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# A combinatorial characterization of the Hermitian surface

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**Abstract**

In this paper we prove that in a projective space of dimension three and square order $q^2$ a $(q^5+q^3+q^2+1)$-set having no external line, of type $(m,n)_2$ and meeting every affine plane in at least $(q^3-q^2)$ points is a Hermitian surface.

## 1 Introduction

The characterizations from a combinatorial point of view of classical algebraic varieties have played an important role in finite geometry since the fifties. By a beautiful result of Barlotti [1] and Panella [6], every $(q^2+1)$-set of type $(0,1,2)_1$, in $PG(3,q)$ with $q$ odd, is an elliptic quadric. This is a generalisation of Segre's famous theorem [8] stating that every $(q+1)$-set of type $(0,1,2)_1$ in $PG(2,q)$, with $q$ odd, is a conic. The theory of $k$-sets analyses the subsets of size $k$ in $PG(r,q)$ with respect to their possible intersections with all the subspaces of a given dimension $d$; see [10]. Let $\theta_d := q^d+q^{d-1}+\cdots+q+1$ denote the number of points of a projective subspace in $PG(r,q)$. Let $K$ denote a $k$-set in $PG(r,q)$. We recall that the *characters* of $K$ with respect to dimension $d$ are the numbers $t_i^d(K)$ of $d$-dimensional subspaces meeting $K$ in exactly $i$ points, $0 \le i \le \theta_d$. The set $K$ is called of *class* $[m_1, m_2, \cdots, m_h]_d$ if $t_i^d(K) \neq 0$ only if $i \in \{m_1, m_2, \cdots, m_h\}$. Moreover the set $K$ is said to be of *type* $(m_1, m_2, \cdots, m_h)_d$ if $t_i^d(K) \neq 0$ if and only if $i \in \{m_1, m_2, \cdots, m_h\}$.

An *affine plane* in $PG(r,q)$ is the difference between a (projective) plane $\pi$ and a line contained in $\pi$. Moreover an *affine line* is a projective line without one point. Let us consider $x \to \overline{x}$ the involutory automorphism of $GF(q^2)$ defined by $\overline{x} = x^q$ and a square matrix $H$ such that $H = \overline{H}^t$.

In $\Pi := PG(2,q^2)$ a *Hermitian curve* is defined to be the set $C = \{X = (x_0,x_1,x_2) \in \Pi \mid XH\overline{X}^t = 0\}$. A Hermitian curve has the following combinatorial properties. The number of points on it is $q^3+1$. Any line of $\Pi$ meets $C$ in either 1 or $q+1$ points. Lines of the first type are called *tangents*, while those of the second type are called *secants* or *chords* of C. There is just one tangent at every point $P \in C$, whereas the remaining $q^2$ lines through $P$ are secants. If $P \notin C$, then through $P$ there are $q+1$ tangents and $q^2-q$ secants. Therefore $C$ is a *unital*, i.e. a $(q^3+1)$-set of type $(1,q+1)_1$ of $\Pi$.

In $\Sigma := PG(3,q^2)$ a *Hermitian surface* is defined to be the set $S = \{X = (x_0,x_1,x_2,x_3) \in \Sigma \mid XH\overline{X}^t = 0\}$. A Hermitian surface has the following combinatorial properties. The number of points on it is $q^5+q^3+q^2+1 = (q^2+1)(q^3+1)$. Any line of $\Sigma$ meets $S$ in either 1 or $q+1$ or $q^2+1$ points. The latter lines are called *generators* of $S$ and they are $(q+1)(q^3+1)$ in number. The intersections of size $q+1$ are called *Baer sublines*, whereas lines meeting $S$ in one point are called *tangent lines*. Through every point $P$ of $S$ there pass exactly $q+1$ generators, and these generators are coplanar. The plane containing these generators, say $\pi$, is called the *polar plane* of $P$ with respect to $S$. The tangent lines through $P$ are precisely the remaining $q^2-q$ lines of $\pi$ incident with $P$, and $\pi$ is called the *tangent plane* to $S$ at $P$. Every plane of $\Sigma$ which is not a tangent plane to $S$ is called a *secant plane* and meets $S$ in a *Hermitian curve* $C$. Therefore $S$ is a $(q^5+q^3+q^2+1)$-set of type $(q^3+1, q^3+q^2+1)_2$ and type $(1,q+1,q^2+1)_1$ of $\Sigma$. Moreover every affine plane meets $S$ in at least $q^3-q$ points. As usual, a line which has no point in common with a set $S$ is called *external line* of $S$.

Hermitian surfaces have been intensively investigated; see [3, 7, 9, 11]. In particular there are several characterizations of Hermitian surfaces as point sets in $PG(3,q^2)$; see Chapter 19.5 of [5]. The first one was given by Barlotti; see [2].

The object of this paper is to characterize Hermitian surfaces in terms of the numbers of points in which a plane can meet them. We prove the following converse.

**Theorem 1.1** *In $PG(3,q^2)$, a $(q^5+q^3+q^2+1)$-set $K$ having no external line, of type $(m,n)_2$ and meeting every affine plane in at least $q^3-q$ points is a Hermitian surface.*

Note that the hypothesis in the theorem that $K$ has no external line cannot be dropped because $PG(3,q^2)$ contains a spread $S$ of $q^4+1$ pairwise skew lines. Any point-set of $q^3+1$ pairwise skew lines of $S$ is a $(q^5+q^3+q^2+1)$-set of type $(m,n)_2$, meeting every affine plane in at least $q^3-q$ points, but it is not a Hermitian surface. The final step uses the following result contained in Section 19.5 of [5].

**Result** *In $PG(3,q^2)$, a $(q^5+q^3+q^2+1)$-set of class $[1,q+1,q^2+1]_1$ is a Hermitian surface.*

## 2 The proof of the Theorem

In this section we prove the theorem in several steps.

Assume that in $PG(3,q^2)$, $K$ is a $(q^5+q^3+q^2+1)$-set having no external line, of type $(m,n)_2$ and meeting every affine plane in at least $q^2-q$ points.

I *$K$ is of type $(q^3+1, q^3+q^2+1)_2$.*
**Proof.** Let $K$ be a $(q^5+q^3+q^2+1)-$set of type $(m,n)_2$ in $PG(3,q^2)$. We recall that if $m=0$ then $K$ is either a point or a complementary set of a plane, cf. [10]; moreover if $n=q^4+q^2+1$, then $K$ is either a plane or a complementary set of a point. In order to avoid these trivial cases we assume

$$1 \le m < n \le q^4+q^2. \tag{1}$$

By counting in a double way the number of planes, the number of pairs $(P,\pi)$, where $P \in K$ and $\pi$ is a plane through $P$, and the number of pairs $(\{P,Q\},\pi)$, where $\{P,Q\} \subset K$ and $\pi$ is a plane through $P$ and $Q$, we get the following equations on the integers $t_i$:

$$\begin{cases} t_m+t_n & = q^6+q^4+q^2+1 \\ mt_m+nt_n & = (q^5+q^3+q^2+1)(q^4+q^2+1) \\ m(m-1)t_m+n(n-1)t_n & = (q^5+q^3+q^2+1)(q^5+q^3+q^2)(q+1). \end{cases} \tag{2}$$

From the first two equations we get

$$\begin{cases} t_m & = \dfrac{n(q^6+q^4+q^2+1)-(q^5+q^3+q^2+1)(q^4+q^2+1)}{n-m} \\ t_n & = \dfrac{(q^5+q^3+q^2+1)(q^4+q^2+1)-m(q^6+q^4+q^2+1)}{n-m} \end{cases} \tag{3}$$

and then by substituting (3) in the third equation of (2) we get

$$(q^5+q^3+q^2+1)^2-(q^5+q^3+q^2+1)\left[1+(m-n+1)\frac{q^4+q^2+1}{q^2+1}\right]+mn(q^4+1)=0. \tag{4}$$

Firstly we show that $m \le q^3+q+1$.

Since $t_n>0$, it follows that $(q^5+q^3+q^2+1)(q^4+q^2+1) > m(q^6+q^4+q^2+1)$. We get

$$\begin{aligned} m &< \frac{(q^5+q^3+q^2+1)(q^4+q^2+1)}{(q^6+q^4+q^2+1)} \\ &= \frac{(q^3+1)(q^4+q^2+1)}{q^4+1} = q^3+q+1+\frac{q^2-q}{q^4+1}. \end{aligned}$$

Hence $m \le q^3+q+1$. So we can write $m=q^3+q+1-a$, with $a$ an integer such that $0 \le a \le q^3+q$.

By (4), we get $n$ as a rational increasing function of $m$:

$$\begin{aligned} n &= \frac{q^{10}+2q^8+3q^7+q^6+4q^5+2q^4+2q^3+2q^2+1}{q^7+q^5+q^4+q^3+q^2+1-m(q^4+1)} \\ &\quad -\frac{m(q^7+q^5+q^4+q^3+q^2+1}{q^7+q^5+q^4+q^3+q^2+1-m(q^4+1)}. \end{aligned}$$

By substituting $m = q^3+q+1-a$ we get

$$n = \frac{(a+1)q^7-q^6+(a+1)q^5+aq^4+(a-1)q^3+(a+1)q^2-q+a}{aq^4+q^2-q+a}.$$

If $m = q^3+q+1$ then $n = q^5+q^3+q^2+1$, but $t_n = \dfrac{q^4-q^3+q^2-q}{q^5+q^2-q}$ is not an integer, a contradiction. Hence $m \leq q^3+q$ and $a \geq 1$.
We get

$$\begin{aligned} n &= \frac{(a+1)q^7-q^6+(a+1)q^5+aq^4+(a-1)q^3+(a+1)q^2-q+a}{aq^4+q^2-q+a} \\ &= q^3+q^2+1-q^2\frac{a(q^4-q^3)-(q^5-q^4)}{a(q^4+1)+(q^2-q)}. \end{aligned} \tag{5}$$

Suppose $a \neq q$. There are two possibilities: either $a < q$ or $a > q$.

- Firstly suppose $a < q$. We write the expression of $n$ as

$$n = q^3+q^2+1+q^3\frac{q^4-aq^3-q^3+aq^2}{aq^4+q^2-q+a}.$$

  Since, for $1 \leq a \leq q-1$, $0 < \dfrac{q^4-aq^3-q^3+aq^2}{aq^4+q^2-q+a} < 1$, if the greatest common divisor of $q^3$ and $aq^4+q^2-q+a$ is 1, then $n$ is not an integer. Thus suppose that the greatest common divisor, say $g$, of $q^3$ and $aq^4+q^2-q+a$ is not 1. Since $g|q^3$ then $g|(q^2-q+a)$. Therefore $g|(q^4-q^3+aq^2)$. Thus $g|(q^4-aq^3-q^3+aq^2)$. Then we can eliminate the greatest common divisor $g$ from the numerator and the denominator of the fraction $\dfrac{q^4-aq^3-q^3+aq^2}{aq^4+q^2-q+a}$. Put $N = (q^4-aq^3-q^3+aq^2)/g$ and $D = (aq^4+q^2-q+a)/g$. Hence we reduce the expression of $n$ to the form $n = q^3+q^2+1+q^3\frac{N}{D}$ in which the greatest common divisor of $q^3$ and $D$ is 1 and $0 < \frac{N}{D} < 1$, so then $n$ is not an integer.

- Now suppose $a > q$. Taking into account (5), since for $q+1 \leq a \leq q^3+q$, $-1 < -\dfrac{a(q^4-q^3)-(q^5-q^4)}{a(q^4+1)+(q^2-q)} < 0$, if the greatest common divisor of $q^2$ and $a(q^4+1)+(q^2-q)$ is 1, then $n$ is not an integer. Thus suppose that the greatest common divisor, say $g$, of $q^2$ and $a(q^4+1)+(q^2-q)$ is not 1. Since $g|q^2$ then $g|(a-q)$. So

$$\frac{a(q^4-q^3)-(q^5-q^4)}{a(q^4+1)+(q^2-q)} = \frac{(a-q)(q^4-q^3)}{aq^4+q^2+(a-q)},$$

and we can eliminate the greatest common divisor $g$ from the numerator and the denominator of the fraction $\frac{(a-q)(q^4-q^3)}{aq^4+q^2+(a-q)}$. Put $N=(a-q)(q^4-q^3)/g$ and $D=(aq^4+q^2+(a-q))/g$. Hence we reduce the expression of $n$ to the form $n=q^3+q^2+1-q^2\frac{N}{D}$ in which the greatest common divisor of $q^2$ and $D$ is 1 and $-1<-\frac{N}{D}<0$, so then $n$ is not an integer.

Therefore $a=q$. Hence we get $m=q^3+1$ and $n=q^3+q^2+1$. Thus $K$ is a $(q^5+q^3+q^2+1)$-set of type $(q^3+1, q^3+q^2+1)_2$ of $PG(3,q)$.

II *Every* $(q^3+1)$*-secant projective plane intersects* $K$ *in a unital.*

**Proof.** Let $\pi$ be a $(q^3+1)$-secant projective plane of $K$ and $K'=K\cap\pi$. The $(q^3+1)$-set $K'$ will be proved to be a unital of $\pi$. Firstly, no line exists in $\pi$ which intersects $K'$ in more than $q+1$ points. Assume on the contrary that $r$ is such a line. Since $K$ meets every affine plane in at least $(q^3-q)$ points $|K\cap(\pi-r)|\geq q^3-q$ implies $|K\cap\pi|>q^3+1$. Therefore, any line in $\pi$ meets $K'$ in at most $q+1$ points. Let $t_i$ be the number of lines meeting $K'$ in $i$ points. Since $K$ has no external line $t_0=0$. Then by counting in two ways the number of lines, the number of pairs $(P,r)$, where $P\in K'$ and $r$ is a line of $\pi$ through $P$, and the number of pairs $(\{P,Q\},r)$, where $\{P,Q\}\subset K'$ and $r$ is a line of $\pi$ through $P$ and $Q$, we get the following equations on the integers $t_i$:

$$\begin{cases} \sum_{i=1}^{q+1}(t_i) & = & q^4+q^2+1 \\ \sum_{i=1}^{q+1}(it_i) & = & (q^3+1)(q^2+1) \\ \sum_{i=2}^{q+1}(i(i-1)t_i) & = & q^3(q^3+1) \end{cases}$$

Multiply the first equation by $-(q+1)$, the second one by $(q+1)$, the third one by $-1$ and add:

$$\sum_{i=2}^{q}((i-1)(q+1-i)t_i) = \\ -(q+1)(q^4+q^2+1)+(q+1)(q^3+1)(q^2+1)-q^3(q^3+1)=0.$$

The coefficients of the $t_i$s in the left-hand side of the equality being positive integers, for the equality to hold the following must be true:

$$t_2=t_3=\cdots=t_q=0.$$

Therefore, $K'$ is a $(q^3+1)$-set of type $(1,q+1)_1$ of $\pi$, i.e. a unital of $\pi$.

III *An $h$-secant line is contained in exactly $h$ $(q^3+q^2+1)$-secant planes.*

**Proof.** Let $r$ denote an $h$-secant line. Let $x$ and $y$ denote the number of $(q^3+1)$-secant and $(q^3+q^2+1)$-secant planes through $r$, respectively. By counting in two ways the total number of planes through $r$ and the number of pairs $(P,\alpha)$ where $P \in K - r$ and $\alpha$ is a plane such that $P \in \alpha$ and $r \subset \alpha$, we get:

$$\begin{cases} x+y & = q^2+1 \\ (q^3+1-h)x+(q^3+q^2+1-h)y & = q^5+q^3+q^2+1-h \end{cases}$$

Multiply the first equation by $-(q^3+1-h)$ and add:

$$q^2y = q^2h$$

Thus $y = h$.

IV *Every $(q^3+q^2+1)$-secant projective plane intersects $K$ in $q+1$ lines through one point of $K$.*

**Proof.** Let $\pi$ be a $(q^3+q^2+1)$-secant projective plane of $K$ and $K' = K \cap \pi$. Let $P$ and $Q$ any two points in $K'$ and $r$ the line through them. By III, if $r$ is not contained in $K$ there is at least one $(q^3+1)$-secant plane through $r$. In view of II, $|r \cap K| = q+1$. Therefore $K'$ is a $(q^3+q^2+1)$-set of class $[1, q+1, q^2+1]_1$ in $\pi$. Take a point $R \in K'$. Through $R$ there is at least one line $s$ contained in $K'$, otherwise $|K'| \le (q^2+1)q+1 < q^3+q^2+1$. If $K'$ contains exactly one line $s$, then $K'-s$ is a set of type $(0,q)_1$ and $|K'-s| = q^2(q-1)+q < q^3$, a contradiction. Thus $K'$ contains at least two lines $s$ and $t$. Let us denote by $S$ the point such that $\{S\} = s \cap t$. A line through any point $T$ of $\pi$ not in $K'$ meets $K'$ in 1 or $q+1$ points. Let us denote by $x$ and $y$ the number of lines through $T$ which meet $K'$ in 1 or $q+1$ points, respectively. We get

$$\begin{cases} x+y & = q^2+1 \\ x+(q+1)y & = q^3+q^2+1 \end{cases}$$

Multiply the first equation by $(q+1)$, the second one by $-1$ and add:

$$qx = q$$

Thus $x = 1$. Hence $ST$ is a unisecant of $K'$. So there are no $(q+1)-$secants through $S$. Hence $K'$ consists of $(q+1)$ lines through $S$.

V *$K$ is of class $[1, q+1, q^2+1]_1$.*

**Proof.** Assume that a projective line $r$ is not contained in $K$. By II, $r$ is contained in a $(q^3+1)-$secant projective plane $\pi$. By III, $K' = K \cap \pi$ is a unital of $\pi$. Thus $|r \cap K| \in \{1, q+1\}$.

Finally, the Theorem follows by V and the result contained in Section 19.5 of [5].

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