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A note on equimatchable graphs

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Abstract

Let $G = (V, E)$ be a graph. A set M of edges is called a matching in G if each vertex in G belongs to at most one edge from M, and M is a maximal matching if any edgeset M', such that $M \subset M'$, is not a matching in G. If all maximal matchings in G have the same cardinality then G is an equimatchable graph. In this paper we characterize the equimatchable graphs of girth at least five. As a consequence we also determine those graphs of girth five or more in which every minimal set of edges dominating edges is minimum.

1 Introduction

For notation and graph theory terminology we in general follow [8]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*. The *girth* of a graph G, denoted $g(G)$, is the length of the shortest cycle or circuit of G.

A set M of edges is called a *matching* in G if each vertex in G belongs to at most one edge in M, and M is a *maximal matching* if any edgeset M', such that M' contains M as a proper subset, $M \subset M'$, is not a matching in G. If all maximal matchings in G have the same cardinality then G is an *equimatchable graph*. For a matching M we define $A(M)$ to be all the edges in M or incident with an edge from M.

Although the equimatchable graphs were characterized in [1] they are not explicitly described. Rather, as stated by the authors in [1] the article shows by using their characterization that membership in this class can be polynomially determined. For further results see [3], [4] as well as [5]. In this note, we give an explicit, easy to recognize, description of such graphs in the case that the girth is 5 or more. Part of the motivation is also the connection to two similar problems. Graphs in which every minimal set of dominating vertices is of one order have been called well-dominated [7]. Noting that the complement of a point cover (set of vertices covering or dominating edges) is an independent set, we observe that the graphs in which every minimal set of vertices dominating edges is of fixed cardinality are called well-covered graphs [2]. In both cases, these graphs have been characterized in the situation that the girth is five or more [6, 7]. Now consider graphs in which every minimal set of edges dominating edges is of one cardinality (call these graphs Class A). Noting that a maximal matching is also a minimal edge set dominating edges, it follows that the equimatchable graphs contain the class A ones as a sub-collection. In particular, if a graph is in the class A collection then all independent as well as non-independent minimal dominating sets are of one cardinality whereas a graph that is equimatchable only requires the independent minimal dominating sets to be equal. As it turns out, for girth at least 5, the two collections are identical while at girth 4 the class A ones are a proper subcollection of the equimatchable ones.

2 Results on equimatchable graphs

We begin by stating some observations on matchings and equimatchable graphs. In addition a number of lemmas are established that will be useful in proving the main result in this note.

Observation 1 *If M is a matching in a graph G, then M can be extended to a maximal matching M'. That is, $M \subseteq M'$.*

Observation 2 *A graph G is equimatchable if and only if each component of G is equimatchable.*

Observation 3 *If M is a matching in an equimatchable graph G, then $G - A(M)$ is an equimatchable graph.*

Lemma 1 *Let G be an equimatchable graph. If M_1 and M_2 are matchings in G and $A(M_1) \subseteq A(M_2)$, then $|M_1| \leq |M_2|$.*

Proof. Let M_2' be a maximal matching such that $M_2 \subseteq M_2'$. Since $A(M_1) \subseteq A(M_2)$ the set $M_1 \cup (M_2' \backslash M_2)$ is a matching in G. Thus there is a maximal matching M_1' such that $M_1 \cup (M_2' \backslash M_2) \subseteq M_1'$. If G is equimatchable then $|M_2'| = |M_1'| \geq |M_1| + |M_2' \backslash M_2| = |M_1| - |M_2| + |M_2'|$ and therefore $|M_1| \leq |M_2|$. □

Lemma 2 *If $G \cong C_k$ then G is equimatchable if and only if $k \in \{3, 4, 5, 7\}$.*

Proof. Let $G \cong C_k : v_1, v_2, \ldots, v_k, v_1$. If $k \leq 7$ the statement can easily be checked. Assume that $k \geq 8$ and let $M_1 := \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$ and $M_2 := \{v_1v_2, v_4v_5, v_7v_8\}$. Then $A(M_1) = A(M_2)$ and Lemma 1 implies that $|M_1| = |M_2|$ if G is equimatchable. Since $|M_1| \neq |M_2|$ this implies that G is not equimatchable. □

Lemma 3 *Let $G \ncong K_2$ be a connected equimatchable graph. Then G does not contain a path $v_1, v_2, \ldots, v_{2k}$ such that v_1 and v_{2k} are stems.*

Proof. Assume that G contains a path $v_1, v_2, \ldots, v_{2k}$ such that v_1 and v_{2k} are stems and let l_1 be a leaf adjacent to v_1 and let l_2 be a leaf adjacent to v_{2k}. If $M_1 := \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ and $M_2 := \{l_1v_1, v_2v_3, \ldots, v_{2k}l_2\}$ then M_1 and M_2 are matchings in G and $A(M_1) = A(M_2)$. From Lemma 1 it follows that $|M_1| = |M_2|$ but by construction $|M_2| = |M_1| + 1$. Since this is a contradiction the statement is verified. □

Lemma 4 *Let G be an equimatchable graph with $g(G) \geq 8$. Then G does not contain a path v_1, v_2, v_3, v_4 such that neither v_1 nor v_4 are stems.*

Proof. Assume otherwise and let $P : v_1, v_2, v_3, v_4$ be a path such that neither v_1 nor v_4 are stems. For each edge e incident to v_1 or v_4 not on P let e' be an edge incident with e where e' does not include a vertex of P. Let M be the set containing all of these edges. Since $g(G) \geq 8$, then M is a matching and $A(M) \cap \{v_1v_2, v_2v_3, v_3v_4\} = \emptyset$. Thus $M_1 := M \cup \{v_1v_2, v_3v_4\}$ and $M_2 := M \cup \{v_2v_3\}$ are matchings in G. By construction $A(M_1) = A(M_2)$ and thus Lemma 1 implies that $|M_1| = |M_2|$ but this contradicts the fact that $|M_1| = 1 + |M_2|$. □

Lemma 5 *Let G be an equimatchable graph with $g(G) \geq 5$. If G contains a path v_1, v_2, v_3 such that v_1 is a stem then v_3 is also a stem.*

Proof. Assume otherwise and let $P: v_1, v_2, v_3$ be a path such that v_1 is a stem and v_3 is not a stem. Let l be a leaf adjacent to v_1. For each edge e incident to v_3 not on P let e' be an edge incident with e but not incident with v_3. Let M be the set containing all of these edges. Since $g(G) \geq 5$, then M is a matching and $A(M) \cap \{lv_1, v_1v_2, v_2v_3\} = \emptyset$. Thus $M_1 := M \cup \{lv_1, v_2v_3\}$ and $M_2 := M \cup \{v_1v_2\}$ are matchings in G. By construction $A(M_1) = A(M_2)$ and thus Lemma 1 implies that $|M_1| = |M_2|$ but this contradicts the fact that $|M_1| = 1 + |M_2|$. □

Let $\mathcal{F}$ be the family of graphs containing K_2 and all connected bipartite graphs G with bipartite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem.

Lemma 6 *Each graph from $\mathcal{F}$ is equimatchable.*

Proof. Consider a graph G from $\mathcal{F}$. If M is a maximal matching in G then for each edge $e = uv$ between a stem u and a leaf v the matching M must contain an edge from $A(e)$. If M' is a set of edges containing exactly one edge from $A(e)$ for each such edge e then $A(M') = E(G)$. Thus it follows that $|M|$ is the number of stems in G or $G \cong K_2$ and thus G is equimatchable. □

Corollary 1 *Let G be a connected equimatchable graph with girth $g(G) \geq 5$. If G has a stem, then $G \in \mathcal{F}$.*

Proof. The statement is trivially true if $G \cong K_2$. If s is a stem in G and $s, v_1, v_2, \ldots, v_{2k}$, $k \geq 1$, is a path in G then Lemma 5 implies that $v_2, v_4, \ldots, v_{2k}$ are stems and by Lemma 3 $v_1, v_3, \ldots, v_{2k-1}$ are not stems. Thus it follows that $G \in \mathcal{F}$. □

Corollary 2 *Let G be a connected equimatchable graph with girth $g(G) \geq 8$. Then $G \in \mathcal{F}$.*

Proof. By Lemma 4 the graph G must contain a stem, and thus Corollary 1 implies that $G \in \mathcal{F}$. □

3 Main Result

By using the results from Section 2 we obtain a simple characterization of the equimatchable graphs with girth at least five.

Theorem 1 *Let G be a connected equimatchable graph with girth $g(G) \geq 5$. Then $G \in \mathcal{F} \cup \{C_5, C_7\}$.*

Proof. Assume that there is a connected equimatchable graph with girth $g(G) \geq 5$ not contained in $\mathcal{F} \cup \{C_5, C_7\}$. Let G be such a graph of minimum size. It follows from Corollary 2 that $g(G) \leq 7$ and by Corollary 1 it can be assumed that G does not have a stem. Assume that $g(G) = 7$ and let C be a 7-cycle in G. Since $G \ncong C_7$ there must be a vertex $u \in V(C)$ such that u is adjacent to a vertex $v \notin V(C)$. For each vertex $z \in N[v] \backslash u$ let e_z be an edge incident with z and not incident with v and let M be all of these edges. Since $g(G) = 7$ the set M is a matching and the vertex u is a stem in $G - A(M)$. Observe that C is still a cycle in the equimatchable graph $G - A(M)$. Since $|E(G - A(M))| < |E(G)|$ a contradiction is obtained and we may assume that $g(G) < 7$.

Now assume that $g(G) = 6$ and let C be a 6-cycle in G. Since C_6 is not equimatchable there must be a vertex $u \in V(C)$ such that u is adjacent to a vertex $v \notin V(C)$. Let z be a vertex from $V(C)$ adjacent to u. For each vertex from $x \in N(\{u, z\}) \backslash V(C)$ let e_x be an edge incident with x and not incident with u or z. If M denotes all of these edges then M is a matching in G. Thus since the girth is 6 it can be observed that $G - A(M)$ has a component that contains C and two adjacent vertices u and z where $deg(u) = deg(z) = 2$. Since this component is not contained in $\mathcal{F} \cup \{C_5, C_7\}$ but is equimatchable a contradiction is reached, and therefore we may assume that $g(G) = 5$.

Let C be a 5-cycle in G. Since $G \ncong C_5$ there must be a vertex $u \in V(C)$ such that u is adjacent to a vertex $v \notin V(C)$. By considering an edge e not incident with a vertex from $V(C)$ it follows that $G - A(e)$ has a component containing C and if the component is not isomorphic to C then we have a contradiction with the choice of G. It follows that G must be the graph obtained from a $C_5 : v_1, v_2, \ldots, v_5, v_1$ by adding two vertices x, y and the edges v_1x, xy and yv_3. Since this graph is not equimatchable a contradiction is obtained. □

Theorem 1 along with Lemmas 2 and 6 complete the characterization.

We conclude by observing that it is easy to verify that the equimatchable graphs of girth 5 or more are also in the class A collection but considering the equimatchable graph $K_{3,2}$ we see that once the girth is 4 the class A ones are a proper subcollection.

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