \notin N(v_1)$, to obtain a 2RDF for G with weight less than $\gamma_{2r}(T)$, a contradiction. So $\deg(w) \leq 3$.

  Next we show that $\deg(v_1) \leq 3$. Suppose that $\deg(v_1) \geq 4$. Let $a_1 \neq v$ be a support vertex adjacent to v_1, and a_2 be the leaf adjacent to a_1. By Theorem 17, there is a γ_{2r}-function h for T such that one of the following holds:
  (I) $\{|h(a_1)|, |h(v)|\} = \{1, 2\}$, and if $|f(a)| = 1$ for $a \in \{a_1, v\}$ then $pn[a, f] = \{a\}$;
  (II) $\{|f(a_1)|, |f(v)|\} = \{1\}$, and there is $a \in \{a_1, v\}$ such that $pn[a, f] = \{a\}$ and $\bigcup_{b \in N(a)} f(b) = \{1, 2\} \setminus f(c)$, where $\{c\} = \{a_1, v\} \setminus \{a\}$.

If h satisfies (I), then $|h(v)| + |h(z)| + |h(a_2)| + |h(a_1)| \geq 4$. If $f(v_1) \neq \emptyset$, then we may assume that $1 \in f(v_1)$, and so we replace $h(a_1)$ and $h(v)$ by $\emptyset$, $h(a_2)$ and $h(z)$ by $\{2\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction. So $f(v_1) = \emptyset$. Then we replace $h(a_1)$ and $h(v)$ by $\emptyset$, $h(a_2)$ and $h(z)$ by $\{2\}$, and $h(v_1)$ by $\{1\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction. Therefore we assume that h satisfies (II). Then $|h(a_1)| + |h(a_2)| + |h(v)| + |h(z)| \geq 4$, and as in Case 1 we obtain $f(v_1) = \emptyset$. Then we replace $h(v_1)$ by $\{2\}$, $h(a_1)$ and $h(v)$ by $\emptyset$, and $h(a_2)$ and $h(z)$ by $\{1\}$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(G)$, a contradiction. We conclude that $\deg(v_1) \leq 3$.

We next show that $\deg(w) = 2$. Suppose that $\deg(w) = 3$. We know that $f(w) = \emptyset$. If w is not a support vertex, then there is a support vertex $a_1 \neq u$ which is adjacent to w. Let a_2 be the leaf adjacent to a_1. We replace $f(w)$ by $\{1\}$, $f(a_1)$, $f(u)$ by $\emptyset$, and $f(a_2)$, $f(y)$ by $\{2\}$ to obtain a 2RDF for G with weight less than $\gamma_{2r}(T)$, a contradiction. So w is a support vertex adjacent to exactly one leaf. If $deg(v_1) = 2$, then $\gamma_{2r}(T) = \gamma_{2r}(T + uv) = 5$, a contradiction. So $\deg(v_1) = 3$. If v_1 is not a support vertex, then there is a support vertex $a_1 \neq v$ adjacent to v_1, and by Lemma 18, $\deg(a_1) = 2$. Then $\gamma_{2r}(T) = \gamma_{2r}(T + a_1 v) = 6$, a contradiction. So v_1 is a support vertex. This time $\gamma_{2r}(T) = \gamma_{2r}(T + uv) = 6$, a contradiction. Hence $\deg(w) = 2$.

If $\deg(v_1) = 2$, then $T = P_6$, which is not γ_{2r}-edge critical, since $\gamma_{2r}(P_6) = \gamma_{2r}(P_6 + uv) = 4$. So $\deg(v_1) = 3$. If v_1 is a support vertex, then $\gamma_{2r}(T) = \gamma_{2r}(T + uv) = 5$, a contradiction. So v_1 is not a support vertex. This time $\gamma_{2r}(T) = \gamma_{2r}(T + yz) = 5$, a contradiction.

- **Case 3.** diam(T) ≥ 6. Let $v_1 \in N(v) \setminus \{z\}$. By Theorem 17 there is a γ_{2r}-function g for T such that one of the following holds:
(i) $\{|g(w)|, |g(v_1)|\} = \{1, 2\}$, and if $|g(a)| = 1$ for $a \in \{w, v_1\}$ then $pn[a, g] = \{a\}$;
(ii) $\{|g(w)|, |g(v_1)|\} = \{1\}$, and there is $a \in \{w, v_1\}$ such that $pn[a, g] = \{a\}$, $\bigcup_{v \in N(a)} g(v) = \{1, 2\} \setminus g(d)$, where $\{d\} = \{w, v_1\} \setminus \{a\}$.
In both cases, without loss of generality we assume that $g(v_1) = \{1\}$, $g(v) \neq \emptyset$, and $g(z) \neq \emptyset$. Now we replace $g(v)$ by $\emptyset$ and $g(z)$ by $\{2\}$ to obtain a 2RDF for T with weight less than $\gamma_{2r}(T)$, a contradiction.

■

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References

[1] B. Bresar and T. K. Sumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007), 2394-2400.

[2] R. C. Brigham, P. Z. Chinn, and R. D. Dutton, Vertex domination-critical graphs. *Networks* **18** (1988), 173–179.

[3] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi, *On Roman domination in graphs, Discrete Math.* **278** (2004), 11-22.

[4] R. D. Dutton, and R. Brigham, *On global domination critical graphs.* Discrete Math. **309** (2009), no. 19, 5894–5897.

[5] W. Goddard, T. W. Haynes, M. A. Henning, and L. C. van der Merwe, The diameter of total domination vertex critical graphs. *Discrete Math.* **286** (2004), 255–261.

[6] P. J. P. Grobler and C. M. Mynhardt, *Secure domination critical graphs.* Discrete Math. **309** (2009), no. 19, 5820–5827.

[7] A. Hansberg, N. Jafari Rad and L. Volkmann, *Vertex and edge critical Roman domination in graphs*, Utilitas Mathematica, To appear.

[8] A. Hansberg, N. Jafari Rad and L. Volkmann, *Characterization of Roman domination critical unicyclic graphs*, Utilitas Mathematica, To appear.

[9] D. Hanson and P. Wang, A note on extremal total domination edge critical graphs. *Util. Math.* **63** (2003), 89–96.

[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, NewYork, 1998.

[11] M. A. Henning and N. Jafari Rad, On total domination vertex critical graphs of high connectivity. *Discrete Appl. Math.* **157** (2009), 1969–1973.

[12] M. A. Henning, and L. C. van der Merwe, *Properties of total domination edge critical graphs.* Discrete Appl. Math. **158** (2010), no. 2, 147–153.

[13] N. Jafari Rad, *On the diameter of a domination dot critical graph*, Discrete Appl. Math. **157** (2009), 1647-1649.

[14] D. P. Sumner, P. Blitch, *Domination critical graphs*, J. Combin. Theory Ser. B **34** (1983) 6576.

[15] C. Wang, Z. Hu, and X. Li, A constructive characterization of total domination vertex critical graphs. *Discrete Math.* **309** (2009), 991–996.

[16] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.

[17] Y. Wu and H. Xing, *Note on 2-rainbow domination and Roman domination in graphs*, Applied Mathematical Letters **23** (2010), 706-709.

[18] G. Xu, 2-*rainbow domination in generalized Petersen graphs* $P(n, 3)$, Discrete Appl. Math. **157** (2009), 2570-2573.

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