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Note: On the second order Eulerian numbers

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Abstract

Let $T(n,k)$ denote the second order Eulerian numbers. In this paper, a new explicit expression for $T(n,k)$ is presented.

1 Introduction

Let $\sigma = [x_1, x_2, \ldots, x_{2n}]$ be a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. An *ascent* of permutation σ is an occurrence of $x_j < x_{j+1}$ for $j = 1, 2, \ldots, 2n-1$. For instance, $\sigma_1 = [3, 3, 2, 1, 2, 1]$ has only one ascent $[1, 2]$. Likewise, $\sigma_2 = [2, 3, 1, 1, 3, 2]$ has 2 ascents $[2, 3]$ and $[1, 3]$, etc.

The *second order Eulerian number* $T(n,k)$, $(0 \le k \le n-1)$, corresponds to the number of permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ with exactly k ascents which have the property that for each $j \in \{1, 2, \ldots, n\}$, all the numbers appearing between the two occurrences of j in the permutation are greater than j. For example, for $n = 3$ there are 15 such permutations, one with no ascents, eight with a single ascent, and six with two ascents:

$[3, 3, 2, 2, 1, 1]$;

$[2, 2, 1, 1, 3, 3]$, $[2, 2, 1, 3, 3, 1]$, $[2, 2, 3, 3, 1, 1]$, $[2, 3, 3, 2, 1, 1]$, $[1, 1, 3, 3, 2, 2]$, $[1, 3, 3, 2, 2, 1]$, $[3, 3, 1, 1, 2, 2]$, $[3, 3, 1, 2, 2, 1]$;

$[1, 1, 2, 2, 3, 3]$, $[1, 2, 2, 1, 3, 3]$, $[1, 1, 2, 3, 3, 2]$, $[1, 2, 3, 3, 2, 1]$, $[1, 3, 3, 1, 2, 2]$, $[1, 2, 2, 3, 3, 1]$.

Hence $T(3, 0) = 1$, $T(3, 1) = 8$, $T(3, 2) = 6$.

Lemma 1.1 ([1]) *The second order Eulerian numbers $T(n,k)$ satisfy the triangular recurrence relation*

$$T(n,k) \;=\; (k+1)T(n-1,k) + (2n-k-1)T(n-1,k-1).$$

See [1] on the other properties of the second order Eulerian numbers. In this paper, we find a new explicit expression for the Eulerian numbers $T(n,k)$.

2 The exact value of the second order Eulerian numbers

Theorem 2.1 *The second order Eulerian numbers $T(n,k)$ have the explicit formula*

$$T(n,k) = \sum_{\substack{t_1+t_2+\cdots+t_{k+1}=n-k-1\\ t_j\geq 0,\ j=1,2,\cdots,k+1}} 1^{t_1}2^{t_2}\dots(k+1)^{t_{k+1}}(2t_1+2)(2(t_1+t_2)+3)\dots \dots(2(t_1+\cdots+t_k)+(k+1)).$$

Proof. We use induction on n.

It is clear that the proposition holds for $n=1$.

Suppose that Theorem 2.1 holds for $n(0\leq k\leq n-1)$. We now consider $T(n+1,k)$, where $0\leq k\leq n$. From Lemma 1.1 and the induction hypothesis, we have

$$T(n+1,k) = (k+1)T(n,k)+(2n-k+1)T(n,k-1)$$

$$= (k+1)\sum_{\substack{t_1+t_2+\cdots+t_{k+1}=n-k-1\\ t_j\geq 0,\ j=1,2,\cdots,k+1}} 1^{t_1}2^{t_2}\cdots(k+1)^{t_{k+1}}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_k)+(k+1))$$

$$+(2n-k+1)\sum_{\substack{t_1+t_2+\cdots+t_k=n-k\\ t_j\geq 0,\ j=1,2,\cdots,k}} 1^{t_1}2^{t_2}\cdots k^{t_k}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_{k-1})+k)$$

$$= \sum_{\substack{t_1+t_2+\cdots+(t_{k+1}+1)=n-k\\ t_j\geq 0,\ j=1,2,\cdots,k+1}} 1^{t_1}2^{t_2}\cdots(k+1)^{t_{k+1}+1}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_k)+(k+1))$$

$$+\sum_{\substack{t_1+t_2+\cdots+t_k=n-k\\ t_j\geq 0,\ j=1,2,\cdots,k}} 1^{t_1}2^{t_2}\cdots k^{t_k}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_{k-1})+k)(2(n-k)+(k+1))$$

$$= \sum_{\substack{h_1+h_2+\cdots+h_{k+1}=n-k\\ h_j\geq 0,\ j=1,2,\cdots,k,\ h_{k+1}\geq 1}} 1^{h_1}2^{h_2}\cdots(k+1)^{h_{k+1}}(2h_1+2)(2(h_1+h_2)+3)\cdots(2(h_1+\cdots+h_k)+(k+1))$$

$$+\sum_{\substack{t_1+t_2+\cdots+t_k+t_{k+1}=n-k\\ t_j\geq 0,\ j=1,2,\cdots,k,\ t_{k+1}=0}} 1^{t_1}2^{t_2}\cdots(k+1)^{t_{k+1}}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_k)+(k+1))$$

$$= \sum_{\substack{t_1+t_2+\cdots+t_k+t_{k+1}=n-k\\ t_j\geq 0,\ j=1,2,\cdots,k+1,}} 1^{t_1}2^{t_2}\cdots(k+1)^{t_{k+1}}(2t_1+2)(2(t_1+t_2)+3)\cdots(2(t_1+\cdots+t_k)+(k+1))$$

So Theorem 2.1 holds for $n+1$. This completes the proof. □

Here is an example illustrating Theorem 2.1.

$$
\begin{aligned}
T(5,2) &= \sum_{\substack{t_1+t_2+t_3=2 \\ t_j\geq 0,\ j=1,2,3}} 1^{t_1}2^{t_2}3^{t_3}(2t_1+2)(2(t_1+t_2)+3) \\
&= 1^2 2^0 3^0(4+2)(4+3) + 1^0 2^2 3^0(0+2)(4+3) + 1^0 2^0 3^2(0+2)(0+3) \\
&\quad +1^1 2^1 3^0(2+2)(4+3) + 1^1 2^0 3^1(2+2)(2+3) + 1^0 2^1 3^1(0+2)(2+3) \\
&= 42+56+54+56+60+60 \\
&= 328.
\end{aligned}
$$

Corollary 2.2 *The second order Eulerian numbers $T(n,k)$ satisfy*

(1) $T(n,n-1)=n!$ $(n\geq 1)$;

(2) $T(n,1)=2^{n+1}-2(n+1)$ $(n\geq 1)$.

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References

[1] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Reading, MA: Addison-Wesley, p. 256, 1994.

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