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A note on partial list coloring

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Abstract

Albertson, Grossman and Haas in [*Discrete Math.* **214** (2000), 235–240] conjecture that if $\mathcal{L}$ is a t-list assignment for a graph G and $1 \leq t \leq \chi_\ell(G)$, then at least $\frac{t|V(G)|}{\chi_\ell(G)}$ vertices of G can be colored from these lists where $\chi_\ell(G)$ is the list chromatic number of G. In this note we investigate the partial list coloring conjecture. Precisely, we show that the conjecture is true for at least half of the numbers in the set $\{1, 2, \ldots, \chi_\ell(G) - 1\}$. In addition, we introduce a new related conjecture and finally we present some results about this conjecture.

1 Introduction and Preliminaries

In this note we only consider simple graphs which are finite and undirected, with no loops or multiple edges. We mention some of the definitions which are referred to throughout the note. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For each vertex $v \in V(G)$ let $\mathcal{L}(v)$ be a list of allowed colors assigned to v. The collection of all lists is called a *list assignment* and is denoted by $\mathcal{L}$. We have a *t-list assignment* if $|\mathcal{L}(v)| = t$ for all $v \in V(G)$. Also, we call $R(\mathcal{L}) = \bigcup_{v \in V(G)} \mathcal{L}(v)$ the *color list* of $\mathcal{L}$.

The graph G is called $\mathcal{L}$*-list colorable* if there is a coloring $c : V(G) \to R(\mathcal{L})$ such that $c(v) \neq c(u)$ for all $uv \in E(G)$ and $c(v) \in \mathcal{L}(v)$ for all $v \in V(G)$. Moreover, G is k*-choosable* if it is $\mathcal{L}$-list colorable for every k-list assignment $\mathcal{L}$. The *list chromatic number* or *choice number* of G, denoted by $\chi_\ell(G)$ or briefly by χ_ℓ, is the smallest number k such that G is k-choosable. List coloring was introduced independently by Vizing [7] and by Erdos, Rubin and Taylor [4].

Also, the notation $\lambda_{\mathcal{L}}(G)$ stands for the maximum number of vertices of G which are colorable with respect to the list assignment $\mathcal{L}$. Moreover, set $\lambda_t(G) \stackrel{\text{def}}{=} \min \lambda_{\mathcal{L}}(G)$,

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where the minimum is taken over all t-list assignments $\mathcal{L}$. For other necessary definitions and notation we refer the reader to the text [2].

Clearly, if $t \geq \chi_\ell(G)$, then $\lambda_t(G) = |V(G)|$. It is therefore of interest to know about $\lambda_t(G)$ when $t < \chi_\ell(G)$.

Conjecture A. (Albertson, Grossman and Haas [1]) Let G be a graph with n vertices. Then, for any $1 \leq t \leq \chi_\ell(G) = \chi_\ell$ we have $\lambda_t(G) \geq \frac{tn}{\chi_\ell}$.

We call this conjecture *AGH conjecture*. The conjecture is clearly correct for $t = 1$ and $t = \chi_\ell(G)$. Albertson et al. [1] proved the following theorem.

Theorem A. *Let G be a graph with n vertices and $1 \leq t \leq \chi_\ell(G)$. Then $\lambda_t(G) \geq (1 - (1 - \frac{1}{\chi(G)})^t)n$, and this number is asymptotically best possible.*

Furthermore, Chappell [3] found the lower bound $\frac{6}{7}\frac{t|V(G)|}{\chi_\ell(G)}$ for $\lambda_t(G)$ for all t between 1 and $\chi_\ell(G)$. In addition, Haas et al. [5] showed that the conjecture holds when $t|\chi_\ell(G)$. For more information see also [6] and [8].

In the next section, we find some new results which show that *AGH conjecture* is true for at least half of the numbers of the set $\{1, 2, \ldots, \chi_\ell(G) - 1\}$. Finally, we introduce a new conjecture and present some results about this conjecture.

2 Main Results

Here we prove a simple and useful inequality about λ_t where $1 \leq t \leq \chi_\ell$.

Theorem 1. (Triangle Inequality) *Let G be a graph and $1 \leq r, s \leq \chi_\ell(G)$. Then*

$$\lambda_r(G) + \lambda_s(G) \geq \lambda_{r+s}(G).$$

Proof. To prove the assertion, it is sufficient to introduce an $(r+s)$-list assignment $\mathcal{L}_{r+s}$ such that $\lambda_r(G) + \lambda_s(G) \geq \lambda_{\mathcal{L}_{r+s}}(G)$. To see this, let $\mathcal{L}_r$ and $\mathcal{L}_s$ be r-list and s-list assignments of G respectively, such that $\lambda_{\mathcal{L}_r}(G) = \lambda_r(G)$, $\lambda_{\mathcal{L}_s}(G) = \lambda_s(G)$, and $R(\mathcal{L}_r) \cap R(\mathcal{L}_s) = \varnothing$. Define an $(r+s)$-list assignment $\mathcal{L}_{r+s}$ as follows:

$$\mathcal{L}_{r+s}(v) := \mathcal{L}_r(v) \cup \mathcal{L}_s(v).$$

Now suppose that $c : V(G) \to R(\mathcal{L}_{r+s})$ is a partial list coloring of G such that $\lambda_{\mathcal{L}_{r+s}}(G)$ vertices of G are colored. Divide the colored vertices of G to the sets $R = \{v \in V(G)|c(v) \in \mathcal{L}_r(v)\}$ and $S = \{v \in V(G)|c(v) \in \mathcal{L}_s(v)\}$. This partition gives $|R| \leq \lambda_{\mathcal{L}_r}(G)$ and $|S| \leq \lambda_{\mathcal{L}_s}(G)$. Finally,

$$\lambda_{r+s}(G) \leq \lambda_{\mathcal{L}_{r+s}}(G) = |R| + |S| \leq \lambda_{\mathcal{L}_r}(G) + \lambda_{\mathcal{L}_s}(G) = \lambda_r(G) + \lambda_s(G).$$

■

The following corollary is a direct consequence of Theorem 1.

Corollary 1. *Let G be a graph and $n, k_i, r_i, r, s \in \mathbb{N}$ when $1 \leq i \leq n$.*

1. *If* $s = \sum_{i=1}^{n} k_i r_i$*, then* $\sum_{i=1}^{n} k_i \lambda_{r_i}(G) \geq \lambda_s(G)$*. In particular, if* $n = 1$, $r_1 = r$ *and* $k_1 = k$*, then we have* $k\lambda_r(G) \geq \lambda_{kr}(G)$*.*

2. *If* $1 \leq r, s \leq \chi_\ell(G)$ *and* $r|s$*, then* $\frac{s}{r}\lambda_r(G) \geq \lambda_s(G)$ *or* $\frac{\lambda_r(G)}{r} \geq \frac{\lambda_s(G)}{s}$*.*

3. [5] *The* AGH conjecture *holds when* $r|\chi_\ell(G)$*.*

4. [5] *If* $1 \leq r \leq \chi_\ell(G) = s$*, then* $\lambda_r(G) \geq \frac{|V(G)|}{\lceil \frac{s}{r} \rceil}$*.*

As a consequence of Theorem 1, one can prove that the *AGH conjecture* holds for at least half of the numbers in the set $\{1, 2, \ldots, \chi_\ell(G) - 1\}$.

Corollary 2. *Let* G *be a graph with* n *vertices and* $1 \leq r \leq \chi_\ell(G) = s$*. Then, the* AGH conjecture *is true for* r *or* $s - r$*. In other words, at least one of the following inequalities holds:*

$$\lambda_r(G) \geq \frac{rn}{s}, \qquad \lambda_{s-r}(G) \geq \frac{(s-r)n}{s}.$$

Proof. On the contrary, assume that $\lambda_r(G) < \frac{rn}{s}$ and $\lambda_{s-r}(G) < \frac{(s-r)n}{s}$. Then

$$\lambda_r(G) + \lambda_{s-r}(G) < \frac{rn}{s} + \frac{(s-r)n}{s} = n.$$

Furthermore, in view of Theorem 1 we have $\lambda_r(G) + \lambda_{s-r}(G) \geq \lambda_{r+s-r}(G) = \lambda_s(G)$. So we conclude that $\lambda_s(G) < n$, a contradiction. ■

Similar to Corollaries 1 and 2, we can prove the following corollary.

Corollary 3. *Let* G *be a graph and* $n, k_i, r_i, r, s \in \mathbb{N}$ *when* $1 \leq i \leq n$*.*

1. *Suppose that* $s = k_1 r_1 + k_2 r_2 + \cdots + k_n r_n$*. If the* AGH conjecture *is true for* s*, then it holds for at least one element of* $\{r_i \mid 1 \leq i \leq n\}$*.*

2. *If* $\chi_\ell(G) = k_1 r_1 + k_2 r_2 + \cdots + k_n r_n$*, then the* AGH conjecture *holds for at least one element of* $\{r_i \mid 1 \leq i \leq n\}$*.*

3. *If* $r|s$ *and the* AGH conjecture *is true for* s*, then it holds for* r*.*

Now, we give a new conjecture which is a generalization of the *AGH conjecture*:

Conjecture 1. *Let* G *be a graph and* $1 \leq r \leq s \leq \chi_\ell(G)$*. Then*

$$\frac{\lambda_r(G)}{r} \geq \frac{\lambda_s(G)}{s}.$$

Remark.

1. Part two of Corollary 2 shows that Conjecture 1 holds when $r|s$.
2. If $s = \chi_\ell(G)$, then $\lambda_s = |V(G)|$. So the *AGH conjecture* is a special case of Conjecture 1.
3. Suppose that $r = 1$. Then, $r|s$ and we have $\lambda_s(G) \leq s\lambda_1(G) = s\alpha(G)$ where $\alpha(G)$ is the independence number of G.

Hereafter, we investigate Conjecture 1.

Definition 1. Let G be a graph. The set of all ordered pairs (r, s) of positive integers with $r \leq s$ such that $\frac{\lambda_r(G)}{r} \geq \frac{\lambda_s(G)}{s}$, is denoted by $Td(G)$.

One can deduce the following theorem whose proof is almost identical to those of Corollaries 1, 2 and 3, and the proof is omitted for the sake of brevity.

Theorem 2. *Let G be a graph, $1 \leq r \leq s \leq \chi_\ell(G)$ and $n, k_i, r_i, r, s \in \mathbb{N}$ when $1 \leq i \leq n$.*

1. *We have $(r, s) \in Td(G)$ or $(s - r, s) \in Td(G)$. So Conjecture 1 is true for at least half of the elements of the set $\{(r, s) \mid 1 \leq r \leq s \leq \chi_\ell(G)\}$.*
2. *If $s = k_1r_1 + k_2r_2 \cdots + k_nr_n$, then $(r_i, s) \in Td(G)$ for some $i \in \{1, 2, \ldots, n\}$.*
3. *In view of Corollary 1, we have $\lceil \frac{s}{r} \rceil \lambda_r(G) \geq \lambda_s(G)$. In other words $\alpha \frac{\lambda_r(G)}{r} \geq \frac{\lambda_s(G)}{s}$, where $\alpha = \frac{r}{s} \lceil \frac{s}{r} \rceil$.*
4. *The set $Td(G)$ induces an order relation on $\{1, 2, \ldots, \chi_\ell(G)\}$.*

Definition 2. Let H be a subgraph of the graph G, $\mathcal{L}'$ be a list assignment of H and $\mathcal{L}$ be a list assignment of G. We say $\mathcal{L}'$ is a *restriction of $\mathcal{L}$ to H* if $\mathcal{L}'(v) = \mathcal{L}(v)$ for all $v \in V(H)$ and we show that by $\mathcal{L}' = \mathcal{L} \mid_H$.

The following lemma is a straightforward consequence of Definition 2.

Lemma 1. *Let H be an induced subgraph of the graph G and $1 \leq t \leq \chi_\ell(G)$. Then $\lambda_t(G) \geq \lambda_t(H)$.*

Now we can prove the following theorem about Conjecture 1, which is similar to Theorem A.

Theorem 3. *Let G be a graph and $1 \leq r \leq s \leq \chi_\ell(G)$. Also, suppose that $\mathcal{L}_s$ is an s-list assignment of G with $\mathcal{L}_s(v) = \{1, 2, \ldots, s\}$ for all $v \in V(G)$. Then*

$$\lambda_r(G) \geq (1 - (1 - \frac{1}{s})^r)\lambda_{\mathcal{L}_s}(G).$$

Proof. We can color at most $\lambda_{\mathcal{L}_s}(G)$ vertices of G from the list color $\{1, 2, \dots, s\}$. Let H be an induced subgraph of G with $|V(H)| = \lambda_{\mathcal{L}_s}(G)$ such that the vertices of H are colorable with respect to the list assignment $\mathcal{L}_s$. At first we show that $\chi(H) = s$. Suppose that $\chi(H) \leq s - 1$. So there is a coloring of $V(H)$ by at most $s - 1$ colors of $\{1, 2, \dots, s\}$. Thus we can assign a color to at least one vertex of $V(G)\backslash V(H)$ from the remaining colors such that the coloring remains proper; this contradicts the definition of H. So $\chi(H) = s$. Considering Theorem 1 we have

$$\lambda_r(H) \geq (1 - (1 - \frac{1}{\chi(H)})^r)|V(H)| = (1 - (1 - \frac{1}{s})^r)\lambda_{\mathcal{L}_s}(G).$$

Also, Lemma 1 yields $\lambda_r(G) \geq \lambda_r(H)$. So $\lambda_r(G) \geq (1 - (1 - \frac{1}{s})^r)\lambda_{\mathcal{L}_s}(G)$. ■

In view of Theorem 3 and by use of $\lambda_{\mathcal{L}_s}(G) \geq \lambda_s(G)$, one can easily conclude the following corollary, which is similar to the result of Theorem A.

Corollary 4.

$$\lambda_r(G) \geq (1 - (1 - \frac{1}{s})^r)\lambda_s(G).$$

Definition 3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two list assignments of G. We say $\mathcal{L}_1 \subseteq \mathcal{L}_2$ if $\mathcal{L}_1(v) \subseteq \mathcal{L}_2(v)$ for all $v \in V(G)$.

The next theorem shows the relation between a partial list coloring of a graph and its subgraphs.

Theorem 4. *Let G be a graph and $1 \leq t \leq \chi_\ell(G)$.*

(1) *Suppose that H is a subgraph of G with maximum number of vertices such that $\chi_\ell(H) = t$. Then, $\lambda_t(G) \geq |H|$.*

(2) *Suppose that $\lambda_t(G) = \lambda_{\mathcal{L}_t}(G)$ where $\mathcal{L}_t$ is a t-list assignment of G. If H is a subgraph of G of order $\lambda_t(G)$, induced by colored vertices in the partial coloring of G by $\mathcal{L}_t$, then $\chi_\ell(H) \geq t$.*

Proof. (1) Suppose, for contradiction, that $\lambda_t(G) \leq |H| - 1$. Also, suppose that $\lambda_t(G) = \lambda_{\mathcal{L}_t}(G)$ where $\mathcal{L}_t$ is a t-list assignment of G. Thus in the coloring of G from the lists of $\mathcal{L}_t$, we can color fewer than $|H|$ vertices. But $\chi_\ell(H) = t$. By restriction of $\mathcal{L}_t$ to H, all vertices of H can be colored. This is a partial list coloring of G which contradicts with $\lambda_{\mathcal{L}_t}(G) < |H|$. Therefore $\lambda_t(G) \geq |H|$.

(2) On the contrary, assume that $\chi_\ell(H) = s \leq t - 1$. Thus for any s-list assignment $\mathcal{L}_s$ of H, there is a proper coloring of H by the lists of $\mathcal{L}_s$. Also, consider the t-list assignment $\mathcal{L}_t$ and select an arbitrary vertex x of $G\backslash H$. Choose an arbitrary color $c \in \mathcal{L}_t(x)$ and assign the color c to the vertex x. Now, since H is s-choosable, one can see that H is $\mathcal{L}'$-colorable where $\mathcal{L}' = \mathcal{L}\backslash\{c\}$. So we can color $|H| + 1$ vertices of G by lists of $\mathcal{L}_t$, a contradiction. Hence $\chi_\ell(H) \geq t$. ■

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