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On connected (γ, k)-critical graphs

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Abstract

A graph G is said to be (γ, k)-critical if $\gamma(G - S) < \gamma(G)$ for any set S of k vertices and domination number γ. Properties of (γ, k)-critical graphs are studied for $k \geq 3$. Ways of constructing a (γ, k)-critical graph from smaller (γ, k)-critical graphs are presented.

1 Introduction

We study domination (k)critical graphs [1], which are graphs whose domination number decreases after removal of any set of k vertices. Let $G = (V, E)$ be a graph. The open neighborhood of a vertex $v \in V$ is $N(v) = \{x \in V \mid vx \in E\}$. The closed neighborhood is $N[v] = N(v) \cup \{v\}$. A set $S \subset V$ is a *dominating set* if every vertex in V is either in S or is adjacent to a vertex in S, that is, $V = \bigcup_{s \in S} N[s]$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$-set. Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. It is useful to write the vertex set of a graph as a disjoint union of three sets according to how their removal affects $\gamma(G)$. Let $V(G) = V^0 \cup V^+ \cup V^-$, where $V^0 = \{v \in V \mid \gamma(G - v) = \gamma(G)\}, V^+ = \{v \in V \mid \gamma(G - v) > \gamma(G)\}, V^- = \{v \in V \mid \gamma(G - v) < \gamma(G)\}$; for more, see [1–9]. We define a graph G to be (γ, k)-*critical*, if the domination number of G is γ and $\gamma(G - S) < \gamma(G)$ for any set S of k vertices. Obviously, a (γ, k)-critical graph G has $\gamma(G) \geq 2$.

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Brigham, Chinn and Dutton [1] introduced domination critical graphs, which are $(\gamma, 1)$-critical graphs (or γ-critical graphs), while Brigham, Haynes, Henning and Rall [2] introduced domination bicritical graphs, which are $(\gamma, 2)$-critical graphs (or γ-bicritical graphs). In the special case $k = 3$, we say the graphs are *domination tricritical* (*γ-tricritical*) or just *tricritical* graphs, and for $k \geq 4$ we say *domination* (k)*critical,* (γ-(k)*critical*) or just (k)*critical* graphs.

The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is k-connected if its connectivity is at least k. A graph is k-edge connected if every disconnecting set has at least k edges. The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set. We denote the distance between two vertices x and y in G by $d_G(x, y)$. The diameter of G, $\mathrm{diam}(G)$, is the maximum $d_G(x, y)$ for any x, y in G. We denote minimum degree by δ and maximum degree by Δ; see [10]. The following are useful.

Observation A. [R. C. Brigham et al. [1]]. If G is any graph and $x, y \in V(G)$ such that $\gamma(G - \{x, y\}) = \gamma(G) - 2$, then $d_G(x, y) \geq 3$.

Proposition B. [1] If G is a critical graph of order n then

$$n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1.$$

Proposition C. [4] If G is a critical graph of order $n = (\Delta(G) + 1)(\gamma(G) - 1) + 1$, then G is regular.

Proposition D. [2] If G is a bicritical graph of order n, then

$$n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 2.$$

Proposition E. [2] If G is a regular bicritical graph of order n, then

$$n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1.$$

2 Elementary results

In this section we study the relationship between (k)criticality of graphs and some other parameters. First we determine the graphs that are not (k)critical for $1 \leq k \leq |V(G)| - 2$.

We have in general:

Example 1. Let $G = K_{m_1, m_2, \ldots m_n}$ be a complete n-partite graph where $m_i \geq 3$ for $1 \leq i \leq n$ and $n \geq 2$. Then $\gamma(G) = 2$ and G is not (k)critical for $1 \leq k \leq |V(G)| - 2$. For $m_i = 3$, the non-(k)criticality is shown as follows. Other cases have a similar proof. Let the ith partite set have vertices $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$ and let $\{v_1, v_2, v_3, \ldots, v_{3n-2}, v_{3n-1}, v_{3n}\}$ be the vertices of G. We show that $\gamma(G - \{x_1, x_2, \ldots, x_k\}) = \gamma(G)$. For $k = 3m$ and $m \leq n - 1$, say $\{x_1, x_2, \ldots, x_k\} = \{v_i \mid i = 1, 2 \ldots 3m\}$. For $k = 3m + 1$ and $m \leq n - 1$, say $\{x_1, x_2, \ldots, x_k\} = \{v_i \mid i = 1, 2, \ldots 3m + 1\}$. For $k = 3m + 2$ and $m \leq n - 2$, say $\{x_1, x_2, \ldots, x_k\} = \{v_i \mid i \in \{1, 2, \ldots 3m + 1\}\} \cup \{v_{3m+4}\}$. Thus G is not (k)critical for $1 \leq k \leq 3n - 2$.

Figure 1 is not (k)critical for $k \leq 10$.

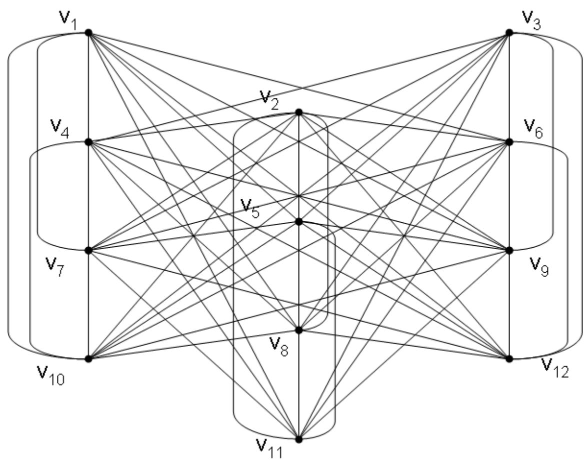


Figure 1: The complete 4-partite graph $K_{3,3,3,3}$

By Observation A, removing v and two neighbors u and w of v from a tricritical graph G reduces the domination number of G by one. Thus adding v to any $\gamma(G-\{u,w,v\})$-set produces a $\gamma(G)$-set. In general one can make the following observation, which has a simple proof.

Observation 1 *For a (γ, k)-critical graph G and $S = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, $\gamma(G) - k \leq \gamma(G) - \gamma_G(S) \leq \gamma(G - S) \leq \gamma(G) - 1$ where $\gamma_G(S)$ is the number of vertices (of G) needed to dominate S.*

Observation 2 *Let G be any graph and $\{x_1, x_2, \ldots, x_k\} \subseteq V(G)$.*
1. If $\gamma(G - \{x_1, x_2, \ldots, x_k\}) = \gamma(G) - k$, then $d_G(x_i, x_j) \geq 3$ for $i \neq j$.
2. If diam$(G) \leq 2$, then $\gamma(G - \{x_1, x_2, \ldots, x_k\}) \geq \gamma(G) - \lceil \frac{k}{2} \rceil$.

Proof. 1. Let D be a $\gamma(G - \{x_1, x_2, \ldots, x_k\})$-set with $|D| = \gamma(G) - k$. Let $d_G(x_i, x_j) \leq 2$ for some $i \neq j$ and y be a common adjacent vertex or be one of vertices x_i, x_j. Then G is dominated by $|D| + k - 1 = \gamma(G) - 1$ vertices; a contradiction.

2. Any two vertices have a common adjacent vertex. So at most $\lceil \frac{k}{2} \rceil$ vertices dominate $\{x_1, x_2, \ldots, x_k\}$. □

Observation 3 *In a connected (γ, k)-critical graph, there is no vertex of degree k.*

Proof. Let G be a connected (γ, k)critical graph, x be a vertex of degree k and $\{x_1, x_2, \ldots, x_k\}$ be $N(x)$. By removing the vertices $x_1, x_2, \ldots, x_k$, the vertex x will be isolated. Let S be a $\gamma(G - \{x_1, x_2, \ldots, x_k\})$-set; then $x \in S$, and since G is (k)critical, $|S| \leq \gamma(G) - 1$, since x dominates $\{x_1, x_2, \ldots, x_k\}$. Thus S dominates G and it is a $\gamma(G)$-set with cardinality $\gamma(G) - 1$, a contradiction. □

Observation 4 *If G is a (γ, k)-critical graph with $\lambda(G) \geq k-1$, then every vertex of G belongs to a $\gamma(G)$-set.*

Proof. Let v be any vertex. Now $\lambda(G) \geq k-1$ implies that $deg(v) \geq k-1$. So deleting the vertex v and its $k-1$ neighbors decreases $\gamma(G)$. Thus v belongs to a $\gamma(G)$-set. $\square$

Lemma 5 *Let G be a (γ, k)-critical but not critical graph. Then for some i, $1 \leq i < k$, there exists a subset $\{v_1, v_2, \ldots, v_i\}$ of $V(G)$ such that $\gamma(G - \{v_1, v_2, \ldots, v_i\}) = \gamma(G)$.*

Proof. Since G is not critical, we know $V^0 \cup V^+ \neq \emptyset$. Form a sequence of k vertices $v_1, v_2, \ldots, v_k$ where $v_1 \in V^0 \cup V^+$. Then $\gamma(G - v_1) \geq \gamma(G)$ and $\gamma(G - \{v_1, v_2, \ldots, v_k\}) < \gamma(G)$. Since

$$\gamma(G - \{v_1, v_2, \ldots, v_{j+1}\}) - \gamma(G - \{v_1, v_2, \ldots, v_j\}) \geq -1,$$

it follows easily that $\gamma(G - \{v_1, v_2, \ldots, v_i\}) = \gamma(G)$ for some $1 \leq i < k$. $\square$

Proposition 6 *If G is a (γ, k)-critical graph ($k \geq 1$) of order n, then $n \leq (\Delta(G)+1)(\gamma(G)-1)+k$.*

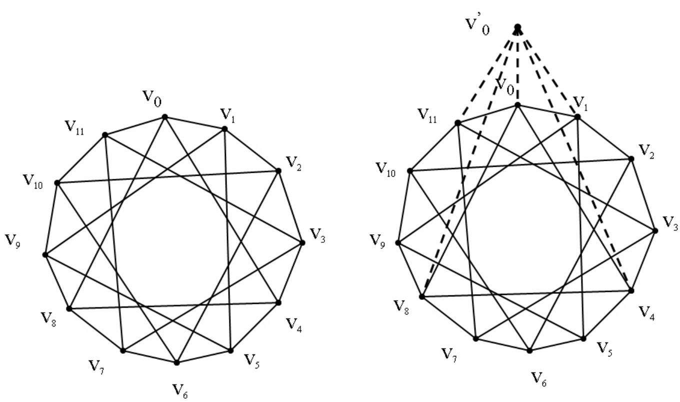
Proof. This will be proven by induction. It has been previously shown to be true for $k = 1$ or $k = 2$. Assume that $k \geq 3$ and that the result holds for any $m \leq k-1$. If G is critical, then the result holds by Proposition B. So suppose that G is not critical. Then by Lemma 5, for some i, $1 \leq i < k$, there exists a subset $\{v_1, v_2, \ldots, v_i\}$ of $V(G)$ such that $\gamma(H) = \gamma(G)$ where $H = G - \{v_1, v_2, \ldots, v_i\}$. So H is $(\gamma, k-i)$-critical. Hence, by induction, we have $n - i \leq (\Delta(H)+1)(\gamma(H)-1)+k-i$. But $\Delta(H) \leq \Delta(G)$ and $\gamma(H) = \gamma(G)$, so $n - i \leq (\Delta(G)+1)(\gamma(G)-1)+k-i$. The result holds by adding i to both sides. $\square$

3 Expansion of graphs

We would like to use a construction of [3] to make it possible to extend a (γ, k)-critical graph to a larger one that is also (k)critical. Let $G = (V, E)$ be any graph, $v \in V$ and $v' \notin V$. The expansion of G via v, $G_{[v]}$ is defined in [3] to be the graph with vertex set $V \cup \{v'\}$ and edge set $E \cup \{v'x \mid x \in N_G[v]\}$. Thus, $G_{[v]}$ is obtained from G by adding a new vertex v' such that $N[v'] = N[v]$.

The circulant graph $C_{n+1}\langle 1, m\rangle$ (see Figure 2 (a), $C_{12}\langle 1, 4\rangle$) is a graph with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{v_i v_{i+j \pmod{n+1}} \mid i \in \{0, 1, \ldots, n\}$ and $j \in \{1, m\}\}$.

Proposition 7 *If v is a vertex of a graph G that is both (k)critical and $(k+1)$critical such that $k < n-1 = |V(G)|-1$, then $G_{[v]}$ is $(k+1)$critical.*



(a): $C_{12}\langle 1, 4\rangle$ (b): $(C_{12}\langle 1, 4\rangle)_{[v0]}$

Figure 2

Proof. Note that $\gamma(G_{[v]}) = \gamma(G)$. Let $\{x_1, x_2, \ldots, x_{k+1}\} \subseteq V(G_{[v]})$. There are three cases to consider depending on whether $|\{x_1, x_2, \ldots, x_{k+1}\} \cap \{v, v^{'}\}|$ is 0, 1 or 2. Let $|\{x_1, x_2, \ldots, x_{k+1}\} \cap \{v, v^{'}\}| = 0$ and let D be a $\gamma(G - \{x_1, x_2, \ldots, x_{k+1}\})$-set. Since G is $(k + 1)$critical, $|D| < \gamma(G)$. Since D dominates v in $G - \{x_1, x_2, \ldots, x_{k+1}\}$, it also dominates $v^{'}$ in $G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\}$. Thus D is a dominating set of $G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\}$, and so $\gamma(G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\}) \leq |D| < \gamma(G) = \gamma(G_{[v]})$. Suppose $|\{x_1, x_2, \ldots, x_{k+1}\} \cap \{v, v^{'}\}| = 1$, since $N_{G_{[v]}}[v] = N_{G_{[v]}}[v^{'}]$; without loss of generality, we assume that $x_1 = v$ and $\{x_2, \ldots, x_{k+1}\} \subseteq V(G) - \{v\}$. But then $G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\} = G_{[v]} - \{v, x_2, \ldots, x_{k+1}\} \cong G_{[v]} - \{v^{'}, x_2, \ldots, x_{k+1}\} = G - \{x_2, \ldots, x_{k+1}\}$. Since G is also a (k)critical graph, it follows that $\gamma(G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\}) = \gamma(G - \{x_2, x_3, \ldots, x_{k+1}\}) \leq \gamma(G) - 1 < \gamma(G_{[v]})$. Suppose, finally $\{x_1, x_2, \ldots, x_{k+1}\} = \{v, v^{'}, x_3, \ldots, x_{k+1}\}$; without loss of generality $x_1 = v, x_2 = v^{'}$, and then

$$\begin{aligned}
\gamma(G_{[v]} - (\{x_1, x_2, \ldots, x_{k+1}\})) &= \gamma(G_{[v]} - (\{v, v^{'}, x_3, \ldots, x_{k+1}\})) \\
&= \gamma(G - (\{v, x_3, \ldots, x_{k+1}\})) \\
&\leq \gamma(G) - 1 < \gamma(G) = \gamma(G_{[v]}).
\end{aligned}$$

Therefore, in all three cases, $\gamma(G_{[v]} - \{x_1, x_2, \ldots, x_{k+1}\}) < \gamma(G_{[v]})$ and so $G_{[v]}$ is $(k + 1)$critical. $\square$

We note that under the assumptions of Proposition 7, the graph $G_{[v]}$ is not critical because $G_{[v]} - v^{'} = G$. Thus $\gamma(G_{[v]} - v^{'}) = \gamma(G) = \gamma(G_{[v]})$ and if G is not a $(k - 1)$critical graph, then $G_{[v]}$ is not (k)critical because $\gamma(G_{[v]} - \{v^{'}, x_1, x_2, \ldots, x_{k-1}\} = \gamma(G - \{x_1, x_2, \ldots, x_{k-1}\}) = \gamma(G) = \gamma(G_{[v]})$.

For example, it is easy to see that $C_{12}\langle 1, 4\rangle$ is 4-critical, 4-bicritical ($(4, 2)$critical) and 4-tricritical ($(4, 3)$critical). So $(C_{12}\langle 1, 4\rangle)_{[v0]}$ is $(4, 2)$critical and $(4, 3)$critical but not critical.

4 Coalescence of two graphs

We would like to use a simple construction from [1] that builds a (k)critical graph from two smaller ones. Let F and H be nonempty graphs and let $u \in F$ and $w \in H$ be non-isolated vertices. The coalescence of F and H via u and w, denoted by $(F \cdot H)(u, w : v)$ is the graph obtained from F and H by identifying u and w in a vertex labeled v. See below.

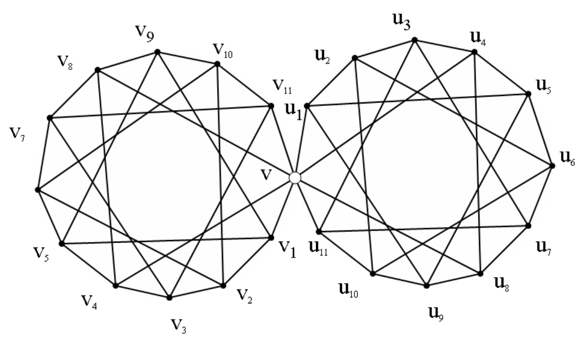


Figure 3: $(C_{12}\langle 1,4\rangle \cdot C_{12}\langle 1,4\rangle)(v_0, u_0 : v)$

Proposition 8 *Let G be a coalescence of two graphs F and H. Then G is critical, bicritical, tricritical, ..., (k)critical if and only if both F and H are critical, bicritical, tricritical, ..., (k)critical.*

Proof. By Propositions 16 and 18 of [2], we have G is critical and bicritical. Furthermore, $\gamma(G) = \gamma(F) + \gamma(H) - 1$. First we show G is (k)critical if F and H are critical, bicritical, tricritical, ..., (k)*critical*. Let $x_1, x_2, \ldots, x_k \in V(G)$ such that i vertices $x_1, x_2, \ldots, x_i \in V(F)$ and $k-i$ vertices $x_{i+1}, x_{i+2}, \ldots, x_k \in V(H)$. If $i \geq 1$, then there is a dominating set D_F of $F - \{x_1, x_2, \ldots, x_i\}$ such that $|D_F| \leq \gamma(F) - 1$ and there is a dominating set D_H of $H - \{x_{i+1}, x_{i+2}, \ldots, x_k, w\}$ such that $|D_H| \leq \gamma(H) - 1$, because F and H are (j)critical for $1 \leq j \leq k$. The set $D_F \cup D_H$ dominates $G - \{x_1, x_2, \ldots, x_k\}$, so $\gamma(G - \{x_1, x_2, \ldots, x_k\}) \leq |D_F| + |D_H| \leq \gamma(G) - 1$. Similarly, it holds if $k - i \geq 1$ or ($i \geq 1$ and $k - i \geq 1$).

For the converse, suppose G is (k)critical. We show that F is (k)critical. By Proposition 16 of [2], $\gamma(G) = \gamma(F) + \gamma(H) - 1$. Let $x_1, \ldots, x_k \in V(F)$. Since G is (k)critical, $\gamma(F) + \gamma(H) - 2 \geq \gamma(G - \{x_1, \ldots, x_k\})$. If $u \in \{x_1, \ldots, x_k\}$, say $u = x_1$, and then since H is critical (by Proposition 18 of [2]), $\gamma(H) + \gamma(F) - 2 \geq \gamma(G - \{x_1, \ldots, x_k\}) = \gamma(F - \{u, x_2, \ldots, x_k\}) + \gamma(H - \{w\}) = \gamma(F - \{x_1, \ldots, x_k\}) + \gamma(H) - 1$. And so $\gamma(F - \{x_1, \ldots, x_k\}) \leq \gamma(F) - 1$. On the other hand, $\{x_1, x_2, \ldots, x_k\} \subset V(F) - \{u\}$. If u is not isolated in $F - \{x_1, \ldots, x_k\}$, then by Lemma 17 of [2] $\gamma(G - \{x_1, \ldots, x_k\}) = \gamma(F - \{x_1, \ldots, x_k\} \cdot H)(u, w : v) \geq \gamma(F - \{\{x_1, \ldots, x_k\}) + \gamma(H) - 1$, and so $\gamma(F - \{x_1, \ldots, x_k\}) \leq \gamma(F) - 1$. Suppose u is isolated in $F - \{x_1, \ldots, x_k\}$. Let $F - \{x_1, \ldots, x_k\} = K \cup \{u\}$. Then $G - \{x_1, \ldots, x_k\} = K \cup H$, and $\gamma(F - \{x_1, \ldots, x_k\}) =$

$\gamma(K)+1$. But then $\gamma(F)+\gamma(H)-2 \geq \gamma(G-\{x_1,\ldots,x_k\}) = \gamma(K)+\gamma(H) = \gamma(F-\{x_1,\ldots,x_k\})-1+\gamma(H)$ and so once again $\gamma(F-\{x_1,\ldots,x_k\}) \leq \gamma(F)-1$. Hence F is (k)critical. Similarly, H is (k)critical. □

As an immediate consequence of Proposition 8, we have.

Corollary 9 *A graph G is critical, bicritical, ..., (k)critical if and only if each block of G is critical , bicritical, ..., (k)critical. Furthermore, if G is critical, bicritical, ..., (k)critical, with blocks $G_1, G_2, \ldots, G_m$, then $\gamma(G) = (\sum_{i=1}^{m}\gamma(G_i)) - m + c(G)$, where $c(G)$ is the number of components of G.*

For example, the graph $(C_{12}\langle 1,4\rangle \cdot C_{12}\langle 1,4\rangle)(v_0, u_0 : v)$ (Figure 3) is $(7,k)$-critical for $k = 1,2,3$, because of $(4,k)$-criticality of the circulant graph $C_{12}\langle 1,4\rangle$.

Definition 10 *Suppose F and H are nonempty graphs. Let u_0, u_1 be two adjacent vertices of F and v_0, v_1 be two adjacent vertices of H. Then $(F\cdot H)(u_0, v_0 : u\ ;\ u_1, v_1 : v)$ denotes the graph obtained from F and H by identifying u_0, v_0 in a vertex labeled u and u_1, v_1 in a vertex labeled v. We call $(F\cdot H)(u_0, v_0 : u\ ;\ u_1, v_1 : v)$ the duality coalescence of F and H.*

The following observation has a simple proof.

Observation 11 *Let u_0, u_1 be two adjacent vertices of F and v_0, v_1 be two adjacent vertices of H, where F and H are distinct nonempty graphs, and let $G = (F\cdot H)(u_0, v_0 : u\ ;\ u_1, v_1 : v)$ be a duality coalescence of F and H. Then $\gamma(F)+\gamma(H)-2 \leq \gamma(G) \leq \gamma(F)+\gamma(H)$.*

Proposition 12 *Let G be a connected duality coalescence of two graphs F and H. Suppose that both F and H are critical, bicritical, tricritical, ..., (k)critical, where D is a $\gamma(G)$-set.*

1)If u or $v \notin D$ then G is (k)critical.

2) If u and $v \in D$ then G is not (k)critical.

Proof. Let $G = (F\cdot H)(u_0, v_0 : u$ and $u_1, v_1 : v)$.

1) If u and $v \notin D$, we say $D \cap V(F) = D_F$ and $D \cap V(H) = D_H$ where D_F and D_H are dominating sets for F and H respectively. So $\gamma(G) = |D| = |D_F| + |D_H| = \gamma(F)+\gamma(H)$. If $u \in D$ and $v \notin D$, we say $D_H = (V(H)\cap D - \{u\}) \cup \{v_0\}$ and $D_F = (V(F)\cap D - \{u\} \cup \{u_0\})$ where D_F and D_H are dominating sets for F and H. So $\gamma(G) = |D| = |D_F| + |D_H| - 1 = \gamma(F)+\gamma(H)-1$.

Let F and H be (j)critical for j, $1 \leq j \leq k$. Let $x_1, x_2, \ldots, x_k \in V(G)$. One can assume that $x_1, x_2, \ldots, x_i \in F$ and $x_{i+1}, x_{i+2} \ldots, x_k \in H$. Let $i \geq 1$, so

$$\begin{aligned}
&\gamma(G-\{x_1,x_2,\ldots,x_k\}) \\
&= \gamma(F-\{x_1,x_2,\ldots,x_i\}) + \gamma(H-\{x_{i+1},x_{i+2}\ldots,x_k,v_0,v_1\}) \\
&\leq \gamma(F)-1+\gamma(H)-1 \\
&= \gamma(F)+\gamma(H)-2 \\
&\leq \gamma(G)-1.
\end{aligned}$$

Similarly, it holds if $k-i \geq 1$ or ($i \geq 1$ and $k-i \geq 1$).

2) If $u, v \in D$, we have $D_F = (V(F) \cap D - \{u, v\}) \cup \{u_0, u_1\}$ and $D_H = (V(H) \cap D - \{u, v\}) \cup \{v_0, v_1\}$. So $\gamma(G) = |D| = |D_F| + |D_H| - 2$. We show G is not (k)critical. By Observation 3, there is no vertex of degree j, $(1 \leq j \leq k$ in F and H. Let x be a vertex of degree $m \geq k+1$ and $\{x_1, x_2, \ldots, x_k\} \subseteq N(x)$ in F. Then $\gamma(G - \{x_1, x_2, \ldots, x_k\}) = \gamma(F - \{x_1, x_2, \ldots, x_k\}) + \gamma(H - \{v_0, v_1\}) = \gamma(F) - 1 + \gamma(H) - 1 = \gamma(F) + \gamma(H) - 2 = \gamma(G)$, a contradiction. So G is not (k)critical. $\square$

The following example explains part 2 of Proposition 12.

Example 2. Let $F = K_4 \square K_4$ and $H = K_4 \square K_4$ with vertices $\{v_{ij} \mid i, j = 1, 2, 3, 4\}$ and $\{u_{ij} \mid i, j = 1, 2, 3, 4\}$, respectively. The graph $K_4 \square K_4$ is (4)critical. Let $G = (F \cdot H)(v_{41}, u_{41} : u\ ;\ v_{43}, u_{43} : v)$ be a duality coalescence of two graphs F and H. We have $\gamma(G) = 6$ and $\gamma(G - \{v_{11}, v_{12}, v_{13}, v_{14}\}) = 6$, which shows that G is not (4)critical, a contradiction.

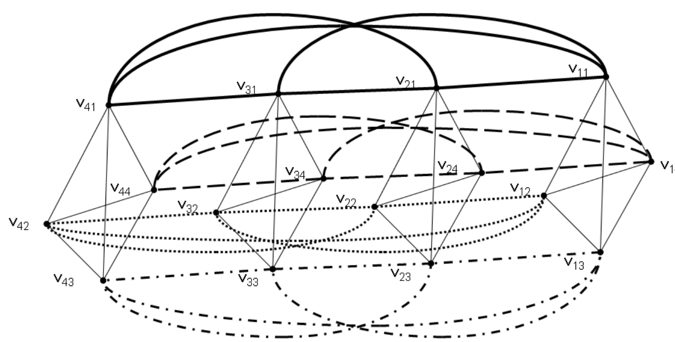


Figure 4: $H = K_4 \square K_4 = F$

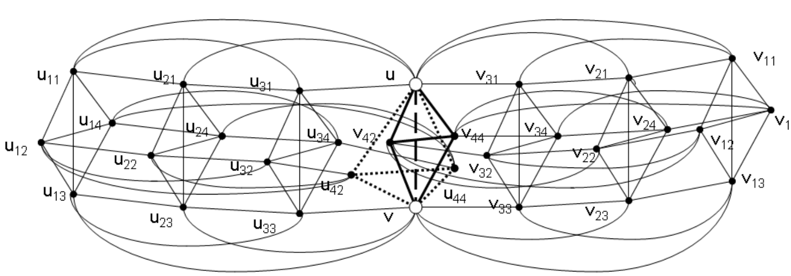


Figure 5: duality coalescence $G = F \cdot H$

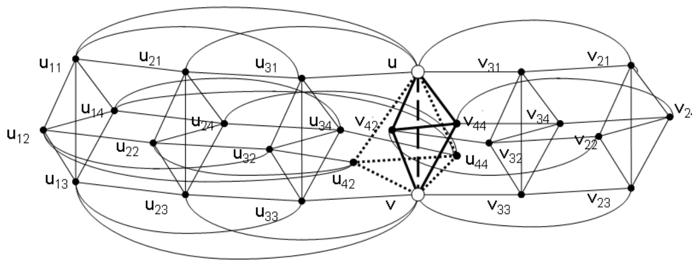


Figure 6: $G - \{v_{11}, v_{12}, v_{13}, v_{14}\}$

5 $(3, k)$-critical graphs

In [2], Brigham et al. showed that if G is a connected 3-bicritical ($(3, 2)$-critical) graph, then $\kappa(G) \geq 3$. But here we show that this result cannot be generalized to connected 3-(k)critical ($(3, k)$-critical) graphs for odd $k \geq 3$. See the following.

Example 3. Let H be a complete graph K_{2n} minus a perfect matching $\{v_{2i-1}v_{2i} | \ 1 \leq i \leq n\ \}$ where $n \geq 1$. Then H is $(2, k)$-critical for odd k, $1 \leq k \leq 2n - 1$. Since k is odd, then $H - \{x_1, \ldots, x_k\}$ has a vertex v_{2i} such that $v_{2i-1} = x_j$ or has a vertex v_{2l-1} such that $v_{2l} = x_j$. So v_{2i} or v_{2l-1} dominates $H - \{x_1, \ldots, x_k\}$.

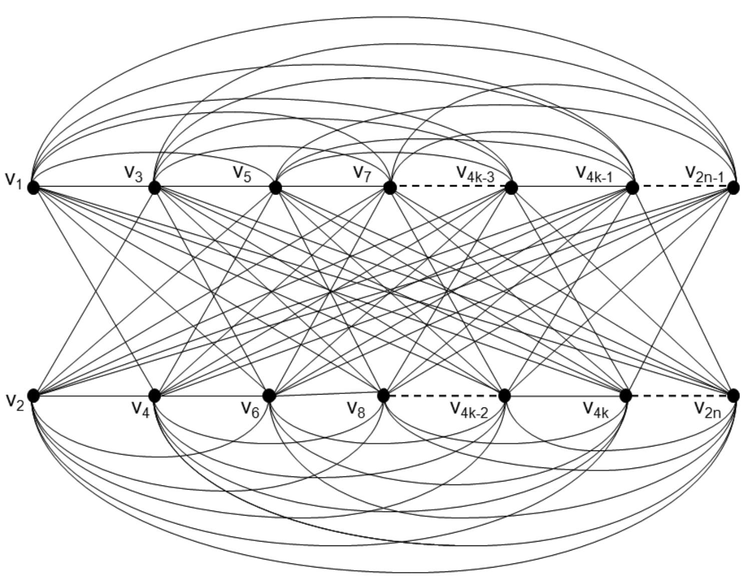


Figure 7: $H = K_{2n} - \{v_{2i-1}v_{2i} | \ 1 \leq i \leq n\}$

Proposition 13 *Let H be a complete graph K_{2n} minus a perfect matching $\{v_{2i-1}v_{2i} \mid 1 \leq i \leq n\}$, where $n \geq 2k$ for odd $k \geq 3$ and G be a graph with vertices $V(H) \cup \{a, b, c\}$ and with edges $E(H) \cup \{ab, ac, bv_1, bv_5, \ldots, bv_{4k-3}, cv_4, cv_8, \ldots cv_{4k}\}$. Then G is $(3, k)$-critical with $\kappa(G) = 2$.*

Proof. It is easy to see that G is connected, $\gamma(G) = 3$ and $\{b, c\}$ is a vertex cut set. Now we show that G is (k)critical graph. If the k vertices are in $V(H)$ then $G - \{x_1, \ldots, x_k\} = \langle H - \{x_1, \ldots, x_k\} \cup \{a, b, c\}\rangle$. Since k is odd, one vertex dominates $V(H)\backslash\{x_1, \ldots, x_k\}$ and the vertex a dominates $\{b, c\}$. If $\{x_2, \ldots, x_k\} \subseteq H$ and $x_1 = a$ then we choose two vertices v_i and v_j such that v_ib and v_jc are in $E(G - \{x_1, \ldots, x_k\})$. If $\{x_2, \ldots, x_k\} \subseteq H$ and $x_1 = b$, or $\{x_3, \ldots, x_k\} \subseteq H$ and $x_1 = a, x_2 = b$, we choose two vertices v_{2j-1} and c such that $v_{2j}c$ is in $E(G - \{x_1, \ldots, x_k\})$. If $\{x_3, \ldots, x_k\} \subseteq H$ and $x_1 = c, x_2 = b$, we choose two vertices v_{2j-1} and a or (v_{2i} and a) such that $v_{2j} \in \{x_3, \ldots, x_k\}$ or ($v_{2i-1} \in \{x_3, \ldots, x_k\}$). Finally, if $\{x_4, \ldots, x_k\} \subseteq H$ and $x_1 = a, x_2 = b, x_3 = c$, we choose two vertices v_{2j} and v_{2j-1} such that $\{v_{2j}, v_{2j-1}\} \cap \{x_4, \ldots, x_k\} = \emptyset$. Thus $\gamma(G - \{x_1, \ldots, x_k\}) = 2 = \gamma(G) - 1$. □

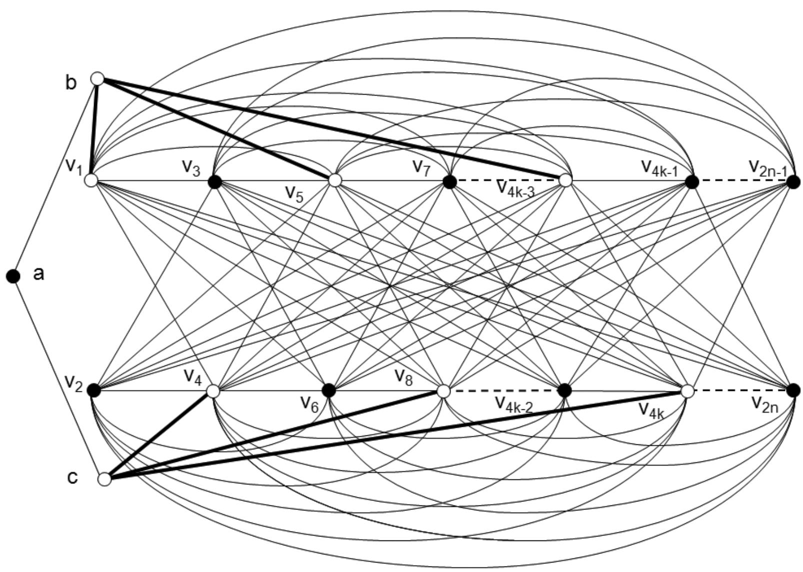


Figure 8: $G = \langle H \cup \{a, b, c\}\rangle$

Questions.

We close this paper with two questions. The first question is a generalization of Proposition E, and the second one is a problem to generalize Example 3 for even $k \geq 4$.

1. If G is a regular connected (γ, k)-critical graph, is it true that
$$n \leq (\Delta(G) + 1)(\gamma(G) - 1) + k - 1\,?$$
2. Let $k \geq 4$ be even. For a $(3, k)$-critical graph, is $\kappa(G) \geq 3$?

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