Pairs of simple balanced ternary designs with prescribed numbers of triples in common

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ABSTRACT: A balanced ternary design of order v with block size three, index 2 and $\rho_2 = 1$ is a collection of multi-subsets of size 3 (of type $\{x, y, z\}$ or $\{x, x, y\}$) called blocks, in which each unordered pair of distinct elements occurs twice — possibly in one block — and in which each element is repeated in just one block. So there are precisely v blocks of type $\{x, x, y\}$. Necessarily $v \equiv 0 \pmod 3$. We denote such a design by BTD($v; \rho_2 = 1; 3, 2$). It is called *simple* if all its blocks are distinct. Let $I_1(v)$ denote the set of numbers k so that there exist two simple BTD($v; \rho_2 = 1; 3, 2$) with precisely k blocks in common. (Here the subscript 1 refers to the value of ρ_2.) In this paper we determine the set $I_1(v)$ for all admissible $v \equiv 0 \pmod 3$.

1 Some preliminaries

1.1 Introduction

A balanced ternary design is one possible generalisation of a balanced incomplete block design, and was essentially first introduced by Tocher [10]. For a survey of such designs see Billington [1] and the update [3]. However, in order to make the present paper self-contained, we now define a *balanced ternary design* or BTD. This is a collection of multi-subsets of size k, chosen from a v-set in such a way that each pair of distinct elements $\{x, y\}$ occurs λ times, each pair of non-distinct elements $\{x, x\}$ occurs ρ_2 times, and each element occurs 0, 1 or 2 times in a block (hence "ternary"). These parameters we denote by BTD($v; \rho_2; k, \lambda$). It is straightforward to show that

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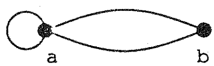
each element must occur singly in a constant number of blocks, say ρ_1 blocks, and so each element occurs altogether $r = \rho_1 + 2\rho_2$ times. Also there are b blocks altogether, where $vr = bk$, and the identity

$$\lambda(v-1) = r(k-1) - 2\rho_2$$

is easily obtained by counting distinct pairs. (Of course in a block such as $\{x, x, y, y, z\}$ the pair $\{x, y\}$ is said to occur 4 times, and the pairs $\{x, z\}$ and $\{y, z\}$ twice each, while $\{x, x\}$ occurs once and $\{y, y\}$ once.) It is clear that $\lambda \geqslant 2$ for a genuine BTD, that is, one with $\rho_2 > 0$. (For further necessary constraints see [**1, 2, 3**].)

Here we shall only consider block size k equal to three, λ (the index) equal to two, and $\rho_2 = 1$. So there are precisely v blocks of type $\{x, x, y\}$ in such a BTD. Altogether, a BTD$(v; 1; 3, 2)$ has $v^2/3$ blocks, and a necessary and sufficient condition for existence is that $v \equiv 0 \pmod 3$. (See, for example, [2].) We can consider such a BTD$(v; 1; 3, 2)$ to be an edge-disjoint decomposition of $2K_v^+$, the complete graph on v vertices with two edges between each pair of vertices and with a loop on each vertex (denoted by the + sign), into triangles and tadpoles. Here a tadpole, corresponding to the block aab, is as follows:



The question "How many blocks may two designs with the same parameters, based on the same set of elements, have in common?" has been considered in the past for Steiner triple systems (Lindner and Rosa [8]), for Steiner quadruple systems (Gionfriddo and Lindner [7] and Fu [6]) and also for group divisible triple systems (Butler and Hoffman [4]). Related work has also appeared in [9]. The difference here, for BTDs, is that the least possible value for the index, λ, is 2. Consequently, to avoid any difficulty concerning interpretation of numbers of common blocks, in the event that some blocks might be repeated, in this paper we shall restrict our attention to BTDs with no repeated blocks; such BTDs are called *simple*. (In the decomposition of $2K_v^+$ there will be no repeated triangles.)

We define

$$I_1(v) = \{k \mid \text{there exist 2 simple BTD}(v; 1; 3, 2) \text{ with } k \text{ blocks in common}\}.$$

(The subscript 1 refers to the value of ρ_2.)

The purpose of this paper is to prove:

MAIN THEOREM

$I_1(v) = \{k \mid 0 \leqslant k \leqslant v^2/3 - 3$ or $k = v^2/3\}$ for all $v \equiv 0 \pmod 3$ with the one exception: $5 \notin I_1(6)$.

For convenience, we define $J_1(v) = \{k \mid 0 \leqslant k \leqslant v^2/3 - 3\} \cup \{v^2/3\}$. Certainly $I_1(v) \subseteq J_1(v)$; we omit the easy proof. So we aim to prove that $I_1(v) = J_1(v)$ for $v \neq 6$, and $I_1(6) = J_1(6) \setminus \{5\}$.

First we deal with the cases $v = 3, 6$ and 9. We shall also need to deal with some of the values in $I_1(12)$ and $I_1(15)$ separately, but we leave those until after our general constructions.

From now on, we write blocks such as $\{x, y, z\}$ and $\{x, x, y\}$ as xyz and xxy for brevity.

1.2 The case $v = 3$

$J_1(3) = \{0, 3\}$.

The two designs

0	0	1		0	0	2
1	1	2		1	1	0
2	2	0		2	2	1

provide us with $I_1(3) = \{0, 3\}$, by taking the same design twice (three common blocks) or these two designs (no common blocks).

1.3 The case $v = 6$

$J_1(6) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 12\}$.

An exhaustive computer search has shown that $5 \notin I_1(6)$. Up to isomorphism, there are precisely two BTD(6; 1; 3, 2) (see Donovan [5]), and only one of these is simple; call it design A:

1	1	2		1	4	5
2	2	3		1	4	6
3	3	1		1	5	6
4	4	3		2	4	5
5	5	3		2	4	6
6	6	3		2	5	6

A

Consider the (isomorphic) designs got from A by permuting elements; let

$$A_1 = (12)A, \quad A_2 = (123)A, \quad A_3 = (14)A, \quad A_4 = (14)(25)(36)A,$$
$$A_5 = (25)A, \quad A_6 = (14)(23)A, \quad A_7 = (24)A.$$

Then it is easy to check that $|A \cap A| = 12$, $|A \cap A_1| = 9$, $|A_5 \cap A_7| = 8$, $|A_3 \cap A_5| = 7$, $|A \cap A_2| = 6$, $|A \cap A_6| = 4$, $|A_1 \cap A_2| = 3$, $|A_2 \cap A_3| = 2$, $|A_3 \cap A_4| = 1$, and $|A_4 \cap A| = 0$. So $I_1(6) = J_1(6) \setminus \{5\}$.

1.4 The case $v = 9$

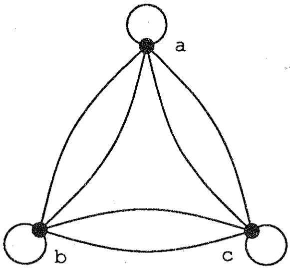
$J_1(9) = \{0, 1, 2, \ldots, 23, 24, 27\}$.

To construct a simple BTD($9; \rho_2 = 1; 3, 2$), we need to partition $2K_9^+$ into triangles and tadpoles, with no triangles repeated. We do this in different ways, so that we can have two such BTDs with a prescribed number of blocks in common. For the vertices of $2K_9^+$ we take $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ and $\{a,b,c\}$. The edges of $2K_9^+$ we partition into three parts:

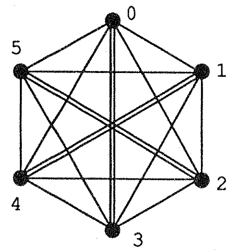
Firstly, the graph G_1 is $2K_3^+$ on vertices a, b and c. Secondly, G_2 is the graph on $\{0,1,2,3,4,5\}$ with edges obtained by joining vertices i and j with one edge if $|i-j| = 1$ or 2 (mod 6) and with two edges if $|i-j| = 3$ (mod 6). Finally, G_3 is the graph consisting of all the remaining edges of $2K_9^+$.

For $i = 1,2,3$, let S_i be the set of possible numbers of common blocks in two partitions of the edges of G_i into triangles and tadpoles. Then $I_1(9)$ contains $S_1 + S_2 + S_3$.

G_1 :



G_2 :



Clearly, $S_1 = I_1(3) = \{0,3\}$.

Now G_2 has 18 edges; it is K_6 together with the edges $\{0,3\}$, $\{1,4\}$, $\{2,5\}$. There is essentially only one way to partition G_2 into six triangles, and we can 'label' this in just two ways:

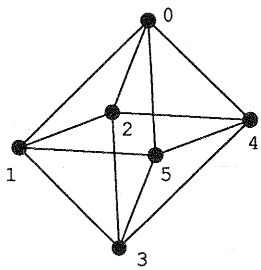
031, 034, 142, 145, 253, 250 or
032, 035, 143, 140, 254, 251.

Thus $S_2 = \{0,6\}$.

So far we have $S_1 + S_2 = \{0,3,6,9\}$.

Now consider the graph G_3 consisting of the remaining edges of $2K_9^+$. It is in fact the octahedron graph G_4, together with a loop on each vertex, with three more vertices, a, b, c, each of which is joined to each vertex of G_4 by two edges.

G_4 :



We claim that S_3 contains $\{0,6,10,14,18\}$. If we take the set

$$\begin{matrix} a13 & b12 & c00 \\ a15 & b23 & c11 \\ a35 & b34 & c22 \\ a02 & b45 & c33 \\ a04 & b50 & c44 \\ a24 & b01 & c55 \end{matrix} \quad = \quad T$$

and then consider $(abc)T = T_1$, $(ab)T = T_2$, we find that $|T \cap T_1| = 0$, $|T \cap T_2| = 6$ and of course $|T \cap T| = 18$.

Now let

$$\begin{matrix} a13 & b21 & c00 \\ a32 & b10 & c11 \\ a20 & b05 & c22 \\ a04 & b53 & c33 \\ a45 & b34 & c44 \\ a51 & b42 & c55 \end{matrix} \quad = \quad U.$$

Then $|T \cap U| = 14$.

Finally, if V is obtained from T by removing the blocks (or triangles) $a15$, $a35$, $a02$, $a24$, $b12$, $b23$, $b45$, $b50$ and replacing them with $a12$, $a32$, $a05$, $a45$, $b15$, $b35$, $b02$, $b24$, we see that $|V \cap T| = 10$. So

$$\begin{aligned} S_1 + S_2 + S_3 &\supseteq \{0,3,6,9\} + \{0,6,10,14,18\} \\ &= \{0,3,6,9,10,12,13,\ldots,21,23,24,27\}. \end{aligned}$$

It remains to show that $\{1,2,4,5,7,8,11,22\} \subseteq I_1(9)$.

Let D_1 be the design generated modulo 9 from the starter blocks $\{112, 136, 137\}$, and let $D_2 = (12)D_1$, $D_3 = (13)D_1$, $D_4 = (124)D_1$, $D_5 = (135)D_1$ and $D_6 = (1357)D_1$. Then $|D_4 \cap D_6| = 2$, $|D_4 \cap D_5| = 4$, $|D_2 \cap D_5| = 7$, $|D_3 \cap D_4| = 8$, and $|D_1 \cap D_5| = 11$.

Now let D^* be the design:

112	223	334	445	551	661	771	882	992
138	139	148	149	246	247	256	257	679
356	359	367	378	468	479	578	589	689.

By removing the five blocks 112, 223, 334, 445, 551 and replacing them by 115, 221, 332, 443, 554, we have $22 \in I_1(9)$. Also $|D^* \cap D_4| = 5$.

Finally, $1 \in I_1(9)$ because $|D_1 \cap (45)(23)D^*| = 1$. Hence $I_1(9) = J_1(9)$.

2 Construction of designs

2.1 v to $2v$, v odd

We shall use the idea of a BTD with a *hole*. A BTD of order w with a hole of size v can be thought of as a triple (Q, P, B) where Q is a w-set, P is a v-subset of Q and B is a collection of blocks (multi-sets of size k) of Q such that:

(i) each pair of distinct elements of Q not both in P occurs exactly λ times in the blocks of B;

(ii) each element of $Q \setminus P$ occurs twice in exactly ρ_2 blocks, and never more than twice in any block;

(iii) each pair of elements of P, distinct or not, occurs in no blocks.

From now on BTD shall mean a BTD with block size three, index two and $\rho_2 = 1$.

In this section we shall describe how to construct a simple BTD of order $2v$ from one of order v in the case that v is odd. We do this by constructing a BTD of order $2v$ with a hole of order v. This means that by changing the BTD of order v that is put in the hole, we can fine-tune the number of blocks in common to two of these BTDs of order $2v$. Moreover, we shall be able to coarse-tune the number of common blocks in two such BTDs of order $2v$ by adjusting the way in which we "sew" the BTD of order v into the hole.

So now let v be odd, and consider the multi-set of v differences

$$D = \{0, 1, 1, 2, 2, \ldots, (v-1)/2, (v-1)/2\}.$$

Each one of these differences corresponds to a 2-factor in $2K_v^+$.

For example, if $v = 9$, we have $D = \{0, 1, 1, 2, 2, 3, 3, 4, 4\}$, and the 2-factor corresponding to 0 is the set of 9 loops on the vertices, while in general the 2-factor corresponding to difference d joins the vertices at distance d. So for difference 3 (when $v = 9$) we have three cycles of length 3: (036), (147), (258), while for difference 4 we have the one cycle of length 9: (048372615).

Let the original BTD of order v (which we put in the hole) be based on the set $\{x_1, x_2, \ldots, x_v\}$. Now to each of the v differences in D we associate one of the v elements $\{x_1, x_2, \ldots, x_v\}$. For each edge ab in the 2-factor corresponding to difference d (where $a = b$ if $d = 0$, in which case the 2-factor consists of v loops) we take the block abx_i if x_i is the element associated with the difference d. Of course we also take the blocks of the simple BTD of order v based on the set $\{x_1, x_2, \ldots, x_v\}$. Since no 2-factor has a repeated edge (since v is odd here), the resulting BTD of order $2v$ has no repeated blocks and so is simple.

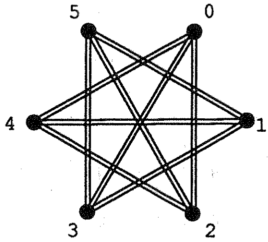
2.2 v to $2v$, v even

In this case we take the multi-set D of differences to be

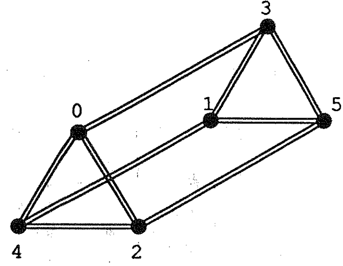
$$\{0, 1, 1, 2, 2, \ldots, (v-2)/2, (v-2)/2\} \setminus \{v/3, v/3\}.$$

This multi-set D contains $v - 3$ differences, each of which gives rise to a 2-factor of $2K_v^+$ with no repeated edges. However, the differences $\{v/3, v/3, v/2\}$ remain; we need to find three simple 2-factors corresponding to these differences. Call these three 2-factors F_1, F_2 and F_3, and their union P. Each component of P (and there are $v/6$ components) is in fact a triangular prism with all edges doubled; such a graph can be partitioned into three cycles of length 6.

For example, when $v = 6$, the graph P is as follows:



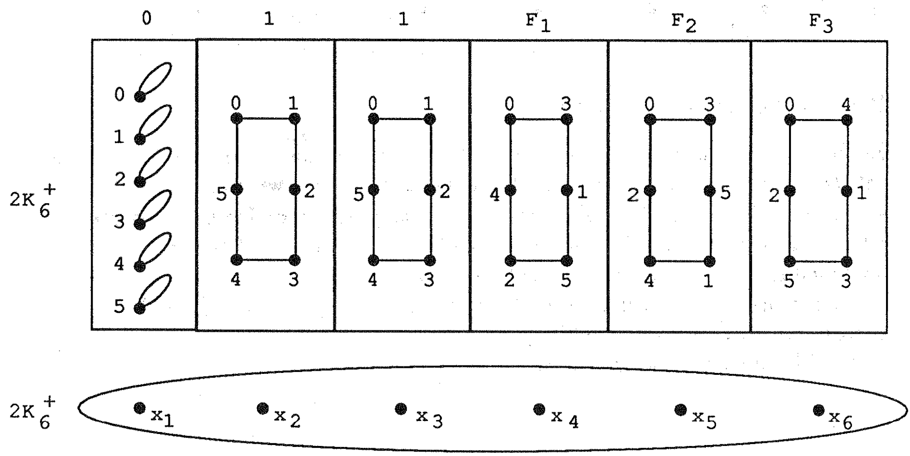
or equivalently



and this can be partitioned into the three 2-factors

$$F_1 = (031524), \quad F_2 = (035142), \quad F_3 = (041352).$$

The differences are $D = \{0, 1, 1\}$ and the differences $\{2, 2, 3\}$ are used in F_1, F_2 and F_3.



Each edge in each 2-factor is joined to the corresponding point x_i, making a triangle (that is, block) in the new design on 12 elements.

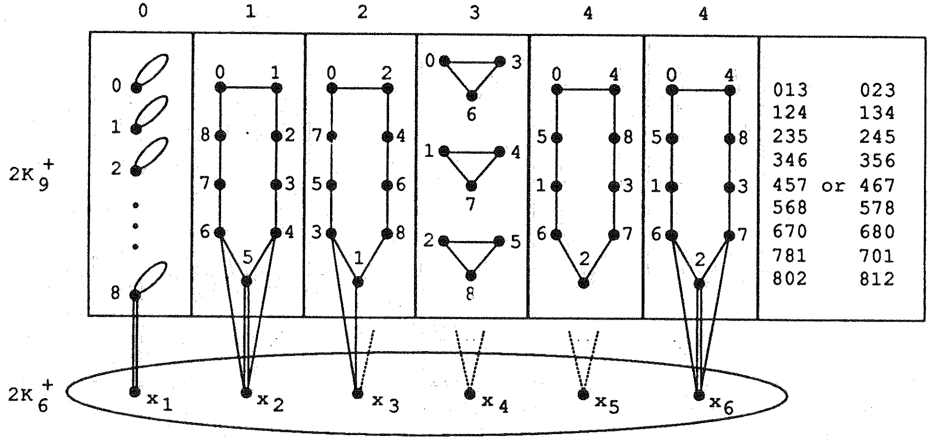
2.3 v to $2v + 3$, v even

First we take the difference triple $\{1, 2, 3\}$ which gives rise to the starter block 013 or 023 (modulo $v + 3$). The remaining differences in this case are, since $v + 3$ is odd,

$$\{0, 1, 2, 3, 4, 4, 5, 5, \ldots, (v/2) + 1, (v/2) + 1\}.$$

These correspond to v 2-factors of $2K^+_{v+3}$, and to each one we associate a vertex from the BTD of order v.

The case $v = 6$ is illustrated below; here the multi-set of differences is $\{0, 1, 2, 3, 4, 4\}$.

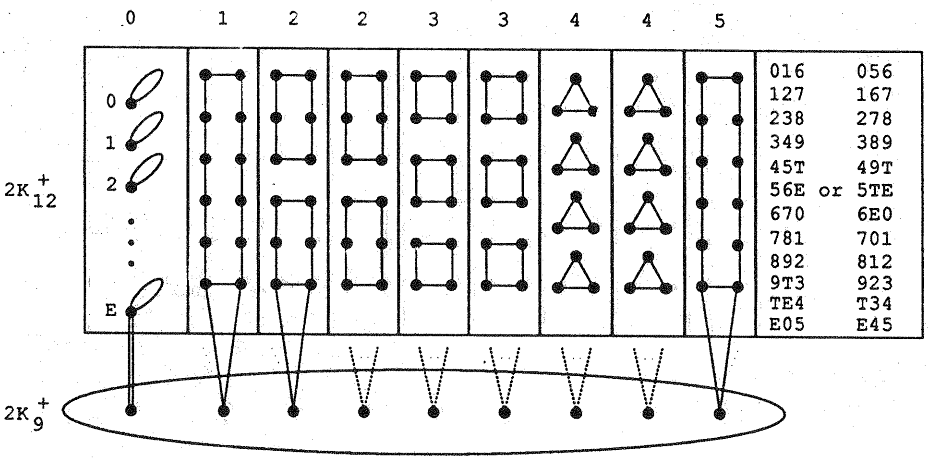


2.4 v to $2v + 3$, v odd

In this case $v + 3$ is even, so we need to avoid the difference $(v + 3)/2$ in our multi-set D in order to avoid repeated blocks arising. So we take the difference triple $\{1, (v + 1)/2, (v + 3)/2\}$. This gives $v + 3$ triples, either from the starter block 0 1 $(v + 3)/2$ or else from 0 $(v + 1)/2$ $(v + 3)/2$. (These $v + 3$ blocks can be traded or switched in the final design of order $2v + 3$.)

The remaining v differences, $\{0, 1, 2, 2, 3, 3, \ldots, (v - 1)/2, (v - 1)/2, (v + 1)/2\}$, correspond to 2-factors in $2K_{v+3}^+$.

We illustrate this with the case $v = 9$:



3 Pairs of designs

Since the order of any BTD(v^*; $\rho_2 = 1$; 3, 2) satisfies $v^* \equiv 0 \pmod 3$, we have $v^* = 6m = 2(3m)$ or else $v^* = 6m + 3 = 2(3m) + 3$, and $3m$ can be either odd or even. So the previous four constructions in Section 2 yield BTDs of all admissible orders (given a BTD of order 3).

We use these constructions to produce two BTDs based on the same set of elements (of size $2v$ or $2v + 3$). Having constructed one BTD, the number of common blocks in the second design can be adjusted by

(i) changing the allocation of the v elements in the hole to the v 2-factors;

(ii) changing the embedded BTD of order v in the hole ($I_1(v)$ having been determined by induction);

(iii) in the v to $2v+3$ cases, possibly trading the single orbit of $v+3$ triples outside the hole.

Explicitly, we have shown the following, for $v \equiv 0 \pmod 3$: If $i \in \{0, 1, \ldots, v-2, v\}$, and $j \in I_1(v)$, then $iv + j \in I_1(2v)$. And if $\ell = 0$ or 1, then $i(v+3) + j + \ell(v+3) \in I_1(2v+3)$.

For the induction to work, we must show that for all $k \in J_1(2v)$ (respectively, $J_1(2v+3)$), the appropriate i and j (respectively, i, j and ℓ), can be found so that $k = iv + j$ (respectively, $k = i(v+3) + j + \ell(v+3)$). This is easy, provided that both $v \geqslant 6$ and $I_1(v) = J_1(v)$. This is not true if $v = 3$, but we have determined $I_1(6)$ and $I_1(9)$. Also it is not quite true if $v = 6$, since $5 \notin I_1(6)$. We fill the gaps in $I_1(12)$ and $I_1(15)$ in the next section.

4 The cases $v = 12$ and $v = 15$

4.1 $v = 12$

Note that $J_1(12) = \{0, 1, 2, \ldots, 45, 48\}$. Also $5 \notin I_1(6)$ and so the construction described above does not quite yield the full set of values in $J_1(12)$. However, we can obtain all except 41 common blocks by using the above construction, if we note the following permutations giving the appropriate assignments of six elements $\{x_1, x_2, \ldots, x_6\}$ to the six 2-factors, three from the differences $\{0, 1, 1\}$ and three, F_1, F_2, F_3, from the differences $\{2, 2, 3\}$ as described in Section 2.2. (Note that F_1, F_2 and F_3 in this case have, pairwise, three edges in common.)

Let $j \in I_1(6) = J_1(6) \setminus \{5\} = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 12\}$.

							number of common blocks
0	1	1	F_1	F_2	F_3		$36+j$
0	1	1	F_2	F_1	F_3	changes 6 blocks	$30+j$
0	1	1	F_2	F_3	F_1	changes 9 blocks	$27+j$
0	1	F_3	F_1	F_2	1	changes 12 blocks	$24+j$
F_1	1	1	F_2	0	F_3	changes 15 blocks	$21+j$
1	1	F_3	F_1	F_2	0	changes 18 blocks	$18+j$
F_1	0	1	F_2	1	F_3	changes 21 blocks	$15+j$
1	0	F_3	F_1	F_2	1	changes 24 blocks	$12+j$
1	0	F_1	F_2	1	F_3	changes 27 blocks	$9+j$
F_2	F_3	0	F_1	1	1	changes 30 blocks	$6+j$
F_1	F_3	0	F_2	1	1	changes 33 blocks	$3+j$
F_1	F_2	F_3	0	1	1	changes 36 blocks	$0+j$

We see from the right-hand column in the above table that this construction yields $J_1(12) \setminus \{41\} \subset I_1(12)$. The following example deals with the case of 41 common blocks.

EXAMPLE 4.1 $41 \in I_1(12)$.

We use the vertex set $\{0, 1, 2, \ldots, 9, T, E\}$. One BTD is as follows.

001*	036*	137*	23T
112*	039	13E	238
220*	047*	146*	249
334	04T	148	24E
445	058	159	257
553	05E	15T	256
667	069	16E	26E
778	07T	179	279
886	08E	18T	28T
99T	369	46T	56T
TTE	37T	47E	57E
EE9	38E	489	589

By replacing the asterisked (*) blocks above by the set

002 110 221 037 046 136 147

we find that $41 \in I_1(12)$.

4.2 $v = 15$

We modify the construction given in Section 2.3 with $v = 6$. Instead of taking the collection of differences $\{0, 1, 2, 3, 4, 4\}$ for the 2-factors, and $\{1, 2, 3\}$ for the starter triple 013 or 023, we take the collection $D = \{0, 1, 1, 2, 3, 4\}$ and use $\{2, 3, 4\}$ for the starter triple 025 or 035 (mod 9). We retain from D the differences 0, 2, 3, which give rise, respectively, to the following three 2-factors

00,	11,	22,	33,	44,	55,	66,	77,	88;
02,	24,	46,	68,	81,	13,	35,	57,	70;
03,	36,	60,	14,	47,	71,	25,	58,	82.

Then we rearrange the edges arising from the differences 1, 1 and 4, which are:
01, 12, 23, 34, 45, 56, 67, 78, 80 (twice each)
and 04, 48, 83, 37, 72, 26, 61, 15, 50,
and take instead the following three 2-factors, each of which is a nine-cycle:

C_1 :	01,	12,	26,	67,	78,	83,	34,	45,	50;
C_2 :	12,	23,	37,	78,	80,	04,	45,	56,	61;
C_3 :	23,	34,	48,	80,	01,	15,	56,	67,	72.

(These use up precisely the same edges.) The advantage of doing this is that C_i and C_j $(i \neq j)$ have 3 common edges. So any two of the six 2-factors arising from $\{0, 2, 3, C_1, C_2, C_3\}$, say X and Y, have:
9 edges in common if $X = Y$;
3 edges in common if $X = C_i$, $Y = C_j$, $i \neq j$;
0 edges in common otherwise.

The following table, exhibiting suitable permutations of 0, 2, 3, C_1, C_2, C_3, shows that $J_1(15) \setminus \{68\} \subset I_1(15)$. (Example 4.2 below deals with the case of 68 common blocks.)
Here $j \in I_1(6) = J_1(6) \setminus \{5\} = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 12\}$.

							number of common blocks ($\ell = 0$ or 1)
0	2	3	C_1	C_2	C_3		$54 + j + 9\ell$
0	2	3	C_1	C_3	C_2	changes 12 blocks	$42 + j + 9\ell$
2	0	3	C_1	C_2	C_3	changes 18 blocks	$36 + j + 9\ell$
0	2	C_1	C_3	C_2	3	changes 24 blocks	$30 + j + 9\ell$
2	3	0	C_1	C_2	C_3	changes 27 blocks	$27 + j + 9\ell$
0	3	2	C_1	C_3	C_2	changes 30 blocks	$24 + j + 9\ell$
0	C_1	2	C_3	C_2	3	changes 33 blocks	$21 + j + 9\ell$
0	C_2	C_3	C_1	2	3	changes 36 blocks	$18 + j + 9\ell$
2	3	0	C_1	C_3	C_2	changes 39 blocks	$15 + j + 9\ell$
2	C_1	0	C_3	C_2	3	changes 42 blocks	$12 + j + 9\ell$
2	3	0	C_2	C_3	C_1	changes 45 blocks	$9 + j + 9\ell$
2	0	C_1	3	C_3	C_2	changes 48 blocks	$6 + j + 9\ell$
C_1	C_2	0	C_3	2	3	changes 51 blocks	$3 + j + 9\ell$
C_1	C_2	C_3	0	2	3	changes 54 blocks	$0 + j + 9\ell$

EXAMPLE 4.2 $68 \in I_1(15)$.

001*	036*	03E	047*	04B
05A	05C	06A	07D	089
08D	09E	0BC	112*	137*
138	146*	14E	15B	15D
169	17E	189	1AC	1AD
1BC	220*	239	23C	24A
24E	258	259	26C	26E
27B	27D	28A	2BD	334
36A	37C	38B	39D	3AD
3BE	445	46B	47A	48C
48D	49C	49D	553	56D
56E	579	57B	58C	5AE
667	69C	6BD	778	79E
7AC	886	8AE	8BE	99A
AAB	BB9	CCD	DDE	EEC

By replacing the asterisked (*) blocks above by the set

037 046 136 147 002 110 221

we find that $68 \in I_1(15)$.

5 Conclusion

We now have our required result:

MAIN THEOREM

There exist two simple balanced ternary designs of order $v \equiv 0 \pmod 3$ with block size 3, index 2 and $\rho_2 = 1$, having k common blocks, for all $k \in \{0, 1, \ldots, v^2/3 - 3, v^2/3\}$, with the one exception that there do not exist two such BTDs of order 6 having 5 common blocks.

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