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Generalized Pell numbers, graph representations and independent sets

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Abstract

In this paper we generalize the Pell numbers and the Pell-Lucas numbers and next we give their graph representations. We shall show that the generalized Pell numbers and the generalized Pell-Lucas numbers are equal to the total number of independent sets in special graphs.

1 Introduction

Consider simple, undirected graphs with vertex set $V(G)$ and edge set $E(G)$. Let $\mathbf{P}_n$ and $\mathbf{C}_n$ denote a path and a cycle on n vertices, respectively. A subset $S \subseteq V(G)$ is an independent set of G if no two vertices of S are adjacent. In addition, a subset containing only one vertex and the empty set also are independent. The number of independent sets in G is denoted $NI(G)$. Prodinger and Tichy [4] initiated the study of the number $NI(G)$ of independent sets in a graph. They called this parameter *the Fibonacci number of a graph* and they proved, that: $NI(\mathbf{P}_n) = F_{n+1}$ and $NI(\mathbf{C}_n) = L_n$, where the Fibonacci numbers F_n are defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ and the Lucas numbers L_n are $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$. Independently, Merrifield and Simmons [1] introduced the number of independent sets to the chemical literature showing connections between this index of a molecular graph and some physicochemical properties. In chemistry $NI(G)$ is called the Merrifield-Simmons index. The Fibonacci numbers in graphs have been investigated in many papers, for example in [3], [4].

The *Pell numbers* are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. The *Pell-Lucas numbers* are defined by the recurrence relation $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ and also by $Q_n = 2P_{n-1} + 2P_n$. The Pell numbers in graphs with respect to the number of k-independent sets, $k \geq 2$, were studied in [5].

In this paper we generalize the Pell numbers and the Pell-Lucas numbers. Firstly we apply this generalization to counting special families of subsets of the set of n integers. Next we give the graph interpretation of the generalized Pell numbers and the Pell-Lucas numbers. Note that some generalizations of the Pell numbers and Pell-Lucas numbers are known; see for example [2], [5].

2 Main results

Let $X = \{1, 2, \ldots, n\}$, $n \geq 3$, be the set of n integers. Let $\mathcal{X}$ be a family of subsets of X such that $\mathcal{X} = \mathcal{X}' \cup \mathcal{X}''$, and $\mathcal{X}' = \mathcal{X}_1 \cup \{\{i\}; i = 2, \ldots, n-1\} \cup \mathcal{X}_n$, where $\mathcal{X}_1$ contains t_1 subsets $\{1\}$ and $\mathcal{X}_n$ contains t_{n-1} subsets $\{n\}$. The family $\mathcal{X}'' = \bigcup_{j=2}^{n-2} \mathcal{X}_{j,j+1}$, where $\mathcal{X}_{j,j+1}$ contains t_j subsets $\{j, j+1\}$, for $j = 2, \ldots, n-2$.

Let $\mathcal{Y} \subset \mathcal{X}$ such that for each $Y, Y' \in \mathcal{Y}$ there exist $i \in Y$ and $j \in Y'$ such that $|i - j| > 1$.

Let $P(n, t_1, \ldots, t_{n-1})$ denote the number of subfamilies $\mathcal{Y}$.

Theorem 1 *Let $n \geq 3$, $t_i \geq 1$, $i = 1, \ldots, n-1$ be integers. Then*

$$\begin{aligned} P(3, t_1, t_2) &= (t_1 + 1)(t_2 + 1) + 1, \\ P(4, t_1, t_2, t_3) &= (t_1 + 1)[(t_2 + 1)(t_3 + 1) + 1] + t_3 + 1, \end{aligned}$$

and for $n \geq 5$ we have

$$P(n, t_1, \ldots, t_{n-1}) = (t_{n-1} + 1)P(n-1, t_1, \ldots, t_{n-2}) + P(n-2, t_1, \ldots, t_{n-3}).$$

Proof: The statement is easily verified for $n = 3, 4$. Hence we may assume $n \geq 5$. Let $\mathcal{Y} \subset \mathcal{X}$ and we recall that for each $Y, Y' \in \mathcal{Y}$ there are $a \in Y$ and $b \in Y'$ such that $|a - b| > 1$. Let $\mathcal{X}_n$ be a subfamily of $\mathcal{X}$ which contains t_{n-1} subsets $\{n\}$. Clearly $|\mathcal{X}_n \cap \mathcal{Y}| \leq 1$. Let $P_{\{n\}}(n, t_1, \ldots, t_{n-1})$ (respectively: $P_{-\{n\}}(n, t_1, \ldots, t_{n-1})$) be the number of subfamilies $\mathcal{Y}$ such that $\mathcal{X}_n \cap \mathcal{Y} \neq \emptyset$ (respectively: $\mathcal{X}_n \cap \mathcal{Y} = \emptyset$). Then $P(n, t_1, \ldots, t_{n-1}) = P_{\{n\}}(n, t_1, \ldots, t_{n-1}) + P_{-\{n\}}(n, t_1, \ldots, t_{n-1})$. Two cases occur now:

(1). $|\mathcal{X}_n \cap \mathcal{Y}| = 1$.
Then the definition of $\mathcal{Y}$ gives that $\{n-1\} \notin \mathcal{Y}$. Let $\mathcal{X}^* \subset \mathcal{X}$ such that $\mathcal{X}^* = \mathcal{X}_1^* \cup \mathcal{X}''$, where $\mathcal{X}_1^* = \mathcal{X}' \setminus (\mathcal{X}_n \cup \{n-1\})$. In the other words $\mathcal{X}_1^* = \mathcal{X}_1 \cup \{\{r\}; r = 2, \ldots, n-2\}$. Clearly $\mathcal{Y} = \mathcal{Y}^* \cup \{n\}$, where $\mathcal{Y}^* \subset \mathcal{X}^*$, and for every $Y, Y' \in \mathcal{Y}^*$ there are $a \in Y$ and $b \in Y'$ such that $|a - b| > 1$. Since in $\mathcal{X}^*$ the integer $n-1$ belongs only to $\mathcal{X}_{n-2,n-1} \subset \mathcal{X}''$, hence the number of considered subfamilies in $\mathcal{X}^*$ is the same as in $\mathcal{X}_1^* \cup \mathcal{X}_{n-1} \cup (\mathcal{X}'' \setminus \mathcal{X}_{n-2,n-1})$, where $\mathcal{X}_{n-1}$ contains t_{n-2} subsets $\{n-1\}$. This implies that there are $P(n-1, t_1, \ldots, t_{n-2})$ subfamilies $\mathcal{Y}^*$. Since the subset $\{n\} \in \mathcal{X}_n$ we can choose on t_{n-1} ways hence $P_{\{n\}}(n, t_1, \ldots, t_{n-1}) = t_{n-1}P(n-1, t_1, \ldots, t_{n-2})$.

(2). $|\mathcal{X}_n \cap \mathcal{Y}| = 0$.
We distinguish the following possibilities:

(2.1). $\{n-1\} \notin \mathcal{Y}$.
Then $\mathcal{Y} \subseteq \mathcal{X} \setminus (\mathcal{X}_n \cup \{n-1\}) = \mathcal{X}_1 \cup \{\{i\}; i = 2, \dots, n-2\} \cup \mathcal{X}''$. Since in $\mathcal{X}_1 \cup \{\{i\}; i = 2, \dots, n-2\} \cup \mathcal{X}''$ the integer $n-1$ belongs only to $\mathcal{X}_{n-2,n-1} \subset \mathcal{X}''$, so we can find the number of subfamilies $\mathcal{Y}$ of $(\mathcal{X}' \setminus \mathcal{X}_n) \cup \mathcal{X}_{n-1} \cup (\mathcal{X}'' \setminus \mathcal{X}_{n-2,n-1})$. Then there are exactly $P(n-1, t_1, \dots, t_{n-2})$ subfamilies $\mathcal{Y}$ in this subcase.

(2.2). $\{n-1\} \in \mathcal{Y}$.
Evidently $\{n-2\} \notin \mathcal{Y}$ and $\mathcal{X}_{n-2,n-1} \cap \mathcal{Y} = \emptyset$. Proving analogously as in above cases we obtain $P(n-2, t_1, \dots, t_{n-3})$ subfamilies $\mathcal{Y}$ in this case.

Consequently we have that $P_{\{-n\}}(n, t_1, \dots, t_{n-1}) = P(n-1, t_1, \dots, t_{n-2}) + P(n-2, t_1, \dots, t_{n-3})$.

Finally, from the above cases $P(n, t_1, \dots, t_{n-1}) = (t_{n-1}+1)P(n-1, t_1, \dots, t_{n-2}) + P(n-2, t_1, \dots, t_{n-3})$.

Thus the theorem is proved. □

The numbers $P(n, t_1, \dots, t_{n-1})$ we will call the *generalized Pell numbers.*

For $t_i = t$, $t \geq 1$ and $i = 1, \dots, n-1$ the numbers $P(n, t, \dots, t)$ create the $(t+1)$-Fibonacci sequence of the form $a_n = (t+1)a_{n-1} + a_{n-2}$ with initial conditions $a_3 = (t+1)^2 + 1$, $a_4 = (t+1)((t+1)^2 + 2)$. In particular if $t = 1$ for $i = 1, \dots, n-1$, then $P(n, 1, \dots, 1)$ is the Pell number P_n with initial conditions $P_3 = 5$ and $P_4 = 12$.

The family $\mathcal{X}$ can be regarded as $V(G_n)$ of the graph G_n of order $n - 2 + \sum_{i=1}^{n-1} t_i$ in Figure 1, where vertices from $V(G_n)$ are labeled by integers belonging to corresponding subsets from $\mathcal{X}$.

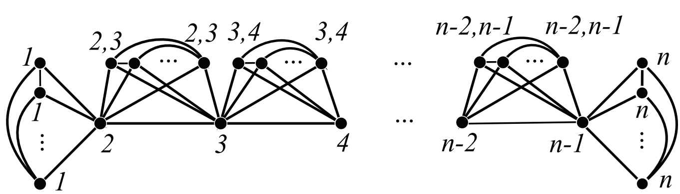

Fig. 1. Graph G_n

Thus in graph terminology, the number $P(n, t_1, \dots, t_{n-1})$, for $n \geq 3$, is equal to the total number of subsets $S \subset V(G_n)$ such that for each two vertices $x_i, x_j \in S$, $x_i x_j \notin E(G_n)$. In other words for $n \geq 3$, $P(n, t_1, \dots, t_{n-1}) = NI(G_n)$.

Let $X = \{1, 2, \dots, n\}$, $n \geq 3$, and let $\mathcal{F}$ be a family of subsets of X such that $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, where $\mathcal{F}' = \{\{i\}; i = 1, \dots, n\}$ and $\mathcal{F}'' = \mathcal{F}_{n,1} \cup \bigcup_{j=1}^{n-1} \mathcal{F}_{j,j+1}$, where $\mathcal{F}_{j,j+1}$ contains t_j subsets $\{j, j+1\}$ and $\mathcal{F}_{n,1}$ contains t_n subsets $\{n, 1\}$.

Let $\mathcal{I} \subset \mathcal{F}$ such that for each $Y, Y' \in \mathcal{I}$ there exist $i \in Y$ and $j \in Y'$ such that $2 \leq |i-j| \leq n-2$.

Let $Q(n, t_1, \dots, t_n)$ denote the number of subfamilies $\mathcal{I}$.

Theorem 2 *Let $n \geq 3$, $t_i \geq 1$, $i = 1, \ldots, n$. Then*

$$Q(3, t_1, t_2, t_3) = (t_1 + 1)[(t_2 + 1)(t_3 + 1) + 1] + t_2 + t_3 + 2,$$

and for $n \geq 4$ we have

$$Q(n, t_1, \ldots, t_n) = P(n + 1, t_1, \ldots, t_n) + P(n - 1, t_1, \ldots, t_{n-2}).$$

PROOF: The statement is easily verified for $n = 3$. Let $n \geq 4$. Assume that $\mathcal{I}$ is a subfamily of $\mathcal{F}$ such that for each $Y, Y' \in \mathcal{I}$ there are $a \in Y$ and $b \in Y'$ and $2 \leq |a - b| \leq n - 2$. We distinguish two possibilities:

1. $\{n\} \in \mathcal{I}$
Then the definition of $\mathcal{I}$ gives that $\mathcal{F}_{n,1} \notin \mathcal{I}$, $\mathcal{F}_{n-1,n} \notin \mathcal{I}$ and $\{1\}, \{n-1\} \notin \mathcal{I}$. Let $\mathcal{F}^* \subset \mathcal{F}$ such that $\mathcal{F}^* = \mathcal{F}_1^{'} \cup \mathcal{F}_2^{''}$ where $\mathcal{F}_1^{'} = \mathcal{F}' \setminus \{\{n\}, \{1\}, \{n-1\}\}$ and $\mathcal{F}_1^{''} = \mathcal{F}' \setminus (\mathcal{F}_{n,1} \cup \mathcal{F}_{n-1,n})$. Clearly $\mathcal{I} = \mathcal{I}^* \cup \{n\}$, where $\mathcal{I}^* \subset \mathcal{F}^*$ and for every $Y, Y' \in \mathcal{I}^*$ there are integers $a \in Y$, $b \in Y'$ such that $2 \leq |a - b| \leq n - 2$. Since in $\mathcal{F}^*$ the integer $n-1$ belongs only to subsets from $\mathcal{F}_{n-2,n-1}$ and the integer 1 belongs only to subsets from $\mathcal{F}_{1,2}$, it follows that the number of considered subfamilies in $\mathcal{F}^*$ is the same as in $\mathcal{X}_1 \cup \{\{i\}; i = 2, \ldots, n-2\} \cup \mathcal{X}_{n-1} \cup \bigcup_{j=2}^{n-3} \mathcal{X}_{j,j+1}$, where $\mathcal{X}_1$, $\mathcal{X}_{n-1}$, $\mathcal{X}_{j,j+1}$ are defined earlier. This implies that $Q(n, t_1, \ldots, t_n) = P(n-1, t_1, \ldots, t_{n-2})$ in this case.

2. $\{n\} \notin \mathcal{I}$
Then $\mathcal{I} \subset \mathcal{F} \setminus \{n\}$, and proving analogously as in Case 1 we have that there are exactly $P(n+1, t_1, \ldots, t_n)$ subfamilies $\mathcal{I}$ in this case.

Finally, from the above cases we have $Q(n, t_1, \ldots, t_n) = P(n + 1, t_1, \ldots, t_n) + P(n - 1, t_1, \ldots, t_{n-2})$. □

The number $Q(n, t_1, \ldots, t_n)$ we will call the *generalized Pell-Lucas number*. If $n \geq 3$ and $t_i = 1$ for $i = 1, \ldots, n$ then $Q(n, 1, \ldots, 1)$ is the Pell-Lucas number Q_n.

The family $\mathcal{F}$ can be regarded as $V(R_n)$ of the graph R_n of order $n + \sum_{i=1}^{n} t_i$ in Figure 2, where vertices from $V(R_n)$ are labeled by integers belonging to corresponding subsets from $\mathcal{F}$.

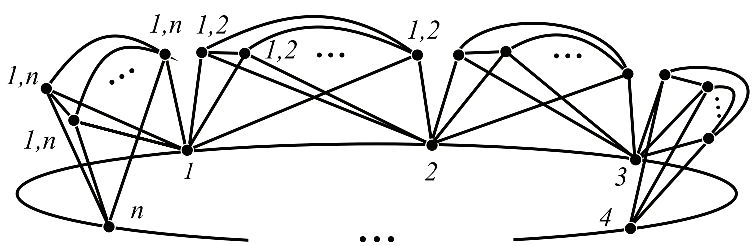

Fig. 2. Graph R_n

In graph terminology, the number $Q(n, t_1, \ldots, t_n)$, for $n \geq 3$, is equal to the total number of subsets $S \subset V(R_n)$ such that for each $x_i, x_j \in S$, $x_i x_j \notin E(R_n)$. Hence for $n \geq 3$, $Q(n, t_1, \ldots, t_n) = NI(R_n)$.

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