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Maximum embedding of an $H(v-w,3,1)$ into a $TS(v,\lambda)$

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Abstract

Let H be a subgraph of a graph G, and let $V \subseteq X$. We say that an H-design $(V,\mathcal{C})$ of order u and index μ is embedded into a G-design $(X,\mathcal{B})$ of order v and index λ, $\mu \leq \lambda$, if there is an injective mapping $f : \mathcal{C} \to \mathcal{B}$ such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$. The mapping f is called the embedding of $(V,\mathcal{C})$ into $(X,\mathcal{B})$. For every pair of positive integers v,λ, we determine the minimum value of w such that there exists a triple system $TS(v,\lambda)$ which embeds a handcuffed path design $H(v-w,3,1)$.

1 Introduction and definitions

Let G be a finite simple graph. A *G-design of order v* and *index* λ is a pair $(V,\mathcal{C})$ where V is the vertex set of K_v (the complete graph on v vertices) and $\mathcal{C}$ is a collection of isomorphic copies of the graph G, called *blocks*, which partition the edges of λK_v (the complete multigraph on v vertices).

A *triple system* $TS(v,\lambda)$ is a K_3-design of order v and index λ. Table 1 shows the spectrum for the existence of a $TS(v,\lambda)$ [7].

$\lambda \pmod 6$	$v \geq 3$
0	$\forall$
1,5	$\equiv 1,3 \pmod 6$
2,4	$\equiv 0,1 \pmod 3$
3	$\equiv 1 \pmod 2$

Table 1: $TS(v,\lambda)$

A *path design* $P(u,s,\lambda)$ is a P_s-design of index λ, where P_s is the simple path with $s-1$ edges, $[a_1,a_2,\dots,a_s] = a_1,a_2,a_2,a_3,\dots,a_{s-1},a_s$. A *handcuffed path design*

$H(u, s, \lambda)$ is a $P(u, s, \lambda)$ balanced, i.e. such that each vertex belongs to exactly the same number of blocks. Hung and Mendelsohn [5] proved that a $H(u, 3, 1)$ exists if and only if $u \equiv 1 \pmod 4$, $u \geq 5$.

Let H be a subgraph of a graph G, and let $V \subseteq X$. We say that an H-design $(V, \mathcal{C})$ of order $v - w$ and index μ is embedded into a G-design $(X, \mathcal{B})$ of order v and index λ, $\mu \leq \lambda$, if there is an injective mapping $f : \mathcal{C} \to \mathcal{B}$ such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$. The mapping f is called the embedding of $(V, \mathcal{C})$ into $(X, \mathcal{B})$. When w attains the minimum possible value we say that f is a maximum embedding.

When $\lambda = \mu = 1$, the embedding of an H-design into a G-design has been studied for many pairs of graphs H and G with H a subgraph of G; see, for example, [2, 3, 4, 6, 8, 9, 11, 13, 14]. Recently Milici [10] proved the existence of a minimum embedding of a non-balanced path design $P(v, 3, 1)$ into a $TS(v + w, \lambda)$ for every $v \equiv 0, 1 \pmod 4$ and every $\lambda \geq 2$. In this paper we wish to consider the reverse problem for balanced path design, i.e. the maximum embedding of an $H(v - w, 3, 1)$ into a $TS(v, \lambda)$. In particular, we will prove the following

Main Theorem: *For all $v \geq 5, \lambda \geq 2$ for which a $TS(v, \lambda)$ exists (see Table 1), the minimum value of w for which there exists a $TS(v, \lambda)$ which embeds a $H(v - w, 3, 1)$ is $w \equiv v - 1 \pmod 4$ and $w \in \{0, 1, 2, 3\}$, with two exceptions: if $\lambda = 2$ and $v = 12$ then $w = 7$, and there is no $TS(7, 2)$ which embeds $H(w, 3, 1)$.*

It is easy to see that the sufficiency for $\lambda = 2, 3, 4, 5, 6$ implies the sufficiency for every λ. For example, the maximum embedding of an $H(v - 2, 3, 1)$ into a $TS(v, 1 + 6k)$, with $v \equiv 3, 7 \pmod{12}$ and $k \geq 1$, is obtained by pasting the blocks of a $TS(v, 5)$ which embeds an $H(v - 2, 3, 1)$ to the blocks of an $TS(v, 2)$ and, for $k \geq 2$, to the blocks of a $TS(v, 6(k - 1))$.

2 Preliminaries

We recall some useful definitions. For terms not defined in this paper or results not explicitly cited the reader is referred to [1]. A 3-*GDD* is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set, $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$ is a partition of V into subsets, the elements of $\mathcal{G}$ are called *groups*, and $\mathcal{B}$ is a collection of isomorphic copies of K_3, called *blocks*, which partition the edges of $K_{g_1, g_2, \ldots, g_n}$, on the vertex set V. If for $i = 1, 2, \ldots, t$, there are k_i groups of size n_i, we say that the 3-GDD is of type $n_1^{k_1} n_2^{k_2} \ldots n_t^{k_t}$. Let $\mathcal{B}$ be the block set of a design or a *GDD*. A *parallel class* or *resolution class* is a collection of blocks which partitions the vertex set of the design or the GDD. A GDD is *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes.

Lemma 2.1. *[1] There exist 3-GDDs of type 6^t, $4^1 6^t$, $8^1 6^t$, $10^1 12^t$, $16^1 12^t$ for $t \geq 3$ and a resolvable 3-GDD of type 4^3 and of type 6^t, $t \geq 4$.*

Lemma 2.2. *[10] There exists a decomposition $(V, \mathcal{C})$ of $\lambda K_{2,2,2}$, $\lambda \geq 2$, into triples which embeds a balanced decomposition $(V, \mathcal{D})$ of $K_{2,2,2}$ into paths with two edges.*

Lemma 2.3. *There exists a decomposition* $(V,\mathcal{C})$ *of* $\lambda K_{2,2,2}$, $\lambda \geq 2$, *into one 2-factor and triples which embeds a balanced decomposition* $(V,\mathcal{D})$ *of* $K_{2,2,2}$ *into paths with two edges.*

Proof Let $V=\{a,b\}\cup\{1,2\}\cup\{x,y\}$. Put:
$\mathcal{D}=\{[1,a,x],[2,b,y],\ [x,1,b],[y,2,a],\ [2,x,b],[1,y,a]\}$,
$\mathcal{M}=\{\{\alpha,\beta,\gamma\}:[\alpha,\beta,\gamma]\in\mathcal{D}\}$,
$F=\{\{1,y\},\{y,b\},\{b,1\},\{2,x\},\{x,a\},\{a,2\}\}$ and
$\mathcal{N}=\{\{a,1,x\},\{a,2,y\},\{b,1,y\},\{b,2,x\}\}$.
Let $\mathcal{T}$ be the set of triples containing a copy of $\mathcal{M}$ and $\lambda-2$ copies of $\mathcal{N}$. Then it is easy to see that F is a 2-factor and $(V,\mathcal{T})$ embeds $(V,\mathcal{D})$. □

Lemma 2.4. *Let* $v\equiv 0 \pmod 3$ *and let* G *be the graph* $(\mathbb{Z}_v, P=P_1\cup P_2)$, *where* $P_1=\{(i,i+1): i\in\mathbb{Z}_v\}$ *and* $P_2=\{(i,i+2): i\in\mathbb{Z}_v\}$. *Then there exists a decomposition of* G *into a set of triples* $\mathcal{T}$ *and one 2-factor* F.

Proof Let $\mathcal{T}=\{(i,i+1,i+2): i\in\mathbb{Z}_v, i\equiv 0 \pmod 3\}$. Let $F=\{\{i-1,i\},\{i-1,i+1\},\{i-2,i\}: i\in\mathbb{Z}_v, i\equiv 0 \pmod 3\}$. It easy to check that $\mathcal{T}$ and F form the required decomposition. □

Theorem 2.1. *[6] For every* $TS(v,2)$ $(V,\mathcal{B})$, $v\equiv 1 \pmod 6$, *there exists a* $TS(v+1,6)$ *which embeds* $(V,\mathcal{B})$.

3 $\lambda=2$

Lemma 3.1. *If there is a* $TS(v,2)$ *which embeds an* $H(v-w,3,1)$, *then*

$$w\leq\frac{2v+1-\sqrt{8v+1}}{6}\quad \textit{or}\quad w\geq\frac{2v+1+\sqrt{8v+1}}{6}.$$

Proof Suppose we embed an $H(v-w,3,1)$ into a $TS(v,2)$. By counting the edges from the $v-w$ vertices to the remaining w vertices, we must have

$$\frac{1}{2}2(v-w)w\leq\frac{v(v-1)}{3}-\frac{(v-w)(v-w-1)}{4}$$

which is equivalent to

$$9w^2-3(2v+1)w+v(v-1)\geq 0$$

and so

$$w\leq\frac{2v+1-\sqrt{8v+1}}{6}\vee w\geq\frac{2v+1+\sqrt{8v+1}}{6}.$$

□

Corollary 3.1. *There is no* $TS(7,2)$ *which embeds an* $H(v,3,1)$.

Proof From the previous theorem it follows that there is no $H(5,3,1)$ embedded into a $TS(7,2)$ and the result follows, since an $H(v,3,1)$ exists only for $v\equiv 1 \pmod 4$, $v\geq 5$. □

Theorem 3.1. *For $v = 12$ the minimum value of w such that there exists a $TS(v, 2)$ which embeds an $H(v - w, 3, 1)$ is $w = 7$.*

Proof From the previous theorem, it follows that there is no $H(9, 3, 1)$ embedded into a $TS(12, 2)$. Case 20 in [12] shows a $TS(12, 2)$ which embeds an $H(5, 3, 1)$. □

Theorem 3.2. *[10] For $v \equiv 1, 9 \pmod{12}$, there exists a $TS(v, 2)$ which embeds an $H(v, 3, 1)$.*

Theorem 3.3. *[10] For $v \equiv 6 \pmod{12}$, there exists a $TS(v, 2)$ which embeds an $H(v - 1, 3, 1)$.*

Theorem 3.4. *For $v \equiv 3 \pmod{12}$, $v \geq 15$, there exists a $TS(v, 2)$ $(V, \mathcal{B})$ which embeds an $H(v - 2, 3, 1)$.*

Proof For $v = 15, 27$ see cases 1,3 in [12]. For $v \geq 39$, let $v = 3 + 12t$, $t \geq 3$, $X = \{\infty, \infty_1, \infty_2\}$, $V = X \cup (\mathbb{Z}_{6t} \times \mathbb{Z}_2)$. On $\mathbb{Z}_{6t}$ place a 3-GDD of type 6^t, $t \geq 3$ (see Lemma 2.1), having groups $G_i, i = 1, 2, \ldots t$. For each block $\{x, y, z\}$, place on $\{x, y, z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.2). Construct a $TS(15, 2)$ on $(G_1 \times \mathbb{Z}_2) \cup X$ which embeds an $H(13, 3, 1)$ on $(G_1 \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 1 in [12]). For $i = 2, 3, \ldots, t$, on $(G_i \times \mathbb{Z}_2) \cup X$ place a K_3-decomposition of $2(K_{15} \setminus K_3)$ having $\{\infty, \infty_1, \infty_2\}$ as the hole which embeds an $H(13, 3, 1)$ on the vertex set $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 2 in [12]). □

Theorem 3.5. *For $v \equiv 4 \pmod{12}$, $v \geq 16$, there exists a $TS(v, 2)$ $(V, \mathcal{B})$ which embeds an $H(v - 3, 3, 1)$.*

Proof For $v = 16, 28, 40$ see cases 4,5,6 in [12]. For $v \geq 52$, let $v = 4 + 12t$, $t \geq 4$, $X = \{\infty, \infty_1, \infty_2, \infty_3\}$, $V = X \cup (\mathbb{Z}_{6t} \times \mathbb{Z}_2)$. On $\mathbb{Z}_{6t}$ place a resolvable 3-GDD of type 6^t, $t \geq 4$ (see Lemma 2.1), having groups $G_i, i = 1, 2, \ldots t$ and resolution classes $\mathcal{C}_i, i = 1, 2, \ldots, 3(t - 1)$. For each $i = 1, 2, \ldots t$, construct a $TS(13, 2)$ on $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ which embeds an $H(13, 3, 1)$ on the same vertex set (Theorem 3.2). For each block $\{x, y, z\} \in \mathcal{C}_i$, $i = 1, 2, 3$, construct on $\{x, y, z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into one 2-factor and a set of triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.3) to obtain a 2-factor F_i and a set of triples $\mathcal{T}_i$. For each block $\{x, y, z\} \in \mathcal{C}_i$, $i = 4, \ldots 3(t-1)$, construct on $\{x, y, z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into a set of triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.2) to obtain a set of triples $\mathcal{T}_i$. Add the set of triples $\bigcup_{i=1}^{3(t-1)} \mathcal{T}_i$, the triples of a $TS(4, 2)$ on $\{\infty, \infty_1, \infty_2, \infty_3\}$ and the sets of triples $\{\{\infty_i, x, y\} : \{x, y\} \in F_i\}$, $i = 1, 2, 3$. It is easy to check that the result is a $TS(v, 2)$ which embeds an $H(v - 3, 3, 1)$ on vertex set $\{\infty\} \cup \{\bigcup_{i=1}^{t} (G_i \times \mathbb{Z}_2)\}$. □

Theorem 3.6. *For $v \equiv 10 \pmod{12}$, there exists a $TS(v, 2)$ $(V, \mathcal{B})$ which embeds an $H(v - 1, 3, 1)$.*

Proof For $v = 10, 22, 34$ see cases 8,9,10 in [12]. For $v \geq 46$, let $v = 10 + 12t$, $t \geq 3$, $X = \{\infty, \infty_1\}$, $V = X \cup (\mathbb{Z}_{4+6t} \times \mathbb{Z}_2)$. On $\mathbb{Z}_{4+6t}$ place a 3-GDD of type

$4^1 6^t$, $t \geq 3$ (see Lemma 2.1), having groups $G_i, i = 0,1,2,\dots t$, $|G_0| = 4$. For each block $\{x,y,z\}$, place on $\{x,y,z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.2). Construct a $TS(10,2)$ on $(G_0 \times \mathbb{Z}_2) \cup X$ which embeds an $H(9,3,1)$ on $(G_0 \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 8 in [12]). For $i = 2,3,\dots,t$, on $(G_i \times \mathbb{Z}_2) \cup X$ place a K_3-decomposition of $2(K_{14} \setminus K_2)$ having $\{\infty, \infty_1\}$ as the hole which embeds an $H(13,3,1)$ on the vertex set $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 7 in [12]). □

Theorem 3.7. *For $v \equiv 7 \pmod{12}$, $v \geq 19$, there exists a $TS(v,2)$ $(V,\mathcal{B})$ which embeds an $H(v-2,3,1)$.*

Proof For $v = 19,31,43$ see cases 11,12,13 in [12]. For $v \geq 55$, let $v = 7+12t, t \geq 4$, $X = \{\infty, \infty_1, \infty_2\}$, $V = X \cup (\mathbb{Z}_{2+6t} \times \mathbb{Z}_2)$. On $\mathbb{Z}_{2+6t}$ place a 3-GDD of type $8^1 6^{t-1}$, $t \geq 4$ (see Lemma 2.1), having groups $G_i, i = 1,2,\dots t$, $|G_1| = 8$. For each block $\{x,y,z\}$, place on $\{x,y,z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.2). Construct a $TS(19,2)$ on $(G_1 \times \mathbb{Z}_2) \cup X$ which embeds an $H(17,3,1)$ on $(G_1 \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 11 in [12]). For $i = 2,3,\dots,t$, on $(G_i \times \mathbb{Z}_2) \cup X$ place a K_3-decomposition of $2(K_{15} \setminus K_3)$ having $\{\infty, \infty_1, \infty_2\}$ as the hole which embeds an $H(13,3,1)$ on the vertex set $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 2 in [12]). □

Theorem 3.8. *For $v \equiv 0 \pmod{12}$, $v \geq 24$ there exists a $TS(v,2)$ $(V,\mathcal{B})$ which embeds an $H(v-3,3,1)$ $(U,\mathcal{D})$.*

Proof For $v = 24,36,48,60,72,84$ see cases 14,15,16,22,23,24 in [12]. For $v \geq 96$, let $v = 24u$ or $v = 24u+12, u \geq 4$, $X = \{\infty, \infty_1, \infty_2, \infty_3\}$, $V = X \cup (\mathbb{Z}_{12u-2} \times \mathbb{Z}_2)$ or $V = X \cup (\mathbb{Z}_{12u+4} \times \mathbb{Z}_2)$, respectively. On $\mathbb{Z}_{12u-2}$ or $\mathbb{Z}_{12u+4}$ place a 3-GDD of type $10^1 12^{u-1}$ or $16^1 12^{u-1}$, $u \geq 4$ (see Lemma 2.1), having groups $G_i, i = 0,1,2,\dots u-1$, $|G_0| = 10$ or $|G_0| = 16$. For each block $\{x,y,z\}$, place on $\{x,y,z\} \times \mathbb{Z}_2$ a decomposition of $2K_{2,2,2}$ into triples which embeds a balanced decomposition of $K_{2,2,2}$ into paths with two edges (see Lemma 2.2). On $(G_0 \times \mathbb{Z}_2) \cup X$ place a $TS(24,2)$ which embeds an $H(21,3,1)$ or a $TS(36,2)$ which embeds an $H(33,3,1)$. For each $i = 1,2,\dots,u-1$, place on $(G_i \times \mathbb{Z}_2) \cup X$ a K_3-decomposition of $2(K_{28} \setminus K_4)$, with a hole of size 4, X, which embeds an $H(25,3,1)$ on $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ (see case 21 in [12]). The result is a $TS(v,2)$ $(V,\mathcal{B})$ which embeds an $H(v-3,3,1)$. □

4 $\lambda = 3$

Theorem 4.1. *There exists a $TS(v,3)$ which embeds an $H(v,3,1)$ for $v \equiv 1,5,9 \pmod{12}$ and an $H(v-2,3,1)$ for $v \equiv 3,7 \pmod{12}$, $v \geq 7$.*

Proof For $v \equiv 1,9 \pmod{12}$ paste a $TS(v,1)$ to an $TS(v,2)$ which embeds an $H(v,3,1)$ (see Theorem 3.2). For $v \equiv 5 \pmod{12}$ the proof is in [10]. For $v = 7$, see case 17 in [12]. For $v \equiv 3,7 \pmod{12}$, $v \geq 15$, paste an $TS(v,1)$ to an $TS(v,2)$ which embeds an $H(v-2,3,1)$ (see Theorems 3.4, 3.7). □

Theorem 4.2. *For $v \equiv 11 \pmod{12}$ there exists a $TS(v,3)$ which embeds an $H(v-2,3,1)$.*

Proof Let $v = 11 + 12t$, $t \geq 0$. Take a $TS(13 + 12t, 1)$ with a block $\{\infty_1, \infty_2, \infty_3\}$. Delete the block $\{\infty_1, \infty_2, \infty_3\}$, replace ∞_i, $i = 1, 2, 3$, with ∞ and paste the blocks so obtained to the blocks of a $TS(10 + 12t, 2)$ which embeds an $H(9, 3, 1)$ (Theorem 3.6). □

5 $\lambda = 4$

Theorem 5.1. *There exists a $TS(v, 4)$ which embeds an $H(v, 3, 1)$ for $v \equiv 1, 9 \pmod{12}$, an $H(v-1, 3, 1)$ for $v \equiv 6, 10 \pmod{12}$, an $H(v-2, 3, 1)$ for $v \equiv 3, 7 \pmod{12}$, an $H(v-3, 3, 1)$ for $v \equiv 0, 4 \pmod{12}$.*

Proof For $v = 7$ paste a $TS(7, 1)$ to a $TS(7, 3)$ which embeds an $H(5, 3, 1)$. For $v = 12$ see case 18 in [12]. For $v \neq 7, 12$ the result follows by doubling the solution for $\lambda = 2$. □

6 $\lambda = 5$

Theorem 6.1. *There exists a $TS(v, 5)$ which embeds an $H(v, 3, 1)$ for $v \equiv 1, 9 \pmod{12}$ and an $H(v-2, 3, 1)$ for $v \equiv 3, 7 \pmod{12}$.*

Proof For $v \equiv 1, 9 \pmod{12}$, the proof follows by pasting an $TS(v, 1)$ to an $TS(v, 4)$ which embeds an $H(v, 3, 1)$ (Theorem 5.1). For $v \equiv 3, 7 \pmod{12}$, we get the proof by pasting an $TS(v, 1)$ to an $TS(v, 4)$ which embeds an $H(v-2, 3, 1)$ (Theorem 5.1). □

7 $\lambda = 6$

Theorem 7.1. *There exists a $TS(v, 6)$ which embeds an $H(v-w, 3, 1)$ where $w = 0$ for $v \equiv 1, 5, 9 \pmod{12}$, $w = 1$ for $v \equiv 6, 10 \pmod{12}$, $w = 2$ for $v \equiv 3, 7, 11 \pmod{12}$, $w = 3$ for $v \equiv 0, 4 \pmod{12}$.*

Proof For $v \equiv 1, 3, 5, 7, 9, 11 \pmod{12}$, the result follows by doubling the solution for $\lambda = 3$. For $v \equiv 0, 4, 6, 10 \pmod{12}$, $v \neq 12$, the result follows by tripling the solution for $\lambda = 2$. For $v = 12$ paste a $TS(v, 2)$ to a $TS(v, 4)$ which embeds an $H(v-3, 3, 1)$. □

Theorem 7.2. *There exists a $TS(v, 6)$ which embeds an $H(v-1, 3, 1)$ for $v \equiv 2 \pmod{12}$ and an $H(v-3, 3, 1)$ for $v \equiv 8 \pmod{12}$.*

Proof For $v = 8$, see case 19 in [12]. For $v = 8+12t$, $t > 0$, embed an $H(5+12t, 3, 1)$ into a $TS(7+12t, 2)$ (Theorem 3.7). Applying Theorem 2.1 we obtain the result. For $v = 2 + 12t$ embed an $H(1 + 12t, 3, 1)$ into a $TS(1 + 12t, 2)$ (Theorem 3.2). Applying Theorem 2.1 we obtain the result. □

References

[1] *The CRC Handbook of Combinatorial Designs*, Second Edition, Eds. C.J. Colbourn and J.H. Dinitz, CRC Press, Boca Raton, FL, 2007.

[2] C.J. Colbourn, A.C.H. Ling and G. Quattrocchi, Embedding path designs into kite systems, *Discrete Math.* **297** (2005), 38–48.

[3] C.J. Colbourn, A.C.H. Ling and G. Quattrocchi, Minimum embedding of P_3-designs into $K_4 - e$-designs, *J. Combin. Des.* **11** (2003), 352–366.

[4] M. Gionfriddo and G. Quattrocchi, Embedding balanced P_3-designs into P_4-designs, *Discrete Math.* **308** (2008), 155–160.

[5] S.H.Y. Hung and N.S. Mendelsohn, Handcuffed designs, *Discrete Math.* **18** (1974), 23–33.

[6] S.P. Hurd, P. Munson and D.G. Sarvate, Minimal Enclosing of Triple Systems I: Adding one point, *Ars Combin.* **68** (2003), 145–159.

[7] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, 1997, 208 pp.

[8] S. Mendelsohn and G. Quattrocchi, Minimum embedding of balanced P_4-designs into 5-cycle systems, *Discrete Math.* **279** (2004), 407–421.

[9] S. Meszka and A. Rosa, Embedding Steiner triple systems into Steiner systems $S(2,4,v)$, *Discrete Math.* **279** (2004), 199–212.

[10] S. Milici, Minimum embedding of P_3-designs into $TS(v,\lambda)$, *Discrete Math.* **308** (2008), 331–338.

[11] S. Milici and G. Quattrocchi, Embedding handcuffed designs with block size 2 or 3 in 4-cycle systems, *Discrete Math.* **208/209** (1999), 443–449.

[12] S. Milici and G. Ragusa, Appendix for "Maximum embedding of an $H(v-w,3,1)$ into a $TS(v,\lambda)$", `http://ajc.maths.uq.edu.au/appendices/AJCvol46pp121-127Appendix.pdf` .

[13] G. Quattrocchi, Embedding path designs in 4-cycle systems, *Discrete Math.* **255** (2002), 349–356.

[14] G. Quattrocchi, Embedding G_1-designs into G_2-designs, a short survey, 6th Workshop on Combinatorics (Messina, 2002), *Rend. Sem. Mat. Messina* **Ser II Suppl. 8** (2002), 129–143.

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