AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **45** (2009), Pages 97–107

Unicyclic radially-maximal graphs on the minimum number of vertices

Martin Knor

Slovak University of Technology
Faculty of Civil Engineering
Department of Mathematics
Radlinského 11, 813 68 Bratislava
Slovakia
knor@math.sk

Abstract

We characterize unicyclic, non-selfcentric, radially-maximal graphs on the minimum number of vertices. Such graphs must have radius $r \geq 5$, and we prove that the number of these graphs is $\frac{1}{48}r^3 + O(r^2)$.

1 Introduction and results

We say that a graph G is **radially-maximal** if adding of any edge from its complement decreases its radius, i.e., if $\mathrm{rad}(G \cup e) < \mathrm{rad}(G)$ for every edge e from $\overline{G}$. A graph is **selfcentric** if its radius equals its diameter, otherwise it is **non-selfcentric**.

Obviously, for every r there is a radially-maximal graph of radius r, as can be shown by complete graphs (in the case $r = 1$) and even cycles (in the case $r > 1$). Both complete graphs and cycles are selfcentric graphs. One may expect that a graph is radially-maximal if it is either a very dense or a balanced (highly symmetric) one. Therefore, it is interesting that for $r \geq 3$ there are non-selfcentric radially-maximal graphs of radius r which are planar. Such graphs are neither symmetric nor dense. In fact, in [1] we have the following conjecture:

Conjecture A *Let G be a non-selfcentric radially-maximal graph with radius $r \geq 3$ on the minimum number of vertices. Then we have*

(a) G has exactly $3r - 1$ vertices;
(b) G is planar;
(c) the maximum degree of G is 3 and the minimum degree of G is 1.

Conjecture A deals with graphs on the minimum number of vertices, since from these one can easily obtain larger ones (see the node-extension in [1]). This conjecture was proved for the case $r = 3$, see [1], and by a computer also for $r = 4$ and 5, see [4]. For $r = 3$, 4 and 5 there are exactly 2, 8 and 22 graphs, respectively, satisfying

Conjecture A. And among the 22 graphs of radius 5 there is one which is unicyclic, see Figure 1. In this paper we present a characterization of unicyclic non-selfcentric radially-maximal graphs on the minimum number of vertices. This characterization is based on the graph depicted in Figure 1.

Definition. Let z be a vertex of degree 3 in a graph G. By $Y_G(z)$ (or by $Y(z)$ when no confusion is likely) we denote a graph operation consisting of subdividing all edges incident with z, each by one vertex.

In the graph in Figure 1, let us denote the vertices of degree 3 by z_1, z_2, z_3 and z_4. In the following, we use the same names for the vertices of degree 3, before as well as after applying the operation Y. This enables us to apply Y several times to a vertex. Now denote by $G_{(a,b,c,d)}$ a graph obtained from the one in Figure 1 by applying a times $Y(z_1)$, b times $Y(z_2)$, c times $Y(z_3)$, and d times $Y(z_4)$. Then the graph in Figure 1 is $G_{(0,0,0,0)}$. We have

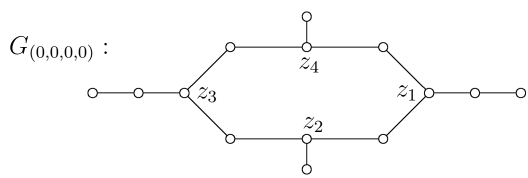

Figure 1

Theorem 1 *For every four-tuple of non-negative integers* (a, b, c, d)*, the graph* $G_{(a,b,c,d)}$ *is unicyclic, non-selfcentric and radially-maximal. Its order is* $3r - 1$*, its radius is* $r = a+b+c+d+5$ *and its unique cycle has length* $2(a+b+c+d)+8 = 2r-2$*. Moreover, its central subgraph contains exactly four edges.*

The following theorem complements Theorem 1.

Theorem 2 *The graphs* $G_{(a,b,c,d)}$ *are the only unicyclic, non-selfcentric radially-maximal graphs on the minimum number of vertices.*

Hence Conjecture A is true in the class of unicyclic graphs.

Theorems 1 and 2 characterize unicyclic non-selfcentric radially-maximal graphs on the minimum number of vertices. However, there are unicyclic non-selfcentric radially-maximal graphs on more than $3r - 1$ vertices, where r is the radius. One of these graphs can be obtained from $G_{(0,0,0,0)}$. The graph $G_{(0,0,0,0)}$ consists of two parts (each having 7 vertices), which are glued together to form a cycle. If one takes three such parts instead of two, then the resulting graph is unicyclic, radially-maximal of radius 7 on 21 vertices. (The fact that this graph is radially-maximal was verified by a computer.)

We conclude with an estimation of the number of graphs $G_{(a,b,c,d)}$ of radius r.

Corollary 3 *There are* $\frac{1}{48}r^3 + O(r^2)$ *unicyclic non-selfcentric radially-maximal graphs of radius* r *on* $3r - 1$ *vertices.*

2 Proofs

By a $u - v$ geodesic we mean a shortest $u - v$ path in a graph.

Proof of Theorem 1. It is obvious that $G_{(a,b,c,d)}$ is a unicyclic graph with a cycle C of length $8 + 2a + 2b + 2c + 2d$. To this cycle, there are attached four paths. Let us denote their endpoints by u_1, u_2, u_3 and u_4, so that the path starting at z_i terminates at u_i, $1 \le i \le 4$. Then the lengths of paths $z_1 - u_1$, $z_2 - u_2$, $z_3 - u_3$ and $z_4 - u_4$ are $a + 2$, $b + 1$, $c + 2$ and $d + 1$, respectively. In the following when we discuss a subpath $x_1 - x_2$, we always mean a clock-wise subpath of C. Then the subpaths $z_1 - z_2$, $z_2 - z_3$, $z_3 - z_4$ and $z_4 - z_1$ have lengths $a + b + 2$, $b + c + 2$, $c + d + 2$ and $a + d + 2$, respectively, see Figure 2.

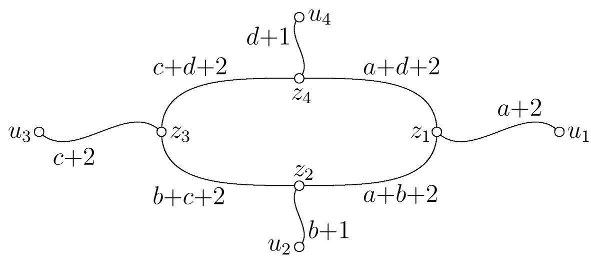

Figure 2

Let us distinguish eight vertices of $G_{(a,b,c,d)}$. By $v_{i,c}$ ($v_{i,a}$) we denote a vertex of C, which is at distance $a+b+c+d+5$ from u_i, and such that when reaching z_i, the $u_i - v_{i,c}$ ($u_i - v_{i,a}$) geodesic continues clockwise (anti-clockwise), see Figure 3. Since the length of C is $2(a + b + c + d) + 8$, the subpath $v_{1,c} - v_{1,a}$ has length

$$2(a + b + c + d) + 8 + 2(a + 2) - 2(a + b + c + d + 5) = 2a + 2.$$

Analogously, the subpaths $v_{2,c} - v_{2,a}$, $v_{3,c} - v_{3,a}$ and $v_{4,c} - v_{4,a}$ have lengths $2b$, $2c+2$ and $2d$, respectively, see Figure 3. Hence, if $b = 0$ then $v_{2,c} = v_{2,a}$ and if $d = 0$ then $v_{4,c} = v_{4,a}$.

The vertices $v_{1,a}$ and $v_{2,c}$ are adjacent, since if we sum the lengths of subpaths $v_{1,a} - z_1$, $z_1 - z_2$ and $z_2 - v_{2,c}$ we obtain $[(a+b+c+d+5) - (a+2)] + (a+b+2) + [(a+b+c+d+5) - (b+1)] = 2(a+b+c+d)+8+1$. This implies that $v_{2,c}$ is on a shortest $v_{1,a} - u_1$ path and $v_{1,a}$ is on a shortest $u_2 - v_{2,c}$ path. Analogously, in a clockwise rotation on C we have the edges $v_{2,a}v_{3,c}$, $v_{3,a}v_{4,c}$ and $v_{4,a}v_{1,c}$, see Figure 3. We remark that z_1 is not necessarily between $v_{3,c}$ and $v_{3,a}$ on C. It can happen that z_1 is between $v_{2,c}$ and $v_{2,a}$ or between $v_{4,c}$ and $v_{4,a}$. For this reason z_1, z_2, z_3 and z_4 are not depicted in Figure 3.

Every interior vertex of the subpath $v_{1,c} - v_{1,a}$ has distance from u_1 greater than $a + b + c + d + 5$. Also, every interior vertex of the subpath $v_{1,a} - v_{1,c}$ has distance from u_1 less than $a + b + c + d + 5$. When analogous considerations are applied for u_2, u_3 and u_4, one can see that $v_{1,c}$, $v_{1,a}$, $\dots$, $v_{4,c}$, $v_{4,a}$ are the only central vertices of

$G_{(a,b,c,d)}$. Since the length of subpath $v_{i,c}{-}v_{i,a}$ is even, $1 \le i \le 4$, the central subgraph of $G_{(a,b,c,d)}$ contains exactly four edges, namely $v_{1,a}v_{2,c}$, $v_{2,a}v_{3,c}$, $v_{3,a}v_{4,c}$ and $v_{4,a}v_{1,c}$. Moreover, by our previous analysis, both $v_{i,c}$ and $v_{i,a}$ have a unique vertex at distance $a{+}b{+}c{+}d{+}5$, namely u_i. As a consequence, the radius of $G_{(a,b,c,d)}$ is $r = a{+}b{+}c{+}d{+}5$. Its number of vertices is $14 + 3(a + b + c + d) = 3(a + b + c + d + 5) - 1 = 3r - 1$.

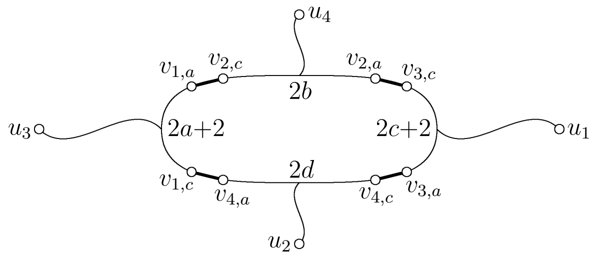

Figure 3

Since any interior vertex of the subpath $v_{1,c}{-}v_{1,a}$ (the length of which is $2a{+}2 \ge 2$) has distance from u_1 greater than $a{+}b{+}c{+}d{+}5$, the graph $G_{(a,b,c,d)}$ is non-selfcentric. Thus, it remains to prove that it is radially-maximal. We have to show that the radius of $G_{(a,b,c,d)}$ decreases after adding of any edge $e = x_1x_2$ from the complement. We proceed by way of contradiction. Suppose that adding of e does not decrease the radius. We distinguish three cases with several subcases each.

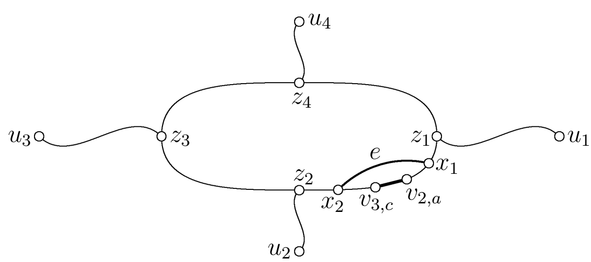

Figure 4

Case (1) Suppose that both endvertices of e are on C.

(1a) Suppose that both x_1 and x_2 are on the subpath $z_1 - z_2$; see Figure 4.

If x_2 is on the subpath $z_3 - v_{3,c}$, then adding of e shortens the $v_{3,c} - u_3$ distance, and hence the radius. Analogously, if x_1 is on the subpath $v_{2,a} - z_2$, then adding of e shortens the $v_{2,a} - u_2$ distance. Therefore, we can assume that the edge $v_{2,a}v_{3,c}$ is on the subpath $x_1 - x_2$. Let us denote the lengths of subpaths $x_1 - v_{2,a}$ and $v_{3,c} - x_2$ by t_1 and t_2, respectively. Then for the very same reason as above we can assume that

$$\begin{aligned} d(v_{2,a}, x_2) &= 1 + t_2 \le d(v_{2,a}, x_1) + 1 \ \le\ t_1 + 1 \quad \text{and} \\ d(v_{3,c}, x_1) &= 1 + t_1 \le d(v_{3,c}, x_2) + 1 \ \le\ t_2 + 1, \end{aligned}$$

which gives $t_1 = t_2$. However, analogously we can obtain that the edge $v_{4,a}v_{1,c}$ is on the subpath $x_1 - x_2$, and in fact that $d(x_1, v_{4,a}) = d(v_{1,c}, x_2)$. Thus, $v_{1,c} = v_{3,c}$ and $v_{4,a} = v_{2,a}$, a contradiction.

(1b) Suppose that x_1 is on the subpath $z_1 - z_2$ and x_2 is on the subpath $z_2 - z_3$.

Analogously as in the previous case the edge $v_{3,a}v_{4,c}$ must be on the subpath $x_1 - x_2$. And analogously as above we obtain that $d(x_1, v_{3,a}) = d(v_{4,c}, x_2)$. Similar considerations for the edge $v_{4,a}v_{1,c}$ yield $v_{4,a} = v_{3,a}$ and $v_{1,c} = v_{4,c}$, a contradiction.

(1c) Suppose that x_1 is on the subpath $z_1 - z_2$ and x_2 is on the subpath $z_3 - z_4$, see Figure 5.

Analogously as above one can see that the edge $v_{2,a}v_{3,c}$ is on the subpath $x_2 - x_1$. Moreover, if the lengths of subpaths $v_{3,c} - x_1$ and $x_2 - v_{2,a}$ are t_1 and t_2, respectively, then $t_1 = t_2$. In a similar way one can see that $v_{4,a}v_{1,c}$ is on the subpath $x_1 - x_2$, and if the lengths of subpaths $v_{1,c} - x_2$ and $x_1 - v_{4,a}$ are l_1 and l_2, respectively, then $l_1 = l_2$. Now consider the position of $v_{1,a}$ and $v_{2,c}$. If $v_{1,a}$ is on the subpath $v_{1,c} - x_2$ then (as $l_1 = l_2$ and $v_{1,c} \neq v_{1,a}$) adding of e decreases the distance from $v_{1,a}$ to u_1. Therefore, $v_{1,a}$ is on the subpath $x_2 - v_{2,a}$. But now $v_{2,c}$ cannot be on the subpath $v_{1,c} - x_2$, so that $v_{2,c} = v_{2,a}$ and $b = 0$ (see Figure 3). Analogously can be shown that $v_{4,c} = v_{4,a}$ and $d = 0$. Now the lengths of subpaths $v_{4,a} - v_{2,c}$ and $v_{2,a} - v_{4,c}$ are $l_1 + t_2$ and $t_1 + l_2$, respectively. As $l_1 + t_2 = t_1 + l_2$ and as the subpaths $z_2 - v_{2,c}$ and $v_{2,a} - z_2$ have the same lengths, we have $z_2 = v_{4,a} = v_{4,c}$ and analogously $z_4 = v_{2,a} = v_{2,c}$. Since the subpaths $z_2 - z_4$ and $z_4 - z_2$ have lengths $2c + 4$ and $2a + 4$, respectively (see Figure 2), we have $a = c$. Thus, all the subpaths $z_1 - z_2$, $z_2 - z_3$, $z_3 - z_4$ and $z_4 - z_2$ have the same length $a + 2$. Since $t_1 = t_2$ and $l_1 = l_2$, we have either $x_2 = z_3$ and z_1x_1 is an edge of C, or $x_1 = z_1$ and z_3x_2 is an edge of C. In the first case adding of e shortens the distance from $v_{3,a}$ to u_3, while in the second one it shortens the distance from $v_{1,a}$ to u_1.

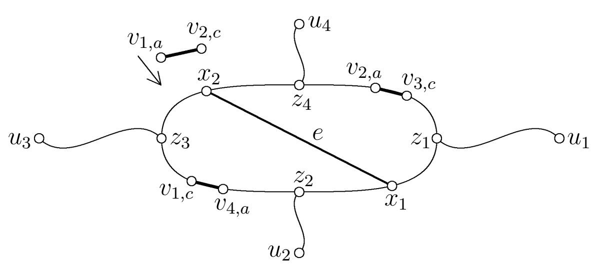

Figure 5

Before proceeding with the other cases, we consider the position of z_1 with respect to the vertices $v_{1,c}, v_{1,a}, \ldots, v_{4,c}$ and $v_{4,a}$. We have mentioned that z_1 is not necessarily on the subpath $v_{3,c} - v_{3,a}$. Obviously, z_1 cannot be on the subpath $v_{1,c} - v_{1,a}$. Thus, z_1 is on the subpath $v_{2,c} - v_{2,a}$ if the length of $v_{2,a} - v_{1,c}$ is not greater than the length of the subpath $z_1 - v_{1,c}$, see Figure 3. This gives

$$1 + (2c + 2) + 1 + 2d + 1 \leq (a + b + c + d + 5) - (a + 2), \text{ i.e., } d + c + 2 \leq b.$$

Analogously, z_1 is on the subpath $v_{4,c} - v_{4,a}$ if $b + c + 2 \le d$. Similar equations hold for the position of vertex z_3. However, with z_2 and z_4 it is slightly different. The vertex z_2 is on the subpath $v_{1,c} - v_{1,a}$ if $c + d + 1 \le a$, and it is on the subpath $v_{3,c} - v_{3,a}$ if $a + d + 1 \le c$.

Case (2) Suppose that x_1 is a vertex of C and x_2 is on $z_1 - u_1$ path. (The case when x_2 is on $z_3 - u_3$ path is symmetric and the cases when x_2 is on $z_2 - u_2$ or $z_4 - u_4$ path are very similar.)

Without loss of generality we may assume that x_1 is on the subpath $z_3 - z_1$. Since adding of e does not decrease the radius, x_1 is inside the subpath $v_{1,c} - v_{1,a}$. Moreover, if we sum the lengths of those paths from u_1 to $v_{1,c}$ and from $v_{1,a}$ to u_1, which contain e, we obtain

$$(2a + 2) + 2 + 2(a + 1) \ge 2(a + b + c + d) + 10, \text{ so that } a \ge b + c + d + 2.$$

(We remark that in the case when x_2 is on $z_2 - u_2$ path, we obtain $b \ge a + c + d + 4$.) This means that $a \ge b + c + 1$ and $a \ge c + d + 1$, so that z_2 and z_4 (and so also z_3) are on the subpath $v_{1,c} - v_{1,a}$, see Figure 6. However, the position of z_1 is not determined yet. Therefore we have three subcases.

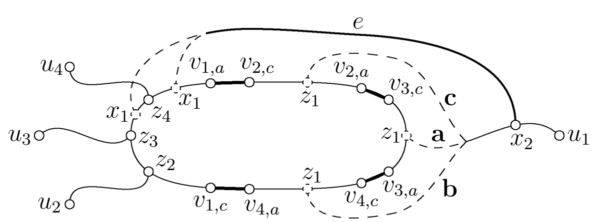

Figure 6

(2a) Suppose that z_1 is on the subpath $v_{3,c} - v_{3,a}$, see Figure 6 **a**.

Denote by t_1 and t_2 the lengths of the subpaths $z_4 - v_{1,a}$ and $z_1 - v_{3,a}$, respectively. We evaluate t_1 and t_2. Since $d(u_3, v_{3,c}) = r = a + b + c + d + 5$ and also $d(u_3, v_{3,c}) = (c+2) + (c+d+2) + t_1 + 1 + 2b + 1$, we have $t_1 = a - b - c - 1$. Since $d(u_1, v_{1,c}) = r = a + b + c + d + 5$ and also $d(u_1, v_{1,c}) = (a+2) + t_2 + 1 + 2d + 1$, we have $t_2 = b + c - d + 1$.

Now if we sum the lengths of those paths from $v_{1,a}$ to u_1 and from $v_{3,a}$ to u_3, which contain e, we obtain

$$[t_1 + (c + d + 2) + (c + 2)] + 2 + [t_2 + (a + 2)] \ge 2r = 2(a + b + c + d) + 10,$$

which gives $0 \ge 2b + 2d + 2$, a contradiction. (In the case when x_2 is on $z_2 - u_2$ path, the constant 2 is replaced by 6.)

Since the proofs of all the other subcases are analogous to (2a), in the next we abbreviate the reasoning. By $d^e(y_1, y_2)$ we denote the length of a shortest $y_1 - y_2$ path containing e.

(2b) Suppose that z_1 is on the subpath $v_{4,c} - v_{4,a}$, see Figure 6 **b**.

Then $t_3 = d(v_{4,c}, z_1) = d-b-c-2$ as $d(v_{1,a}, u_1) = r = 1+2b+1+(2c+2)+1+t_3+(a+2)$. But $2r \le d^e(v_{1,a}, u_1)+d^e(v_{3,a}, u_3) = [t_1+(c+d+2)+(c+2)]+2+[1+t_3+(a+2)]$ gives $0 \ge 4b + 2c + 4$, a contradiction.

(2c) Suppose that z_1 is on the subpath $v_{2,c} - v_{2,a}$, see Figure 6 **c**.

Then $t_4 = d(z_1, v_{2,a}) = b-c-d-2$ as $d(u_1, v_{1,c}) = r = (a+2)+t_4+1+(2c+2)+1+2d+1$. But $2r \le d^e(v_{1,a}, u_1)+d^e(u_3, v_{3,c}) = [t_1+(c+d+2)+(c+2)]+2+[1+t_4+(a+2)]$ gives $0 \ge 2b + 2c + 2d + 4$, a contradiction.

(In cases (2b) and (2c) when x_2 is on $z_2 - u_2$ path, the constant 4 is replaced by 6.)

Case (3) Suppose that e connects vertices outside C. Since the radius trivially decreases if both x_1 and x_2 are on one path attached to C, there are just two cases to consider. First we discuss the case when x_1 is on $z_1 - u_1$ path and x_2 is on $z_4 - u_4$ path. Suppose that $d(z_1, x_1) \ge d(z_4, x_2)$. (The case $d(z_1, x_1) < d(z_4, x_2)$ can be solved similarly.) Since adding of e does not decrease the radius, z_4 must lie inside the subpath $v_{1,c} - v_{1,a}$. Now $2r \le d^e(u_1, v_{1,c}) + d^e(v_{1,a}, u_1) \le (2a+2)+2+2(a+2)$ gives $a \ge b+c+d+1$. (In the case when $d(z_1, x_1) \le d(z_4, x_2)$ we have $d \ge a+b+c+3$.) Consequently, $a \ge b+c+1$ and $a \ge c+d+1$, so that z_2 and z_4 (and so also z_3) are on the subpath $v_{1,c} - v_{1,a}$, see Figure 7. Analogously as above, we have three cases. (Distances t_1, t_2, t_3 and t_4 are defined in Case (2).)

(3a) Suppose that z_1 is on the subpath $v_{3,c} - v_{3,a}$, see Figure 7 **a**.

Then $2r \le d^e(v_{1,a}, u_1) + d^e(u_4, v_{4,c}) = [t_1 + (d+1)] + 2 + [1 + t_2 + (a+2)]$ gives $0 \ge 2b + 2c + 2d + 4$, a contradiction.

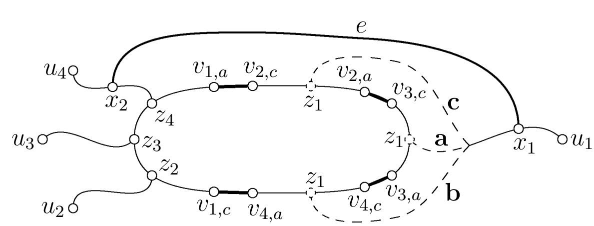

Figure 7

(3b) Suppose that z_1 is on the subpath $v_{4,c} - v_{4,a}$, see Figure 7 **b**.

Then $2r \le d^e(v_{1,a}, u_1) + d^e(u_4, v_{4,c}) = [t_1 + (d+1)] + 2 + [t_3 + (a+2)]$ gives $0 \ge 4b + 4c + 8$, a contradiction.

(3c) Suppose that z_1 is on the subpath $v_{2,c} - v_{2,a}$, see Figure 7 **c**.

Then $2r \le d^e(v_{1,a}, u_1)+d^e(u_4, v_{4,c}) = [t_1+(d+1)]+2+[1+(2c+2)+1+t_4+(a+2)]$ gives $0 \ge 2b + 2c + 2d + 4$, a contradiction.

(In cases (3a), (3b) and (3c) if we have $d(z_1, x_1) \le d(z_4, x_2)$, then the constants 4, 8 and 4 are replaced by 6, 8 and 6, respectively.)

It remains to consider the case when x_1 is on z_1-u_1 path and x_2 is on z_3-u_3 path. (The case when x_1 is on z_2-u_2 path and x_2 is on z_4-u_4 path is similar.) Assume that $d(z_1,x_1) \ge d(z_3,x_2)$. Since adding of e does not decrease the radius, z_3 is on the subpath $v_{1,c}-v_{1,a}$. Then $2r \le d^e(u_1,v_{1,c}) + d^e(v_{1,a},u_1) = (2a{+}2)+2+2(a{+}2)$ gives $a \ge b+c+d+1$. Hence, both z_2 and z_4 are on the subpath $v_{1,c}-v_{1,a}$. But now this case can be solved in the very same way as Case (2). □

In the proof of Theorem 2 we use several former results. By $G-z$ we denote a graph obtained from G by deleting the vertex z and all edges incident with this vertex. Let S be a set of vertices of G (generally $S \neq V(G)$). By $\langle S\rangle$ we denote a subgraph of G induced by the vertices in S. In [1] we have

Theorem B. *Let G be a radially-maximal graph of radius $r \ge 3$ containing a cut-vertex z. Then the graph $G-z$ has exactly two components, say A' and B'. Let $A = \langle V(A') \cup \{z\}\rangle$ and $B = \langle V(B') \cup \{z\}\rangle$, and let the eccentricities of z satisfy $e_A(z) \ge e_B(z)$. Then $e_A(z) \ge r$, $e_B(z) \le r-2$, and B is a diametrically-maximal graph with diameter $e_B(z)$.*

Recall that a graph is diametrically-maximal if its diameter decreases after adding of any edge from its complement. These graphs have been characterized by Ore in [5]:

Theorem C. *A graph with diameter d is diametrically-maximal if and only if it has form $K_1 + K_{a_1} + K_{a_2} + \cdots + K_{a_{d-1}} + K_1$ for some positive integers $a_1, a_2, \ldots, a_{d-1}$.*

Here K_n denotes a complete graph on n vertices, and $G_1 + G_2 + \cdots + G_l$ arises from $G_1 \cup G_2 \cup \cdots \cup G_l$ by adding edges uv, with $u \in V(G_i)$ and $v \in V(G_{i+1})$, $1 \le i \le l{-}1$.

A cycle C in G is **geodesic**, if for any two vertices of C their distance on C equals their distance in G. In [2] Haviar, Hrnčiar and Monoszová proved:

Theorem D. *Let G be a graph with radius r, diameter $d \le 2r-2$, on at most $3r-2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r+1$.*

In [3] we have:

Lemma E. *Let G be a radially-maximal graph of radius r and diameter d. Then $d \le 2r-2$.*

Theorem D and Lemma E are used to prove the (a) part of Conjecture A for unicyclic graphs.

Proof of Theorem 2. Let G be a unicyclic non-selfcentric radially-maximal graph of radius r on the minimum number of veritices. Let C be its unique cycle. Since G is non-selfcentric, it contains at least one vertex outside C. Therefore, G contains cut-vertices. Denote by $z_1, z_2, \ldots, z_k$ all cut-vertices lying on C. (We remark that in all this proof, when considering subpaths, then we always mean clock-wise subpaths of C. Therefore, $z_1, z_2, \ldots, z_k$ determine a clockwise rotation of C.) By Theorem B, $G-z_i$ has exactly two components, say A'_i and B'_i. Denote $A_i = \langle V(A'_i) \cup z_i\rangle$, $B_i = \langle V(B'_i) \cup z_i\rangle$ and assume that $e_{A_i}(z_i) \ge e_{B_i}(z_i)$. By Theorem B, B_i is a diametrically-maximal graph. Since G has the minimum number of vertices (and is unicyclic), B_i is a path attached to C by its endvertex. Hence, G consists of a cycle C and a collection of paths attached by their endpoints to different vertices of C.

Denote by c the length of C and suppose that $c \geq 2r$. Let x_1 and x_2 be vertices adjacent to z_1, such that $x_1 \in V(C)$ and $x_2 \notin V(C)$. Since adding of x_1x_2 to G does not decrease the distances between the vertices of C, every vertex v of C has a partner u on C such that $d_{G\cup x_1x_2}(u,v) \geq r$. Since G is unicyclic, $r(G \cup x_1x_2) = r$ and G is not radially-maximal, a contradiction. Thus, $c \leq 2r - 1$. By Lemma E and Theorem D, G has at least $3r - 1$ vertices. This (together with Theorem 1) implies that Conjecture A is true in the class of unicyclic graphs. Moreover as $c \leq 2r - 1$, every vertex of C has an eccentric vertex outside C. By Theorem B, central vertices of G must be on C.

Now we introduce notation analogous to the one used in the proof of Theorem 1. At every vertex z_i, there is attached a path to C. Denote by u_i the other endvertex of this path and by l_i its length. Moreover, denote by $v_{i,c}$ and $v_{i,a}$ two vertices of C. Both these vertices are at distance r from u_i, but the $u_i - v_{i,c}$ geodesic contains the subpath $z_i - v_{i,c}$ and the $u_i - v_{i,a}$ geodesic contains the subpath $v_{i,a} - z_i$. Observe that our definition is correct. The reason is that if $2r > 2l_i + c$, then every vertex of C has distance smaller than r to u_i. Therefore adding of x_1x_2 to G, where x_1 and x_2 are neighbours of z_i, $x_1 \in V(C)$ and $x_2 \notin V(C)$, does not decrease the radius, a contradiction. Thus, in the worst case, when $2r = 2l_i + c$, we have $v_{i,c} = v_{i,a}$.

Now suppose that there are i and j such that $v_{j,c}$ (the case of $v_{j,a}$ is symmetric) is a vertex of the subpath $v_{i,c} - v_{i,a}$. Denote by x_1 a clockwise neighbour of z_j on C and denote by x_2 a neighbour of z_j outside C. The edge x_1x_2 connects vertices at distance 2 in G. Therefore, if the radius decreases after adding of x_1x_2, then it decreases by one and the central vertices of $G \cup x_1x_2$ must be central in G. We know that every vertex of C has an eccentric vertex outside C. This eccentric vertex can be only u_t for some t. But adding of x_1x_2 can decrease only distances to u_j. And there is only one vertex of C whose distance to u_j is r in G and whose distance to u_j is smaller than r in $G \cup x_1x_2$, namely $v_{j,c}$. However, since v_j is on the subpath $v_{i,c} - v_{i,a}$, we have $d_{G\cup x_1x_2}(v_{j,c}, u_i) \geq r$. This implies that G is not a radially-maximal graph, a contradiction. Therefore, the vertices $v_{i,c}$ and $v_{j,a}$ are on C in (clockwise) order $v_{1,c}$, $v_{1,a}$, $v_{2,c}$, $v_{2,a}$, $\ldots$, $v_{k,c}$, $v_{k,a}$, and although it can happen that $v_{i,c} = v_{i,a}$ for some i, we always have $v_{i,a} \neq v_{i+1,c}$. (By $v_{k+1,c}$ we mean $v_{1,c}$.)

Now suppose that there is i such that $v_{i,a}$ and $v_{i+1,c}$ are not adjacent. Then there is a vertex, say v, on the subpath $v_{i,a} - v_{i+1,c}$. Since v is outside all the subpaths $v_{j,c} - v_{j,a}$, its distance to all vertices u_t, $1 \leq t \leq k$, is smaller than r, a contradiction. Thus, $v_{i,a}$ and $v_{i+1,c}$ are adjacent for all $i = 1, 2, \ldots, k$.

Since the vertex eccentric to z_1 must lie outside C and since $d_G(z_1, u_1) \leq r - 2$ by Theorem B, we have $k \geq 2$.

Now we derive an identity which plays a key role in this proof. For every i, $1 \leq i \leq k$, the sum of distances from $v_{i,a}$ to u_i, from u_{i+1} to $v_{i+1,c}$ and the length of subpath $z_i - z_{i+1}$ is $2r + d(z_i, z_{i+1})$ as well as $c + l_i + l_{i+1} + 1$. (Observe that here we use $k \geq 2$.) Summing these equalities for all i we get $k{\cdot}2r + c = k{\cdot}c + 2\sum_{i=1}^{k} l_i + k$, i.e.,

$$k(2r-1) = (k-1)c + 2\sum_{i=1}^{k} l_i. \qquad (1)$$

By (1), k must be even. Having k even, c must be also even by (1). Suppose that $k = 2$. Then since $v_{1,a}v_{2,c}$ and $v_{2,a}v_{1,c}$ are edges of C, the vertices z_1 and z_2 are opposite on C (i.e., their distance in G is $\frac{1}{2}c$). Denote by x_1 and x_2 the neighbours of z_1 on C. Then adding of x_1x_2 can decrease the distances to neither u_1 nor u_2, so that G is not radially-maximal, a contradiction. Hence, $k \geq 4$.

For the number of vertices of G we have $|V(G)| = c + \sum_{i=1}^{k} l_i$. By (1), $c + \sum_{i=1}^{k} l_i = \frac{1}{2}[k(2r-1) - (k-3)c]$, so that

$$|V(G)| = \tfrac{1}{2}[k(2r-1) - (k-3)c]. \qquad (2)$$

If $c \leq 2r-3$, then (2) gives $|V(G)| \geq \frac{1}{2}(6r + 2k - 9)$, and since $k \geq 4$, we have $|V(G)| \geq 3r - \frac{1}{2}$. This contradicts the fact that $|V(G)| = 3r - 1$. Since c is even and we already proved that $c < 2r$, we have $c = 2r - 2$. Substituting $c = 2r - 2$ into (2) we get $|V(G)| = 3r - 1 = \frac{1}{2}[k(2r-1) - (k-3)(2r-2)]$, which gives $k = 4$.

To conclude the proof it suffices to show that the subpath $z_i - z_{i+1}$ has length $l_i + l_{i+1} - 1$ and that $l_i + l_{i+1} \geq 3$, $1 \leq i \leq 4$. However, we already derived that the length of the subpath $z_i - z_{i+1}$ is $c + l_i + l_{i+1} + 1 - 2r$ (see the identities producing (1)). And $c + l_i + l_{i+1} + 1 - 2r = l_i + l_{i+1} - 1$.

Finally, suppose that $l_i + l_{i+1} \leq 2$ for some i. By definition, $l_j \geq 1$ for every j, so that $l_i = l_{i+1} = 1$ and z_iz_{i+1}, z_iu_i and $z_{i+1}u_{i+1}$ are edges of G. Now add to G the edge u_iu_{i+1}. This edge cannot decrease distances between any vertex of C and u_t, $1 \leq t \leq 4$. Since G is a radially-maximal graph, the center of $G \cup u_iu_{i+1}$ is outside C. But as $c = 2r - 2$, every vertex outside C has a partner on C at distance at least r in both G and $G \cup u_iu_{i+1}$. This contradicts the fact that G is radially-maximal. □

Proof of Corollary 3. By Theorems 1 and 2, unicyclic non-selfcentric radially-maximal graphs of radius r on $3r - 1$ vertices are characterized by the lengths of their four paths attached to the cycle. Since $2r - 2$ vertices are used for the cycle, for the paths we have $r + 1$ vertices. A set of $r + 1$ elements can be decomposed into 4 parts in $\binom{r+4}{3}$ ways. Only a few of the decompositions have two parts of the same size. (More precisely, their number is at most $O(r^2)$.) Analogously, only a few of them contain a part of size at most 1. This means that there are $\frac{1}{6}r^3 + O(r^2)$ decompositions of $r + 1$ elements into 4 sets of different sizes, all with at least 2 elements. However, every one such decomposition has $4! = 24$ different reorderings, which yield 3 different graphs, namely $G_{(a,c,b,d)}$, $G_{(a,b,c,d)}$ and $G_{(a,b,d,c)}$. Thus, the number of graphs satisfying Theorems 1 and 2 is $\frac{3}{24}[\frac{1}{6}r^3 + O(r^2)]$, i.e., $\frac{1}{48}r^3 + O(r^2)$. □

Acknowledgement

This research was supported by Slovak research grants VEGA 1/0489/08, APVV 0040-06 and 0104-07, and APVV LPP 0203-06.

References

[1] F. Gliviak, M. Knor, L'. Šoltés, On radially maximal graphs, *Australas. J. Combin.* **9** (1994), 275–284.

[2] A. Haviar, P. Hrnčiar, G. Monoszová, Eccentric sequences and cycles in graphs, *Acta Univ. M. Belii Math.* **11** (2004), 7–25.

[3] M. Knor, Minimal non-selfcentric radially-maximal graphs of radius 4, *Discussiones Mathematicae Graph Theory* **27** (2007), 603–610.

[4] M. Knor, Minimal non-selfcentric radially-maximal graphs of radii 4 and 5, (submitted).

[5] O. Ore, Diameter in graphs, *J. Combin. Theory* **5** (1968), 1245–1246.

(Received 27 June 2008; revised 28 Nov 2008)