THE CHARACTERIZATION OF EDGE-MAXIMAL CRITICALLY K-EDGE CONNECTED GRAPHS

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ABSTRACT:

Let G be a simple graph on n vertices having edge-connectivity $\kappa'(G) > 0$. We say G is k-critical if $\kappa'(G) = k$ and $\kappa'(G - e) < k$ for every edge e of G. We denote by $\mathscr{C}(n,k)$ the set of all k-critical graphs on n vertices. In this paper we prove that the maximum number of edges of a graph G in $\mathscr{C}(n,k)$ to be: $k(n-k)$ if $n \geq 3k$; and $\lfloor \frac{1}{8}(n+k)^2 \rfloor$, if $k + 1 \leq n < 3k$. Further, we characterise the extremal graphs in $\mathscr{C}(n,k)$.

1. INTRODUCTION

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [2]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, edge-connectivity $\kappa'(G)$, and minimum degree $\delta(G)$. K_n denotes the complete graph on n vertices, $K_{m,n}$ the complete bipartite graph with bipartitioning sets of order m and n; C_n a cycle of length n. The join of disjoint graphs G and H, denoted by $G \vee H$, is the graph obtained by joining each vertex of G to each vertex of H. However, we denote the complement of G by $\bar{G}$.

We say a graph G is **k-critical** if $\kappa'(G) = k$ and $\kappa'(G-e) < k$ for every edge e of G. Observe that : every tree is 1-critical; C_n is 2-critical; $K_1 \vee C_n$ is 3-critical; K_n is (n-1)-critical; and $K_{m,n}$ is k-critical, where $k = \min\{m,n\}$. For fixed positive integers n and k, k-critical graphs on n vertices may not be unique. We denote by $\mathscr{C}(n,k)$ the set of all k-critical graphs on n vertices. Let $\mathscr{A}(n,k)$ denote the members of $\mathscr{C}(n,k)$ that have every edge incident to at least one vertex of degree k.

Define $\mathscr{B}(n,k) = \mathscr{C}(n,k) - \mathscr{A}(n,k)$. We will establish that $\mathscr{B}(n,k) = \phi$ for $n \leq 2k + 1$ and that $\mathscr{B}(n,k) \neq \phi$ for $n \geq 2(k+1)$. An edge $e = xy$ of $G \in \mathscr{C}(n,k)$ will be called a **distinguished edge** if $d_G(x) \geq k + 1$ and $d_G(y) \geq k + 1$. Thus every graph G in $\mathscr{B}(n,k)$ contains at least one distinguished edge.

A graph $G \in \mathscr{C}(n,k)$ is called **edge-minimal (maximal)** if there is no other graph in $\mathscr{C}(n,k)$ having **less (more)** edges than G. We call G **r-semi-regular graph** if every vertex of G has degree r except one which has degree r + 1. We denote by H(n,t) a t-edge connected, t-regular (semi-regular) graph on n vertices for nt even (odd). Clearly this graph has $\lceil \frac{nt}{2} \rceil$ edges.

In [3] we proved that if $G \in \mathscr{C}(n,k)$ then $\delta(G) = k$. We also proved that $G \in \mathscr{C}(n,k)$ if and only if there are exactly k edge-disjoint paths joining any two adjacent vertices of G. So it is obvious that for $k \neq 1$, a graph $G \in \mathscr{C}(n,k)$ is edge-minimal if and only if $G = H(n,k)$. The edge-maximal members of $\mathscr{C}(n,k)$ are not as easily described. Indeed, their structure is much more complex.

In [3] we considered the problem of determining the maximum number of edges for a graph $G \in \mathcal{A}(n,k)$. We proved that this number is equal to : $k(n-k)$ for $n \geq 3k$; and $\lfloor \frac{1}{8}(n+k)^2 \rfloor$, for $k + 1 \leq n < 3k$. In this paper we will prove that this result is true for an edge-maximal graph in $\mathcal{C}(n,k)$. We also prove that there is no distinguished edge in an edge- maximal graph of $\mathcal{C}(n,k)$ for $k \neq 1$. The edge-maximal graphs in $\mathcal{C}(n,k)$ are completely characterised.

2. FUNDAMENTAL LEMMAS

We define an **edge-cut** of a graph G as a subset of E(G) of the form $(V_1, \bar{V}_1)$, where V_1 is a nonempty proper subset of V(G). A **k-edge cut** is an edge cut of k elements.

Suppose G is a k-edge connected graph having two k-edge-cut sets, say E_1 and E_2. Removing $E_1 \cup E_2$ from G yields a graph G′ having three or four components. We will show that if G′ has four components then every component is separated by a k-edge cut set from G; but if it has three components then at least two of them are separated by k-edge cut sets from G.

Lemma 2.1 : Let G be a k-edge connected graph on n vertices with k-edge cuts E_1 and E_2. If $G - (E_1 \cup E_2)$ consists of four components, G_1, G_2, G_3 and G_4, then for every i, $1 \leq i \leq 4$, $(V_i, \bar{V}_i)$ is a k-edge cut in G, where $V_i = V(G_i)$.

Proof : Without loss of generality we assume that

$$E_1 = (V_1 \cup V_2, V_3 \cup V_4)$$

and

$$E_2 = (V_1 \cup V_4, V_2 \cup V_3).$$

We let e_{ij} denote the number of edges in G between G_i and G_j (see Figure 2.1).

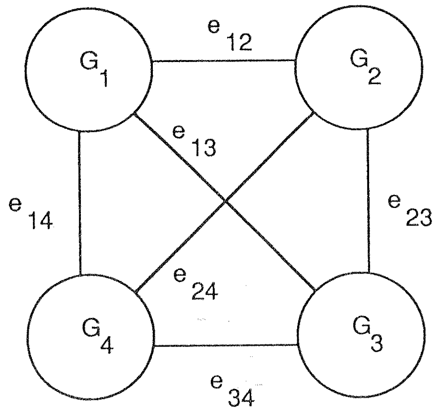

Figure 2.1

Observe that

$$k = |E_1| = e_{13} + e_{14} + e_{23} + e_{24} \quad (1)$$

and

$$k = |E_2| = e_{12} + e_{13} + e_{24} + e_{34} \quad (2)$$

Further, since $\kappa'(G) = k$ we must have

$$e_{12} + e_{13} + e_{14} \geq k , \quad (3)$$

$$e_{12} + e_{23} + e_{24} \geq k , \quad (4)$$

$$e_{13} + e_{23} + e_{34} \geq k , \quad (5)$$

and

$$e_{14} + e_{24} + e_{34} \geq k . \quad (6)$$

Now (1) and (3), (2) and (4), (1) and (5), and (2) and (6) respectively imply :

$$e_{12} \geq e_{23} + e_{24} , \quad (7)$$

$$e_{23} \geq e_{13} + e_{34} , \quad (8)$$

$$e_{34} \geq e_{14} + e_{24} , \quad (9)$$

and $$e_{14} \geq e_{12} + e_{13} . \quad (10)$$

Since $e_{ij} \geq 0$, inequalities (7) to (10) together imply that $e_{12} = e_{23} = e_{34} = e_{14}$ and hence $e_{13} = e_{24} = 0$. Now equations (1) and (2) give

$$e_{14} = e_{23} = e_{12} = e_{34} = \frac{1}{2} k.$$

This completes the proof of Lemma 2.1. □

Note that the above proof yields the following result.

Corollary 2.1 : Let G be a k-edge connected graph with k-edge cuts E_1 and E_2. If $G - (E_1 \cup E_2)$ consists of four components, then k is even and $E_1 \cap E_2 = \phi$. Further, if k is odd $G - (E_1 \cup E_2)$consists of three components. □

Our next lemma considers the case when $G - E_1 \cup E_2$ has 3 components and k is odd or even.

Lemma 2.2 : Let G be a k-edge-connected graph on n vertices with k-edge cuts E_1 and E_2 having t edges in common. If $G - (E_1 \cup E_2)$ consists of 3 components G_1, G_2 and G_3, then $t \leq \frac{1}{2}$ k with equality holding only if $(V_i, \bar{V}_i)$ is a k-edge-cut set in G for each i, $1 \leq i \leq 3$, where $V_i = V(G_i)$. Furthermore, if $t < \frac{1}{2}k$ then $(V_i, \bar{V}_i)$ is a k-edge cut set in G for exactly two of the i's.

Proof : Without loss of generality we assume that

$$E_1 = (V_1, V_2 \cup V_3) \text{ and } E_2 = (V_1 \cup V_2, V_3).$$

As in the above proof we let e_{ij} denote the number of edges in G between G_i and G_j (see Figure 2.2). Note that $t = e_{13}$.

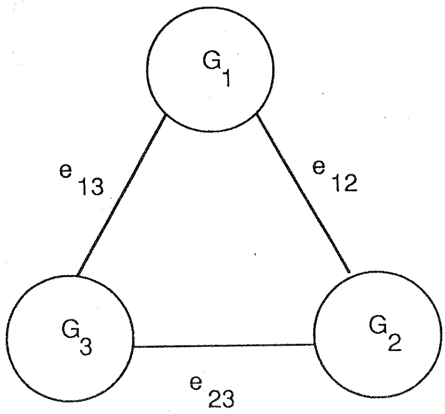

Figure 2.2

We have $e_{12} + e_{13} = k$ and $e_{13} + e_{23} = k$, hence $e_{12} = e_{23}$. Now since $\kappa'(G) = k$ we have

$$e_{12} + e_{23} \geq k$$

and thus $e_{12} = e_{23} \geq \frac{1}{2}k$. This implies that $e_{13} \leq \frac{1}{2}k$. Furthermore, if $e_{13} = \frac{1}{2}k$, then

$$e_{12} = e_{13} = e_{23} = \frac{1}{2}k.$$

Observe that if $e_{13} < \frac{1}{2}k$, then $e_{12} + e_{23} > k$. This completes the proof of the lemma. □

Our next lemma is essential for proving our main result.

Lemma 2.3 : Let G be a graph on n vertices and let k be a positive integer less than n. If every edge of G is incident to at least one vertex of degree k or less, then G has at most $\varepsilon(n,k)$ edges, where

$$\varepsilon(n,k) = \begin{cases} k(n-k) & , \text{ if } n \geq 3k \\ \left\lfloor \frac{1}{8}(n+k)^2 \right\rfloor & , \text{ if } k+1 \leq n < 3k \end{cases} \tag{11}$$

Proof : Let G be a graph on n vertices with every edge incident to at least one vertex of degree k or less. We denote by $A = \{v \in V(G) : d_G(v) \le k\}$ and $\bar{A} = V(G) - A$. Let $|A| = n_1$ and $|\bar{A}| = n_2 = n - n_1$. Since every vertex x of G in $\bar{A}$, has degree at least k+1, and $E(G[\bar{A}]) = \phi$, we have $n_1 \ge k+1$.

If $n_1 \le n-k$, then $n_2 = n-n_1 \ge k$, so the maximum number of edges of G is obtained when there is no edge e = xy of G such that $x,y \in A$. Hence

$$\varepsilon(G) \le n_1 k \le (n-k)k .$$

If $n_1 \ge n-k$, then $n_2 \le k$. Simple counting gives

$$\varepsilon(G) \le n_1(n-n_1) + \left\lfloor \frac{1}{2} n_1 \ (k-n+n_1) \right\rfloor$$

$$= \left\lfloor n_1(n-n_1) + \frac{1}{2} n_1(k - n+n_1) \right\rfloor$$

$$= \left\lfloor \hat{g}(n_1) \right\rfloor = g(n_1) .$$

For $n \ge 3k$, $\hat{g}(n_1)$ is a decreasing function of n_1, and so

$$\max_{n_1} \{\hat{g}(n_1)\} = \left\lfloor \hat{g}(n-k) \right\rfloor$$

$$= (n-k)k$$

Since $\hat{g}(n_1)$ is monotonically increasing when $n_1 \le \left\lfloor \frac{n+k}{2} \right\rfloor$ and monotonically decreasing when $n_1 \ge \left\lceil \frac{n+k}{2} \right\rceil$, we have for $n < 3k$,

$$\max_{n_1} \{ G(n_1)\} = \max \left\{ g\left(\left\lfloor \frac{n+k}{2} \right\rfloor\right), g\left(\left\lceil \frac{n+k}{2} \right\rceil\right)\right\}$$

$$= \left\lfloor \frac{1}{8} (n+k)^2 \right\rfloor .$$

Now since

$$(n+k)^2 - 8k(n-k) = (n-3k)^2 \ge 0$$

and k(n-k) is integer, we have

$$\left\lfloor \frac{1}{8} (n+k)^2 \right\rfloor \ge k(n-k) .$$

Therefore

$$\varepsilon(G) \le \begin{cases} k(n-k) & , \text{ if } n \ge 3k , \\ \lfloor \frac{1}{8}(n+k)^2 \rfloor & , \text{ if } k+1 \le n < 3k . \end{cases}$$

This completes the proof of Lemma 2.3. □

The following result was proved in [3]. It is actually a special case of Lemma 2.3.

Corollary 2.2 : If $G \in \mathcal{A}(n,k)$, then the maximum number edges of G is $\varepsilon(n,k)$, where $\varepsilon(n,k)$ is given by (11). □

Our next result considers the class $\mathcal{B}(n,k)$.

Lemma 2.4 : Let $G \in \mathcal{B}(n,k) = \mathcal{C}(n,k) - \mathcal{A}(n,k)$. Then

$$\nu(G) \ge \begin{cases} 7 & , \text{ if } k = 2 , \\ 2(k+1) & , \text{ otherwise} . \end{cases}$$

Furthermore, this bound is sharp.

Proof : By definition, the graph $G \in \mathcal{B}(n,k)$ contains a distinguished edge e_1, say. Now let E_1 be a k-edge cut of G containing e_1. Let G_1 and G_2 be the components of $G - E_1$ and suppose that $|V(G_i)| = n_i$, $i = 1,2$. Without loss of generality suppose $n_1 \le n_2$. We will show that $n_1 \ge k+1$. In fact, since e_1 is a distinguished edge of G joining G_1 and G_2, there is a vertex, x say, of G_1 in G with $d_G(x) \ge k+1$. Since G is k-critical then $\delta(G) = k$. Hence

$$\sum_{v \in G_1} d_G(v) \ge n_1 k+1 .$$

Now if $n_1 \le k$, we have

$$\sum_{v \in G_1} d_G(v) = \sum_{v \in G_1} d_{G_1}(v) + k$$

$$\le n_1(n_1-1) + k$$

$$\le k(n_1-1) + k = n_1 k ,$$

contradicting the above fact. Hence $n_1 \ge k+1$.

Now since $n_2 \ge n_1$, we must have

$$n = n_1 + n_2 \ge 2(k+1) .$$

For $k = 2$, a straight forward case analysis establishes that, the graph G formed from K_3 and C_4 by adding two edges joining a vertex of K_3 and two nonadjacent vertices of C_4, is the member of $\mathcal{B}(n,k)$ with the smallest number of vertices. This establishes the lower bound on $\nu(G)$. For $k \ne 2$ we establish the sharpness of this bound by construction.

For k odd, we construct the graph $G \in \mathcal{B}(2k+2,k)$ as follows. Take $G_1 = K_1 \vee H(k,k-2)$. Since $k(k-2)$ is odd, there is a vertex, x say, of $H(k,k-2)$ with $d_H(x) = k-1$. We form G by taking two copies G_1' and G_1'' of G_1 and adding a perfect matching between the vertices of H' and H'' with $x'x''$ an edge in this matching. Observe that G is a k-critical graph, on $2(k+1)$ vertices, and $d_G(x') = d_G(x'') = k+1$. So $x'x''$ is a distinguished edge of G. Thus $G \in \mathcal{B}(n = 2k+2,k)$.

For k even, $k \ne 2$, we construct the graph $G \in \mathcal{B}(2k+2,k)$ as follows. Take $G_1 = \bar{K}_2 \vee K_{k-1}$, and denote the vertices of $\bar{K}_2$ by x and y. Let $G_2 = K_2 \vee H(k-1,k-3)$. Since $(k-1)(k-3)$ is odd there is a vertex, z say, of $H(k-1,k-3)$ with $d_H(z) = k-2$. Form the graph G from G_1 and G_2 by joining z to x and y, and joining all other vertices of H to exactly one of x or y so that x and y have the same degree in G. Observe that G is a k-critical graph on 2k+2 vertices, with $d_G(x) = d_G(y) = \lceil \frac{k-1}{2} \rceil + k-1 \geq k+1$ and $d_G(z) = k+2$. Therefore $G \in \mathcal{B}(n = 2k+2, k)$. This completes the proof of the lemma. □

Lemma 2.5 : For positive integers n and k with k > 1 let

$$a(n,k) = \begin{cases} \frac{1}{2} n(n-1) & , \quad \text{if } 1 \leq n \leq k \\ \varepsilon(n,k) & , \quad \text{otherwise.} \end{cases}$$

where $\varepsilon(n,k)$ is given by (11). If $n_1 \geq k+1$, then

$$a(n_1,k) + a(n_2,k) + k \leq a(n_1 + n_2, k) \qquad (12)$$

unless $k+1 \leq n_1 < 3k-1$ and $n_2 = 1$ in which case

$$a(n_1,k) + a(n_2,k) + \frac{1}{2}k \leq a(n_1+n_2,k) .$$

Proof : We define the function

$$f(n_1,n_2,k) = a(n_1,k) + a(n_2,k) + k.$$

We consider two cases according to the value of n_1.

Case (a): $n_1 \geq 3k$

If $n_2 \geq 3k$, then

$$\begin{aligned} f(n_1,n_2,k) &= k(n_1-k) + k(n_2-k) + k \\ &= (n_1 + n_2 - k)k - (k^2 - k) \\ &< (n_1 + n_2 - k)k = a(n_1+ n_2,k) . \end{aligned}$$

This proves the lemma for the case $n_2 \geq 3k$.

If $k+1 \leq n_2 < 3k$, then

$$\begin{aligned} f(n_1,n_2,k) &= k(n_1- k) + \lfloor \frac{1}{8}(n_2+ k)^2 \rfloor + k \\ &\leq k(n_1 - k) + \frac{1}{8}(n_2 + k)^2 + k \end{aligned}$$

$$= a(n_1 + n_2,k) + \frac{1}{8}(n_2 + k)^2 + k - n_2k .$$

Now it is a simple algebraic exercise to show that

$$\frac{1}{8}(n_2 + k)^2 + k - n_2k < 0 .$$

Hence

$$f(n_1,n_2,k) < a(n_1 + n_2,k) \text{ for } k + 1 \le n_2 < 3k .$$

The only remaining case is $n_2 \le k$.

For $n_2 \le k$, we have

$$f(n_1,n_2,k) = k(n_1 - k) + \frac{1}{2} n_2(n_2 - 1) + k$$

$$= a(n_1 + n_2,k) + \frac{1}{2} n_2(n_2 - 1) + k - kn_2 .$$

Now for $n_2 \le k$ the function

$$h(n_2,k) = \frac{1}{2} n_2(n_2 - 1) + k - kn_2$$

is monotonically decreasing in n_2. Hence

$$h(n_2,k) \le h(1,k) = 0 .$$

Therefore

$$f(n_1,n_2,k) \le a(n_1 + n_2,k)$$

with equality possible only if $n_2 = 1$.

Case (b): $k + 1 \le n_1 < 3k$.

We may assume that $n_2 < 3k$ as otherwise we can, by interchanging n_1 and n_2, apply the above argument.

If $n_2 \ge k + 1$, then

$$f(n_1,n_2,k) = \left\lfloor \frac{1}{8}(n_1 + k)^2 \right\rfloor + \left\lfloor \frac{1}{8}(n_2 + k)^2 \right\rfloor + k$$

$$\le \left\lfloor \frac{1}{8} \{(n_1 + k)^2 + (n_2 + k)^2\} + k \right\rfloor$$

We first consider the case when $n_1 + n_2 \geq 3k$. In this case $a(n_1 + n_2, k) = k(n_1 + n_2 - k)$. Now the function

$$h(n_1,n_2,k) = \frac{1}{8}\{(n_1 + k)^2 + (n_2 + k)^2\} + k - k(n_1 + n_2 - k)$$

is for $k + 1 \leq n_2 < 3k$, monotonically decreasing in n_2 so it is maximum when n_2 is as small as possible. Hence, since $n_2 \geq k + 1$

$$\max_{n_2} \{h(n_1,n_2,k)\} = h(n_1, k + 1, k)$$

$$= \frac{1}{8}\{(n_1 + k)^2 + (2k + 1)^2\} + k - k(n_1 + 1)$$

$$= \frac{1}{8}(n_1 - 3k)^2 - (2k - 1)^2 + 2\} .$$

Since for $k + 1 \leq n_1 < 3k$, we have $(n_1 - 3k)^2 \leq (1 - 2k)^2$ we conclude that

$$\max_{n_1,n_2} \{h(n_1,n_2,k)\} = \frac{1}{8}\{(1 - 2k)^2 - (2k - 1)^2 + 2\}$$

$$\leq \frac{1}{4} .$$

Thus we have

$$\frac{1}{8}\{(n_1 + k)^2 + (n_2 + k)^2\} + k - k(n_1 + n_2 - k) \leq \frac{1}{4} .$$

Now since $a(n_1 + n_2,k) = k(n_1 + n_2 - k)$ is an integer, we have

$$\left\lfloor \frac{1}{8}\{(n_1 + k)^2 + (n_2 + k)^2\} + k \right\rfloor - k(n_1 + n_2 - k) \leq 0.$$

Hence

$$f(n_1,n_2,k) \leq a(n_1 + n_2,k) .$$

This completes the proof for the case $n_1 + n_2 \geq 3k$.

We now consider the case $n_1 + n_2 < 3k$. For this case we have

$$a(n_1 + n_2, k) = \lfloor \tfrac{1}{8}(n_1 + n_2 + k)^2 \rfloor .$$

The function

$$\begin{aligned} h(n_1,n_2,k) &= \tfrac{1}{8} \{(n_1 + k)^2 + (n_2 + k)^2\} + k - \tfrac{1}{8} (n_1 + n_2 + k)^2 \\ &= \tfrac{1}{8} \{k^2 + 8k - 2n_1n_2\} \\ &\le \tfrac{1}{8} \{k^2 + 8k - 2(k + 1)^2\} \\ &= \tfrac{1}{8} \{- k^2 + 4k - 2\} \\ &\le \tfrac{1}{8} \text{ (as } k \ge 3). \end{aligned}$$

(Note that this case does not happen for $k = 2$). Again as $f(n_1,n_2,k)$ and $a(n_1 + n_2,k)$ are integers

$$f(n_1,n_2,k) - a(n_1 + n_2,k) \le 0$$

as required.

The only remaining case is $n_2 \le k$. We have

$$\begin{aligned} f(n_1,n_2,k) &= \lfloor \tfrac{1}{8}(n_1 + k)^2 \rfloor + \tfrac{1}{2} n_2(n_2 - 1) + k \\ &\le \tfrac{1}{8}(n_1 + k)^2 + \tfrac{1}{2} n_2(n_2 - 1) + k. \end{aligned}$$

Now

$$a(n_1 + n_2,k) = \begin{cases} k(n_1 + n_2 - k), & \text{if } n_1 + n_2 \ge 3k \\ \lfloor \tfrac{1}{8}(n_1 + n_2 + k)^2 \rfloor, & \text{otherwise} \end{cases}$$

We first consider the case $n_1 + n_2 \ge 3k$. Then the function

$$h(n_1,n_2,k) = \tfrac{1}{8}(n_1 + k)^2 + \tfrac{1}{2} n_2(n_2 - 1) + k - k(n_1 + n_2 - k)$$

is, for $1 \le n_2 \le k$, monotonically decreasing in n_2. Thus, since $n_2 \ge 3k - n_1 \ge 1$,

$$\max_{n_2} \{h(n_1,n_2,k)\} = h(n_1,\ 3k - n_1,\ k)$$

$$= \frac{1}{8}\{5n_1^2 - (22k - 4)n_1 + 17k^2 - 4k\}$$

$$= \hat{h}(n_1,k)\ ,$$

and

$$\max_{n_1} \{\hat{h}(n_1 k)\} = \max\{\hat{h}(k+1,\ k),\ \hat{h}(3k-1,\ k)\}$$

$$= \hat{h}(k+1,\ k)$$

$$= \frac{1}{8}(-12k + 9) < -1\frac{7}{8}\ .$$

Hence

$$f(n_1,n_2,k) < a(n_1 + n_2,k)\ .$$

Now we consider the case $n_1 + n_2 < 3k$.

The function

$$h(n_1,n_2,k) = \frac{1}{8}(n_1 + k)^2 + \frac{1}{2}n_2(n_2 - 1) + k - \frac{1}{8}(n_1 + n_2 + k)^2$$

$$= \frac{3}{8}n_2^2 - \frac{1}{4}(n_1 + k + 2)n_2 + k$$

is, for $k + 1 \le n_1 < 3k$, monotonically decreasing in n_1. Hence

$$\max_{n_1} \{h(n_1,n_2,k)\} = h(k+1,n_2,\ k)$$

$$= \frac{3}{8}n_2^2 - \frac{1}{4}(2k+3)n_2 + k$$

$$\le 0 \quad \text{for} \quad n_2 \ge 2.$$

Thus for $n_2 \ge 2$ we have proved the lemma.

The only remaining case is $n_2 = 1$ and $k+1 \le n_1 < 3k-1$. For this case we have

$$a(n_1 + n_2, k) = \lfloor \tfrac{1}{8}(n_1 + 1 + k)^2 \rfloor$$

and

$$f(n_1, n_2, k) = \lfloor \tfrac{1}{8}(n_1 + k)^2 \rfloor + k \ .$$

Now consider

$$\begin{aligned} f^*(n_1, n_2, k) &= f(n_1, n_2, k) - \tfrac{1}{2}k \\ &= \lfloor \tfrac{1}{8}(n_1 + k)^2 \rfloor + \tfrac{1}{2}k \\ &\le \tfrac{1}{8}(n_1 + k)^2 + \tfrac{1}{2}k. \end{aligned}$$

We have

$$\begin{aligned} h(n_1, n_2, k) &= \tfrac{1}{8}(n_1 + k)^2 + \tfrac{1}{2}k - \tfrac{1}{8}(n_1 + 1 + k)^2 \\ &= \tfrac{1}{2}k - \tfrac{1}{4}(n_1 + k) + \tfrac{1}{8} \\ &\le \tfrac{1}{2}k - \tfrac{1}{4}(2k + 1) + \tfrac{1}{8} < 0 \ . \end{aligned}$$

Hence

$$f^*(n_1, n_2, k) \le a(n_1 + n_2, k) \text{ as required.}$$

This completes the proof of the lemma. □

Lemma 2.6 : For $3 \le k+1 \le n_i < 3k$, $i = 1,2$

$$\varepsilon(n_1, k) + \varepsilon(n_2, k) + 2k \le \varepsilon(n_1 + n_2 + 1, k) \qquad (13)$$

where $\varepsilon(n,k)$ is given by (11).

Proof : Let

$$h(n_1, n_2, k) = \varepsilon(n_1, k) + \varepsilon(n_2, k) + 2k - \varepsilon(n_1 + n_2 + 1, k).$$

We distinguish two cases according to the value of $n_1 + n_2 + 1$.

Case (a) : $n_1 + n_2 + 1 \ge 3k$.

We have

$$h(n_1,n_2,k) = \lfloor \frac{1}{8}(n_1 + k)^2 \rfloor + \lfloor \frac{1}{8}(n_2 + k)^2 \rfloor + 2k - (n_1 + n_2 + 1 - k)k$$

$$\le \frac{1}{8}\{(n_1 + k)^2 + (n_2 + k)^2\} + 2k - k(n_1 + n_2 + 1 - k)$$

$$= \hat{h}(n_1,n_2,k).$$

The function $\hat{h}(n_1,n_2,k)$ is, for $k+1 \le n_1 < 3k$, monotonically decreasing in n_1. So it is maximum when n_1 is as small as possible. Hence, since $n_1 \ge k+1$

$$\max_{n_1} \{\hat{h}(n_1,n_2,k)\} = \hat{h}(k+1, n_2,k)$$

$$= \frac{1}{8}\{(2k + 1)^2 + (n_2 + k)^2\} + 2k - k(n_2 + 2)$$

$$= \frac{1}{8}\{(n_2 - 3k)^2 - (2k - 1)^2 + 2\} .$$

Now for $k + 1 \le n_2 < 3k$, we can show that

$$(n_2 - 3k)^2 \le (1 - 2k)^2 .$$

Hence we have

$$\max_{n_1,n_2} \{\hat{h}(n_1,n_2,k) = \frac{1}{8}\{(1-2k)^2 - (2k-1)^2 + 2\} = \frac{2}{8} .$$

Since $h(n_1,n_2,k)$ is an integer, we conclude that $h(n_1,n_2,k) \le 0$. This proves the lemma for this case.

Case (b) : $n_1 + n_2 + 1 < 3k.$

Note that this case occurs only when $k > 3$. Here we have

$$h(n_1,n_2,k) = \lfloor \frac{1}{8}(n_1 + k)^2 \rfloor + \lfloor \frac{1}{8}(n_2 + k)^2 \rfloor + 2k - \lfloor \frac{1}{8}(n_1 + n_2 + 1 + k)^2 \rfloor .$$

Consider the function

$$\hat{h}(n_1,n_2,k) = \frac{1}{8}\{(n_1 + k)^2 + (n_2 + k)^2 - (n_1 + n_2 + 1 + k)^2\} + 2k$$

$$= \frac{1}{8}\{ - 2 (n_1 + n_2) - 2n_1n_2 + k^2 + 14k - 1\} .$$

Since $\hat{h}(n_1,n_2,k)$ is a monotonically decreasing function in n_2, and $n_2 \geq k+1$, we have

$$\max_{n_2} \{\hat{h}(n_1,n_2,k)\} = \hat{h}(n_1,k+1,k)$$

$$= \frac{1}{8}\{k^2 + 12k - 3 - 2(2+k)n_1\}$$

$$\leq \frac{1}{8}\{ - k^2 + 6k - 7\} \text{ (since } n_1 < 3k)$$

$$\leq \frac{1}{8} \quad \text{(as } (k \geq 4).$$

Consequently $h(n_1,n_2,k) \leq 0$, as required. This completes the proof of Lemma 2.6. □

3. MAIN RESULTS

The following terminology is useful in the proof of our main theorem. Let $G \in \mathscr{C}(n,k)$. For a subset U of V(G) we write $\varepsilon(U,\bar{U})$ for the number of edges between U and $\bar{U}$.

When $\varepsilon(U,\bar{U}) = k$ we call U a **segment** of G (note that $\bar{U}$ is also a segment). A consequence of Lemmas 2.1 and 2.2 is the following result.

Lemma 3.1: If A and B are two segments such that $B \cap \bar{A} \neq \phi$, then either $A \subseteq B$ or $B \cap \bar{A}$ is a segment. □

For $X \subseteq V(G)$ we denote the subgraph of G induced by the vertices in X by $G[X]$.

Theorem 3.1 : Let G be an edge maximal graph in $\mathscr{E}(n,k)$. Then

$$\varepsilon(G) = \begin{cases} k(n-k), & \text{if } n \geq 3k \quad , \\ \left\lfloor \frac{1}{8}(n+k)^2 \right\rfloor, & \text{if } k+1 \leq n < 3k \ . \end{cases}$$

Proof : If G has no distinguished edge, then the result coincides with Corollary 2.2 and we have nothing to prove. So suppose G contains at least one distinguished edge.

Choose a k-edge cut E_1 containing a distinguished edge, e_1 say, such that $G - E_1$ contains a component, G_1 say, having no distinguished edge. That such an E_1 and e_1 exists follows from Lemmas 2.1 and 2.2. Let G_2 be the other component of $G - E_1$ and let $n_i = |V(G_i)|$, $i = 1,2$. As in the proof of Lemma 2.4, since G_1 and G_2 each contain a vertex of degree k+1 we have $n_1 \geq k+1$ and $n_2 \geq k+1$. Since G_1 has no distinguished edge of G, we have by Lemma 2.3, $\varepsilon(G_1) \leq \varepsilon(n_1,k)$. Now if $\varepsilon(G_2) \leq \varepsilon(n_2,k)$, then

$$\begin{aligned} \varepsilon(G) &= \varepsilon(G_1) + \varepsilon(G_2) + k \\ &\leq \varepsilon(n_1,k) + \varepsilon(n_2,k) + k \\ &\leq \varepsilon(n_1 + n_2,k) \qquad\qquad \text{(Lemma 2.5)} \\ &= \varepsilon(n,k) \ , \end{aligned}$$

as required. Thus we may assume that $\varepsilon(G_2) > \varepsilon(n_2,k)$. Then, by Lemma 2.3, G_2 contains at least one distinguished edge.

Our strategy is to partition the vertices of G_2 into sets and then apply Lemmas 2.5 and 2.6 to count the edges of G. Observe that if e′ is a distinguished edge of G in G_2 and E′ is a k-edge cut of G containing e′, then by Lemma 3.1 there exists a segment S′ such that $S' \cap V(G_1) = \phi$. Further, Lemmas 2.1 and 2.2 ensure that we can choose e′ and E′ such that G[S′] contains no distinguished edge of G. Note that $|S'| \geq k+1$ and by Lemma 2.3 $\varepsilon(G[S']) \leq \varepsilon(|S'|, k)$.

Let T denote the largest segment of G such that $T \cap V(G_1) = \phi$ and $\varepsilon(G[T]) \leq \varepsilon(|T|,k)$. That such a T exists follows from the existence of S′. Since $\varepsilon(G_2) > \varepsilon(n_2,k)$, $T \neq V(G_2)$. Let $T' = V(G_2)\setminus T$. If $\varepsilon(G[T'] \leq a(|T'|,k)$, then

$$\begin{aligned}\varepsilon(G) &\leq \varepsilon(G_1) + \varepsilon(G[T]) + \varepsilon(G[T']) + 2k \\ &\leq a(n_1,k) + a(|T|,k) + a(|T'|,k) + 2k \\ &\leq a(n_1 + |T| + |T'|,k) \quad \text{(Lemmas 2.5 \& 2.6)} \\ &= \varepsilon(n,k),\end{aligned}$$

as required. Hence we may assume that $\varepsilon(G[T'],k) > a(|T'|,k)$. The definition of a(n,k) then implies that $|T'| \geq k+1$. We now partition the set T′.

We can find a distinguished edge e_1 of G in G[T′] and a segment S_1 of G such that e_1 is in the cut $(S_1,\bar{S}_1)$, $S_1 \cap V(G_1) = \phi$ (so $S_1 \cap \bar{T} = S_1 \cap T'$) and G $[S_1 \cap \bar{T}]$ does not contain a distinguished edge. That this can be done follows from lemmas 2.1, 2.2 and 3.1. Continuing in this way we partition T′ into sets $B_1,B_2,\ldots,B_t$ such that

$$B_1 = S_1 \cap \bar{T}\ , \quad B_i = (S_i \setminus \bigcup_{j=1}^{i-1} S_j) \cap \bar{T}$$

and each subgraph $G[B_i]$ contains no distinguished edge of G_1.

Hence, by Lemma 2.3, $\varepsilon(G[B_i] \le \varepsilon(|B_i|,k)$ when $|B_i| \ge k+1$. Obviously $\varepsilon(G[B_i]) \le \frac{1}{2}|B_i|(|B_i| -1)$. Hence $\varepsilon(G[B_i]) \le a(|B_i|,k)$ for every i. If $|B_i| \ge k+1$ for some i, with no loss of generality say i=1, then since $|V(G_1)| + |T| + |B_1| \ge 3(k+1)$, we have by Lemma 2.5

$$\varepsilon(G) \le \varepsilon(G_1) + \varepsilon(G[T]) + \varepsilon(G[B_1] + 2k + \sum_{i=2}^{t} \varepsilon(G[B_i]) + (t-1)k$$

$$\le a(|V(G_1)| + |T| + |B_1|,k) + \sum_{i=2}^{t} a(|B_i|,k) + (t-1)k$$

$$\le a(|V(G_1)| + |T| + |B_1| + \dots + |B_t|,k)$$

$$= \varepsilon(n,k),$$

as required. Thus assume that $|B_i| \le k$ for every i. Then B_i is not a segment. Now by Lemma 3.1 $T \subseteq S_1$ and $S_1 = T \cup B_1$.

Suppose that $|B_i| \ge 2$ for some i. With no loss of generality let $|B_1| \ge 2$. Then

$$\varepsilon(G[T \cup B_1]) \le \varepsilon(G[T],k) + \varepsilon\ (G[B_1],k) + k$$

$$\le a(|T|,k) + a(|B_1|,k) + k$$

$$\le a(|T \cup B_1|,k) \qquad \text{(Lemma 2.5)}$$

contradicting the choice of T as $T \cup B_1$ is a segment of G. Thus $|B_i| = 1$ for each i and further no B_i is a segment. Hence, since $|T'| \ge k+1$, $t \ge 2$. By Lemma 3.1, $S_1 \cup B_2$ is a segment of G. Now since $|B_1 \cup B_2| = 2$ we have

$$\varepsilon(G[S_1 \cup B_2]) = \varepsilon(G[T \cup B_1 \cup B_2])$$

$$\leq \varepsilon(G[T]) + \varepsilon(G[B_1 \cup B_2]) + k$$

$$\leq a(|T|,k) + a(|B_1 \cup B_2|,k) + k$$

$$\leq a(|T| + |B_1 \cup B_2|,k) , \qquad \text{(Lemma 2.5)}$$

$$= a(|T \cup B_1 \cup B_2|,k) = a(| S_1 \cup B_2|,k),$$

contradicting the choice of T. This completes the proof of the theorem. □

In the following result we will prove that an edge maximal graph in $\mathcal{C}(n,k\neq 1)$ can not have a distinguished edge. We will make use of the following remarks in the proof of our result.

Remark 3.1 : It can be shown that equality in (12) holds only if one of the following conditions is satisfied :

(i)	$n_1 \geq 3k-1$	and	$n_2 = 1$; for every k
(ii)	$n_1 = n_2 = k+1$	and	$2 \leq k \leq 4$
(iii)	$n_1 = k+1$	and	$n_2 = 2$; for every k
(iv)	$n_1 = 4$	and	$n_2 = k = 3$
(v)	$n_1 = k+2$	and	$n_2 = 2$; k is odd.

Remark 3.2 : Equality in (13) holds only if $n_1 = n_2 = k+1$ and $2 \leq k \leq 5$.

Theorem 3.2 : For $k \neq 1$, there is no edge-maximal graph in $\mathcal{C}(n,k)$ having a distinguished edge.

Proof : Let $G \in \mathscr{C}(n,k)$, $k \neq 1$, be a graph containing a distinguished edge. To prove the theorem it is sufficient to show that G has less than $\varepsilon(n,k)$ edges, where $\varepsilon(n,k)$ is given by (11).

As in the proof of Theorem 3.1, we select a k-edge cut E_1 containing a distinguished edge e_1 such that $G - E_1$ contains a component, G_1 say, having no distinguished edge. Let G_2 be the other component of $G - E_1$ and let $n_i = |V(G_i)|$, $i = 1,2$. Since E_1 contains a distinguished edge, then $n_1 \geq k+1$, and of course $n_2 \geq k+1$.

Since G_1 has no distinguished edge of G Lemma 2.3 implies $\varepsilon(G_1) \leq \varepsilon(n_1,k)$. Consequently, if $\varepsilon(G_2) \leq \varepsilon(n_2,k)$ then

$$\begin{aligned}
\varepsilon(G) &= \varepsilon(G_1) + \varepsilon(G_2) + k \\
&\leq \varepsilon(n_1,k) + \varepsilon(n_2,k) + k, \\
&\leq \varepsilon(n_1 + n_2,k) = \varepsilon(n,k). \qquad \text{(Lemma 2.5)}
\end{aligned}$$

Since $k \geq 2$ and $n_i \geq k+1$ for $i=1,2$, it follows from Remark 3.1 that the above holds with equality only when $n_1 = n_2 = k+1$ and $2 \leq k \leq 4$. Since $\varepsilon(k+1,k) = \lfloor \frac{1}{8}(2k+1)^2 \rfloor = \frac{1}{2}k(k+1)$, we conclude that $G_1 = G_2 = K_{k+1}$. Now since there are k edges between G_1 and G_2 we can assume, without loss of generality, that G_1 contains two vertices, v_1 and v_2 say, joined to vertices in G_2. But then, since $\kappa'(G_1) = k$, there are at least k+1 edge-disjoint (v_1,v_2) - paths in G, contradicting the fact that G is k-critical. Hence $\varepsilon(G) < \varepsilon(n,k)$ when $\varepsilon(G_2) \leq \varepsilon(n_2,k)$.

Assume then that $\varepsilon(G_2) > \varepsilon(n_2,k)$. Then G_2 contains at least one distinguished edge. As in the proof of Theorem 3.1, let T be the largest segment of G such that $T \cap V(G_1) = \phi$ and $\varepsilon(G[T]) \le \varepsilon(|T|,k)$. Let $T' = V(G_2)\backslash T$. Now $T' \ne \phi$ since $\varepsilon(G_2) > \varepsilon(n_2,k)$. As in the proof of Theorem 3.1, if $\varepsilon(G[T'] \le a(|T'|,k)$, then

$$\varepsilon(G) \le \varepsilon(G_1) + \varepsilon(G[T]) + \varepsilon(G[T']) + 2k$$

$$\le a(n_1,k) + a(|T|,k) + a(|T'|,k) + 2k.$$

If $|T'| \ge 2$, then by Lemma 2.5

$$\varepsilon(G) \le a(n,k) + a(|T| + |T'|,k) + k .$$

Since $n_1 \ge k+1$ and $|T| + |T'| \ge k+3$ we have by Lemma 2.5 and Remark 3.1

$$a(n_1,k) + a(|T| + |T'|,k) + k$$

$$< a(n_1 + |T| + |T'|,k),$$

and hence

$$\varepsilon(G) < a(n_1 + |T| + |T'|,k) = \varepsilon(n,k),$$

as required.

Consider now $|T'| = 1$. Then $a(|T'|,k) = 0$, and hence

$$\varepsilon(G) \le a(n_1,k) + a(|T|,k) + 2k.$$

If $n_1 \ge 3k$ or $|T| \ge 3k$, then by Lemma 2.5 and Remark 3.1,

$$a(n_1,k) + a(|T|,k) + 2k < a\ (n_1 + |T| + 1,k).$$

Hence

$$\varepsilon(G) < a(n_1 + |T| + 1,k) = \varepsilon(n,k),$$

as required. So suppose that $n_1 < 3k$ and $|T| < 3k$. Then, by Lemma 2.6

$$a(n_1,k) + a(|T|,k) + 2k \le a(n_1 + |T| + 1,k)$$

with equality holding only if $n_1 = |T| = k+1$ and $2 \le k \le 5$ (by Remark 3.2). As before, since $a(k+1,k) = \frac{1}{2}k(k+1)$, $G_1 = G[T] = K_{k+1}$. Let v be the single vertex T'. Since $d_G(v) \ge k+1$ (≥ 3), v is joined to at least two vertices, say x and y in the same segment G_1 or T of G. But then, there are at least k+1 edge-disjoint (x,y) - paths in G, contradicting the criticallity of G. Consequently

$$\begin{aligned}\varepsilon(G) &< a(n_1 + |T| +1,k)\\ &= a(n,k) = \varepsilon(n,k),\end{aligned}$$

as required. This completes the proof of the theorem for the case $\varepsilon(G[T']) \le a(|T'|,k)$.

Now suppose that $\varepsilon(G[T']) > a(|T'|,k)$. Then $|T'| \ge k+1$. As in the proof of Theorem 3.1, we partition T' into sets $B_1,B_2,\ldots,B_t$ such that each subgraph $G[B_i]$ contains no distinguished edge of G and hence

$$\varepsilon(G[B_i] \le a(|B_i|,k).$$

If $|B_i| \ge k+1$ for some i, say i=1, then

$$\begin{aligned}\varepsilon(G) &\le \varepsilon(G_1) + \varepsilon(G[T]) + \varepsilon(G[B_1]) + 2k + \sum_{i=2}^{t} \varepsilon(G[B_i]) + (t-1)k\\ &\le a(n_1,k) + a(|T|,k) + a(|B_1|,k) + 2k + \sum_{i=2}^{t} a(|B_i|,k) + (t-1)k\\ &\le a(n_1,k) + a(|T|+ |B_1|,k) + k + \sum_{i=2}^{t} a(|B_i|,k) + (t-1)k\end{aligned}$$

(Lemma 2.5).

Since $n_1 \geq k+1$ and $|T| + |B_1| \geq 2(k+1)$, then by Lemma 2.5 and Remark 3.1 we have

$$a(n_1,k) + a(|T| + |B_1|,k) + k < a(n_1 + |T| + |B_1|,k),$$

and hence since $n_1 + |T| + |B_1| \geq 3(k+1)$,

$$\varepsilon(G) < a(n_1,+ |T| + |B_1|,k) + \sum_{i=2}^{t} a(|B_i|,k) + k(t-1)$$

$$\leq a(n,k) = \varepsilon(n,k), \qquad \text{(Lemma 2.5)}$$

as required.

Finally, when $|B_i| \leq k$ for every i, $1 \leq i \leq t$, the desired contradiction is obtained by applying the arguement use for the corresponding case in the proof of Theorem 3.1. This completes the proof of the Theorem. □

Before proceeding to the characterisation, we need to describe the following graphs. Let $H'(n,r)$ be a r-regular (r-semi-regular) graph on n vertices for even (odd) n.

Let n and k be two integers with $2 \leq k+1 \leq n < 3k$. We define

$$n_1 = \left\lfloor \frac{n+k+1}{2} \right\rfloor$$

and, for $n > k+1$, we construct the graph G' such that

$$G' = \bar{K}_{n-n_1} \vee H'(n_1,\ k + n_1 - n).$$

If $n_1(k + n_1 - n)$ is even, then we define $G_1^* = G'$; otherwise we define

$G_1^* = G' - \{e\}$, where $e = xy \in E(G')$ with $x \in V(\bar{K}_{n-n_1})$ and $y \in V(H'(n_1, k + n_1 - n))$ such that $d_{H'}(y) = k-n+n_1+1$. For $n=k+1$, we take $G_1^* = H(n,k)$.

Now we define $\hat{n}_1 = \lceil \frac{n+k+1}{2} \rceil$, if $n = 3k - 2(2i-1)$, where $i = 1,2,\dots,\frac{1}{4}(2k+1)$.

We construct the graph G_2^* as follows :

$$G_2^* = H(\hat{n}_1,k) \text{ , if } n = \hat{n}_1, \text{ and } G_2^* = \bar{K}_{n-\hat{n}_1} \vee H'(\hat{n}_1, k+\hat{n}_1-n) \text{ , if } n > \hat{n}_1.$$

Now for

$$n = 2i - k + 1; \qquad k + 1 \le i \le 2k - 1$$

or

$$n = 2i - k; \qquad k + 2 \le i \le i \le 2k - 1; \quad i \text{ is odd,}$$

let

$$\underline{n}_1 = \lfloor \frac{n+k-1}{2} \rfloor$$

Construct graph G such that

$$G = \bar{K}_{n-\underline{n}_1} \vee H'(\underline{n}_1,\ k + \underline{n}_1 - n)$$

We define $G_3^* = G$ if $\underline{n}_1(k - n + \underline{n}_1)$ is even and $G_3^* = G - \{e\}$ if $\underline{n}_1(k - n + \underline{n}_1)$ is odd where $e = xy \in E(G)$, such that $x \in V(\bar{K}_{n-\underline{n}_1})$ and $y \in V(H')$ with $d_G(y) = k + 1$.

Observe that for $k + 1 \le n < 3k$ the graphs G_1^*, G_2^* and G_3^*, are in the class $\mathcal{A}(n,k)$ and have $\lfloor \frac{1}{8}(n+k)^2 \rfloor$ edges and hence are edge-maximal in the class $\mathcal{A}(n,k)$.

Our next result provides us a characterisation of the edge-maximal graph in $\mathcal{C}(n,k)$.

Theorem 3.3 : G is an edge-maximal graph in $\mathscr{C}(n,k)$, $k \neq 1$ if and only if

(i) $G = K_{k,n-k}$, if $n \geq 3k$

(ii) $G = K_{k,n-k}$ or $G = G_1^*$, if $n = 3k-1$

(iii) $G = K_{k,n-k}$ or $G = G_1^*$ or $G = G_2^*$, if $n = 3k-2$

(iv) $G = G_1^*$ or $G = G_2^*$, if $n = 3k - 2\ (2i-1)$, $i = 2,3,\ldots,\frac{1}{4}(2k+1)$

(v) $G = G_1^*$ or $G = G_3^*$, if $n = 2i - k + 1$, $k + 1 \leq i \leq 2k - 1$ or $n = 2i - k$, i is odd, $k + 2 \leq i \leq 2k - 1$

(vi) $G = G_1^*$, otherwise.

Proof : Let G be an edge-maximal graph in $\mathscr{C}(n,k)$; $k \neq 1$. Then by Theorem 3.2, G has no distinguished edge and so $G \in \mathscr{A}(n,k)$. In other words, every edge e of G is incident to at least one vertex of degree k. Since $G \in \mathscr{C}(n,k)$, then $\delta(G) = k$. Now let $G \in \mathscr{A}(n,k)$. We denote by $X = \{v \in V(G) : d_G(v) = k\}$ and $\bar{X} = V(G)\backslash X$. Let $|X| = n_1$ and $n_2 = |\bar{X}| = n - n_1$. Now since every vertex v of G in $\bar{X}$ has $d_G(v) > k$, and $G \in \mathscr{A}(n,k)$, $n_1 \geq k+1$.

If $n_1 \leq n-k$ then it is obvious that

$$\varepsilon(G) \leq n_1 k = g(n_1) .$$

Clearly

$$\max_{n_1} \{g(n_1)\} = g(n-k) = k(n-k) .$$

Now $n_1 = n-k$, if and only if $n_2 = n - n_1 = k$ and so $K_{k,n-k}$ is the only edge-maximal graph for the case $n_1 = n-k$.

If $n_1 \geq n-k$, then $n_2 = n - n_1 \leq k$ simple counting gives :

$$\varepsilon(G) \leq \begin{cases} n_1\,(n-n_1) + \lceil \frac{1}{2}n_1(k - n \quad n_1)\rceil & \text{, if } n_1(k - n+n_1) \text{ even} \\ n_1(n - n_1) - 1 + \lceil \frac{1}{2}n_1(k - n + n_1)\rceil, & \text{if } n_1(k - n+n_1) \text{ odd.} \end{cases}$$

That is

$$\varepsilon(G) \leq g(n_1) = n_1(n - n_1) + \lfloor \frac{1}{2}n_1(k - n+n_1) \rfloor .$$

Since for $n \geq 3k$, $g(n_1)$ is decreasing in $n_1 \geq n - k$ we have

$$\begin{aligned} \max_{n_1} \{g(n_1)\} &= g(n - k) \\ &= (n - k)\ k + \lfloor \frac{1}{2}(n-k)(0) \rfloor \\ &= (n-k)k. \end{aligned}$$

Again for this case $K_{k,n-k}$ achieves this bound. Now for $n < 3k$, it is easy to verify that $g(n_1)$ is increasing in n_1 for $n_1 \leq \lfloor \frac{n+k-1}{2} \rfloor$ and decreasing in n_1 for $n_1 \geq \lfloor \frac{n+k+1}{2} \rfloor$. Thus the maximum of $g(n_1)$ is attained when

$$n_1 = \lfloor \frac{n+k+1}{2} \rfloor .$$

Some simple algebra gives

$$g(\ \lfloor \frac{n+k+1}{2} \rfloor\ ) = \lfloor \frac{1}{8}(n + k)^2 \rfloor .$$

Observe that the graph G achieving this bound is G_1^* .

Now when $n_1 = \lfloor \frac{n+k-1}{2} \rfloor$, we have

$$g(\lfloor \frac{n+k-1}{2} \rfloor) = \lfloor \frac{1}{8}(n+k)^2 \rfloor$$

only if $n = 2i - k + 1$ for $k + 1 \le i \le 2k - 1$ or $n = 2i - k$ for $k + 2 \le i \le 2k - 1$ and i is odd. So for this case the graph G_3^* achieves this bound. It can be shown that

$$g(\lceil \frac{n+k+1}{2} \rceil) = \lfloor \frac{1}{8} (n+k)^2 \rfloor$$

only if $n = 3k - 4i+2$, $i = 1,2,\dots,\frac{1}{4}(2k+1)$. Clearly for this case G_2^* achieves this bound. Now since

$$(n-3k)^2 = (n+k)^2 - 8k(n-k) \ge 0$$

and $k(n-k)$ is integer, we have

$$\lfloor \frac{1}{8}(n+k)^2 \rfloor \ge k(n-k)$$

for $n < 3k$, with equality holding only if $n = 3k-1$ or $n = 3k-2$. So for this case if G is an edge maximal then G could be $K_{k,n-k}$ as well.

This completes the proof of the Theorem 3.3. □

Note that for $k = 1$, every tree on n vertices belongs to $\mathcal{C}(n,k)$, having $n-1$ edges. Thus it is obvious that every tree on n vertices is an edge-maximal graph in $\mathcal{C}(n,1)$.

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