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The metric chromatic number of a graph

Gary Chartrand

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008
U.S.A.

Futaba Okamoto

Mathematics Department
University of Wisconsin — La Crosse
La Crosse, WI 54601
U.S.A.

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008
U.S.A.

Abstract

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same and let $V_1, V_2, \ldots, V_k$ be the resulting color classes. For a vertex v of G, the metric color code of v is the k-vector

$$\text{code}(v) = (d(v, V_1), d(v, V_2), \cdots, d(v, V_k)),$$

where $d(v, V_i)$ is the minimum distance between v and a vertex in V_i. If $\text{code}(u) \neq \text{code}(v)$ for every two adjacent vertices u and v of G, then c is a metric coloring of G. The minimum k for which G has a metric k-coloring is called the metric chromatic number of G and is denoted by $\mu(G)$. The metric chromatic numbers of some well-known graphs are determined and characterizations of connected graphs of order n having metric chromatic number 2 and $n-1$ are established. We present several bounds for the metric chromatic number of a graph in terms of other graphical parameters and study the relationship between the metric chromatic number of a graph and its chromatic number.

1 Introduction

The primary goal of vertex colorings of a graph G is to distinguish the two vertices in each pair of adjacent vertices of G by using as few colors as possible. Of course, this can be accomplished by proper colorings, where adjacent vertices are required to be assigned distinct colors. The minimum number of colors in a proper coloring of G is then the *chromatic number* $\chi(G)$ of G. There are other methods, however, that can be used to distinguish every two adjacent vertices in G by means of vertex colorings and which may require using fewer than $\chi(G)$ colors.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path. For a set $S \subseteq V(G)$ and a vertex v of G, the *distance* $d(v, S)$ *between* v *and* S is defined as

$$d(v, S) = \min\{d(v, x) : \; x \in S\}.$$

Then $0 \le d(v, S) \le \operatorname{diam}(G)$, where $d(v, S) = 0$ if and only if $v \in S$. Suppose that $c : V(G) \to \{1, 2, \ldots, k\}$ is a k-coloring of G for some positive integer k where adjacent vertices may be colored the same and let $V_1, V_2, \ldots, V_k$ be the resulting color classes. With each vertex v, we can associate a k-vector

$$\operatorname{code}(v) = (a_1, a_2, \cdots, a_k) = a_1 a_2 \cdots a_k$$

called the *metric color code* of v, where for each i with $1 \le i \le k$, $a_i = d(v, V_i)$. If $\operatorname{code}(u) \ne \operatorname{code}(v)$ for every two adjacent vertices u and v of G, then c is called a *metric coloring* of G. The minimum k for which G has a metric k-coloring is called the *metric chromatic number* of G and is denoted by $\mu(G)$. Clearly, $\mu(G)$ is defined for every connected graph G and $\mu(G) \ge 2$ for every nontrivial connected graph G.

Let c be a proper k-coloring of a nontrivial connected graph G with resulting color classes $V_1, V_2, \ldots, V_k$ and let u and v be two adjacent vertices of G. Then $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \ldots, k\}$ with $i \ne j$. Suppose that $\operatorname{code}(u) = (a_1, a_2, \ldots, a_k)$ and $\operatorname{code}(v) = (b_1, b_2, \ldots, b_k)$. Then $a_i = b_j = 0$ and $a_j = b_i = 1$. Thus $\operatorname{code}(u) \ne \operatorname{code}(v)$ and c is also a metric coloring of G. Thus $\mu(G) \le \chi(G)$. Consequently,

$$2 \le \mu(G) \le \chi(G) \le n \tag{1}$$

for every nontrivial connected graph G of order n.

To illustrate these concepts, consider the graph $G = C_7 + K_1$ (the wheel of order 8). The chromatic number of G is $\chi(G) = 4$. We show that its metric chromatic number is $\mu(G) = 3$. Since the 3-coloring shown in Figure 1 is a metric coloring, it follows that $\mu(G) \le 3$. It remains to show that $\mu(G) \ge 3$. Assume, to the contrary, that $\mu(G) = 2$. Then there exists a metric 2-coloring c of G using the colors 1 and 2. Necessarily, two adjacent vertices x and y of G are colored differently, say $c(x) = 1$ and $c(y) = 2$. These vertices belong to a common triangle T. Let z be the third vertex of T. We may assume that $c(z) = 1$. However then $\operatorname{code}(x) = \operatorname{code}(z) = (0, 1)$, which is a contradiction.

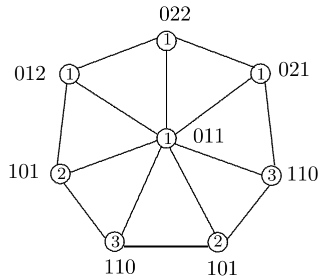


Figure 1: A metric 3-coloring of a graph

The *partition dimension* of a connected graph G was introduced in [2, 3] and defined as the minimum k for which G has a k-coloring such that $\text{code}(u) \neq \text{code}(v)$ for each pair u, v of distinct vertices of G. This concept has also been studied in [1], [5], and [6], among other papers. While the partition dimension deals with *vertex-distinguishing* colorings of G (in which every two vertices of G have distinct color-induced labels), a metric coloring is a *neighbor-distinguishing* coloring of G (in which every two *adjacent* vertices of G have distinct color-induced labels). The following observations will be useful to us.

Observation 1.1 *To show that a given coloring c is a metric coloring, it suffices to show that* $\text{code}(u) \neq \text{code}(v)$ *for adjacent vertices u and v with $c(u) = c(v)$.*

Observation 1.2 *Let c be a metric coloring of a connected graph G and $uv \in E(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$.*

It is convenient to introduce some notation. For a coloring c of a graph G and a set $S \subseteq V(G)$, the *set $c(S)$ of colors of S* is defined by

$$c(S) = \{c(v) : \ v \in S\}.$$

For each positive integer k, let

$$\mathbb{N}_k = \{1, 2, \ldots, k\}.$$

We refer to the book [4] for graph theory notation and terminology not described in this paper.

2 On Graphs with Prescribed Order and Metric Chromatic Number

We noted in (1) that $2 \leq \mu(G) \leq n$ for every nontrivial connected graph G of order n. It is obvious that a connected graph G of order n has metric chromatic number n if and only if $G = K_n$. We now characterize connected graphs having metric chromatic number 2.

Proposition 2.1 *A nontrivial connected graph G has metric chromatic number 2 if and only if G is bipartite.*

Proof. If G is bipartite, then $\chi(G) = 2$ and so the result follows by (1). For the converse, suppose that there is a connected graph G with metric chromatic number 2 that is not bipartite. Let $C : v_1, v_2, \ldots, v_\ell, v_{\ell+1} = v_1$ be an odd cycle in G and let c be a metric 2-coloring of G. Consider the (cyclic) color sequence

$$s : \; c(v_1), c(v_2), \ldots, c(v_\ell), c(v_{\ell+1}) = c(v_1).$$

By a *block* of s, we mean a maximal subsequence of s consisting of terms of the same color. We claim that each block of s must be of odd length. Suppose, to the contrary, that there is a block of even length, say $c(v_1), c(v_2), \ldots, c(v_{2k})$ for some positive integer k where $2k < \ell$. We may assume that $c(v_i) = 1$ for $1 \le i \le 2k$ and so $c(v_\ell) = c(v_{2k+1}) = 2$. Thus $\text{code}(v_i) = (0, d_i)$, where d_i is the distance between v_i and the nearest vertex colored 2. Since $\text{code}(v_1) = (0, 1)$ and $\text{code}(v_i) \neq \text{code}(v_{i+1})$ for $1 \le i \le 2k-1$, it follows that $|d_i - d_{i+1}| = 1$. Consequently, d_i is odd if and only if i is odd for $1 \le i \le 2k-1$. However, $\text{code}(v_{2k}) = (0, 1)$, producing a contradiction. Thus, as claimed, every block of s has odd length. Hence we may assume that either (i) s consists of a single block in which $c(v_i) = 1$ for $1 \le i \le \ell$ or (ii) s contains an even number of blocks, each having an odd number of terms. If (i) occurs, then $\text{code}(v_i) = (0, d_i)$ for some $d_i \ge 1$ and $|d_i - d_{i+1}| = 1$ for $1 \le i \le \ell$, which implies that $d_1, d_2, \ldots, d_\ell, d_1$ alternate between even and odd integers, which is impossible since ℓ is odd. If (ii) occurs, then ℓ is even, which contradicts our assumption. ∎

As immediate consequences of Proposition 2.1, we have the following.

Corollary 2.2 *Let G be a connected graph. If $\chi(G) = 3$, then $\mu(G) = 3$.*

Corollary 2.3 *For each integer $n \ge 3$,*

$$\mu(C_n) = \chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

By Proposition 2.1, if G is a complete bipartite graph, then $\mu(G) = \chi(G) = 2$; while by Corollary 2.2, if G is a complete 3-partite graph, then $\mu(G) = \chi(G) = 3$. This fact can be extended to all complete multipartite graphs, as we show next.

Proposition 2.4 *For every complete k-partite graph G where $k \ge 2$, $\mu(G) = k$.*

Proof. We proceed by induction on $k \ge 2$. For $k = 2$, the result follows by Proposition 2.1. Suppose that the metric chromatic number of every complete $(k-1)$-partite graph is $k-1$ for some integer $k \ge 3$. Let G be a complete k-partite graph with partite sets $U_1, U_2, \ldots, U_k$. Certainly, the metric chromatic number of G is at most k. Assume, to the contrary, that $\mu(G) \le k-1$. Let there be given a metric $(k-1)$-coloring c of G using the colors in $\mathbb{N}_{k-1}$. We claim that for each partite set U_i $(1 \leq i \leq k)$, the coloring c_i that is the restriction of c to $V(G) - U_i$ in the complete $(k-1)$-partite graph $G - U_i$ is a metric coloring. To see this, let u and v be two adjacent vertices in $G - U_i$. Then $\text{code}_c(u) \neq \text{code}_c(v)$. Since $d(u, x) = d(v, x) = 1$ for all $x \in U_i$, it follows that $\text{code}_{c_i}(u) \neq \text{code}_{c_i}(v)$. This implies that c_i is a metric coloring of $G - U_i$. Since $\mu(G - U_i) = k - 1$ by the induction hypothesis, it follows that $c(V(G) - U_i) = \mathbb{N}_{k-1}$ for every partite set U_i of G. Because $c(V(G)) = \mathbb{N}_{k-1}$, there are vertices x and y in G belonging to different partite sets of G such that $c(x) = c(y)$, say $c(x) = c(y) = 1$. We may assume that $x \in U_1$ and $y \in U_2$. Since $c(V(G) - U_1) = c(V(G) - U_2) = \mathbb{N}_{k-1}$, it follows that $\text{code}(x) = \text{code}(y) = (0, 1, 1, \cdots, 1)$, a contradiction. ■

Since the metric chromatic number of the complete k-partite graph $K_{1,1,\ldots,1,n-(k-1)}$ of order n is k, the following result is a consequence of Proposition 2.4.

Corollary 2.5 *For each pair k, n of integers with $2 \leq k \leq n$, there is a connected graph G of order n with $\mu(G) = k$.*

We next determine those connected graphs of order $n \geq 3$ having metric chromatic number $n - 1$.

Theorem 2.6 *A connected graph G of order $n \geq 3$ has metric chromatic number $n - 1$ if and only if $G = K_{n-2} + \overline{K}_2$ or $G = (K_{n-2} \cup K_1) + K_1$.*

Proof. It is easy to see that $\mu(G) = n-1$ if $G = K_{n-2}+\overline{K}_2$ or $G = (K_{n-2} \cup K_1)+K_1$. For the converse, assume that G is a connected graph of order $n \geq 3$ with $\mu(G) = n-1$. Thus $G \neq K_n$ and $\chi(G) = n-1$. Then the clique number of G is $\omega(G) = n-1$. Let $H = K_{n-1}$ be a clique of order $n-1$ in G with $V(H) = \{v_1, v_2, \ldots, v_{n-1}\}$ and let $V(G) - V(H) = \{v\}$. If v is adjacent to exactly $n-2$ vertices of H, then $G = K_{n-2} + \overline{K}_2$; while if v is adjacent to exactly one vertex of H, then $G = (K_{n-2} \cup K_1) + K_1$. Thus we may assume that v is adjacent to $v_1, v_2, \ldots, v_k$ in G where $2 \leq k \leq n-3$. Define a coloring c of G by $c(v_i) = i$ for $1 \leq i \leq n-3$, $c(v_{n-2}) = 1$, $c(v_{n-1}) = 2$, and $c(v) = n-2$. Since v_1 and v_2 are adjacent to v and v_{n-2} and v_{n-1} are not, it follows that $\text{code}(v_1) \neq \text{code}(v_{n-2})$ and $\text{code}(v_2) \neq \text{code}(v_{n-1})$. By Observation 1.1, c is a metric $(n-2)$-coloring of G and so $\mu(G) \leq n-2$, which contradicts our assumption. ■

3 Bounds for the Metric Chromatic Number of a Graph

We have already noted that if G is a nontrivial connected graph of order n, then $2 \leq \mu(G) \leq \chi(G) \leq n$. With the aid of other graphical parameters, we present in this section improved lower and upper bounds for $\mu(G)$.

The *clique number* $\omega(G)$ of a graph G is the largest order of a clique (complete subgraph) in G. It is well known that $\chi(G) \geq \omega(G)$ for every graph G. It need not occur that $\mu(G) \geq \omega(G)$ however. For the graph G of Figure 2, $\chi(G) = \omega(G) = 4$.

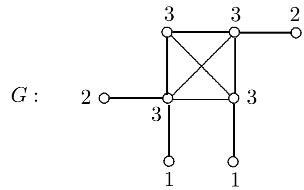


Figure 2: A graph G with $\mu(G) = 3$ and $\omega(G) = 4$

Since the 3-coloring of G shown in Figure 2 is a metric coloring, it follows that $\mu(G) \leq 3$ and so $\mu(G) = 3$ by Proposition 2.1.

There is a lower bound for the metric chromatic number of a graph in terms of its clique number, however. This bound is presented next.

Theorem 3.1 *For every nontrivial connected graph G,*

$$\mu(G) \geq 1 + \lceil \log_2 \omega(G) \rceil. \tag{2}$$

Proof. Let H be a clique of order $\omega = \omega(G)$ in G with $V(H) = \{v_1, v_2, \ldots, v_\omega\}$. Suppose that $\mu(G) = k$ and c is a metric k-coloring of G using colors in the set $\mathbb{N}_k$. Suppose that $|c(V(H))| = r$, say $c(V(H)) = \mathbb{N}_r$, where $1 \leq r \leq k$.

Let $v \in V(H)$ with $\text{code}(v) = (a_1, a_2, \cdots, a_k)$. Observe that $a_i \in \{0, 1\}$ for $1 \leq i \leq r$ and exactly one of $a_1, a_2, \ldots, a_r$ is 0. Thus there are r possibilities for the r-tuple $(a_1, a_2, \ldots, a_r)$. For each j with $r + 1 \leq j \leq k$, let d_j be the minimum distance between a vertex colored j in $V(G)-V(H)$ and a vertex in $V(H)$. Then $a_j \in \{d_j, d_j + 1\}$ and so there are two choices for each coordinate a_j when $r + 1 \leq j \leq k$. Thus there are 2^{k-r} possibilities for the $(k - r)$-tuple $(a_{r+1}, a_{r+2}, \ldots, a_k)$. Therefore, there are $r \cdot 2^{k-r}$ possible metric color codes for the vertices of H and so $\omega \leq r \cdot 2^{k-r}$. Since $r \leq 2^{r-1}$ for each positive integer r, it follows that $\omega \leq 2^{r-1} \cdot 2^{k-r} = 2^{k-1}$ and so $k - 1 \geq \log_2 \omega$. Therefore, $k = \mu(G) \geq 1 + \log_2 \omega$, producing the desired result. ■

The lower bound for the metric chromatic number of a graph in Theorem 3.1 is sharp. To see this, we construct a connected graph G with $\omega(G) = 2^{k-1}$ and $\mu(G) = k$ for each integer $k \geq 2$. We start with the complete graph $H = K_{2^{k-1}}$ of order 2^{k-1}, where $V(H) = \{v_1, v_2, \ldots, v_{2^{k-1}}\}$. Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\mathbb{N}_{k-1}$, where $S_1 = \emptyset$. For each integer i with $2 \leq i \leq 2^{k-1}$, we add $|S_i|$ pendant edges at the vertex v_i, obtaining the connected graph G with $\omega(G) = 2^{k-1}$. The graph G shown in Figure 2 is constructed in this way for $k = 3$. It remains to show that $\mu(G) = k$. By Theorem 3.1, $\mu(G) \geq k$. Define a k-coloring of G by assigning (i) the color k to each vertex of H and (ii) the colors in S_i to the $|S_i|$ end-vertices adjacent to v_i for $2 \leq i \leq 2^{k-1}$. Since c is a metric k-coloring of G, it follows that $\mu(G) = k$.

For a connected graph G, the *diameter* $\text{diam}(G)$ of G is the greatest distance between two vertices of G. If G is a connected graph with $\text{diam}(G) = d$ and $\mu(G) = k$, then for a metric k-coloring of G and a vertex v of G, exactly one coordinate of code(v) is 0, while each of the remaining coordinates of code(v) is an element of the set $\{1, 2, \ldots, d\}$, resulting in at most $k \cdot d^{k-1}$ distinct metric color codes for the vertices of G. Note that each metric k-coloring c of a graph G provides a proper coloring c^* of G, where $c^*(v) = \text{code}(v)$ for each $v \in V(G)$. Since c^* uses at most $k \cdot d^{k-1}$ different colors, $\chi(G) \leq k \cdot d^{k-1}$. This observation is stated below.

Observation 3.2 *If G is a connected graph of order n, diameter d, and metric chromatic number k, then*

$$\chi(G) \leq k \cdot d^{k-1}.$$

By the well-known inequality $\chi(G) \leq n - d + 1$ for every connected graph G of order n and diameter d (see [7]), we obtain the following result that presents an upper bound for the metric chromatic number of a graph in terms of its order and diameter.

Proposition 3.3 *If G is a nontrivial connected graph of order n and diameter d, then*

$$\mu(G) \leq n - d + 1.$$

The upper bound in Proposition 3.3 is sharp. To see this, we construct a connected graph G of order n and diameter d such that $\mu(G) = n - d + 1$ for each pair n, d of integers with $1 \leq d \leq n - 1$. Let G be the graph obtained from the graph K_{n-d+1} and the path $P_d : v_1, v_2, \ldots, v_d$ by identifying a vertex of K_{n-d+1} and the vertex v_1 of P_d and denoting the identified vertex by v_1. Then the order of G is n and the diameter of G is d. It remains to show that $\mu(G) = n-d+1$. By Proposition 3.3, $\mu(G) \leq n-d+1$. By Observation 1.2, the $n-d$ vertices in $V(K_{n-d+1}) - \{v_1\}$ must be assigned different colors in every metric coloring of G and so $\mu(G) \geq n - d$. Assume, to the contrary, that there is a metric $(n - d)$-coloring c of G. Then $c(v_1) = c(x)$ for some $x \in V(K_{n-d+1}) - \{v_1\}$. However then code($v_1$) = code($x$), which is impossible. Thus $\mu(G) = n - d + 1$, as claimed.

4 On the Chromatic Number and Metric Chromatic Number of a Graph

It is well known that if v is a vertex of a nontrivial graph G, then either $\chi(G - v) = \chi(G)$ or $\chi(G-v) = \chi(G)-1$. This, however, is not the case for the metric chromatic number. In fact, it is possible for a graph G to contain a vertex v such that the metric chromatic number of $G - v$ is greater than the metric chromatic number of G. On the other hand, $\mu(G - v)$ can never exceed $\mu(G)$ by more than $\deg v$. For a vertex v, let $N[v] = N(v) \cup \{v\}$ denote the *closed neighborhood* of v.

Theorem 4.1 *If v is a vertex that is not a cut-vertex of a connected graph G, then*

$$\mu(G - v) \leq \mu(G) + \deg v.$$

Proof. Suppose that $\mu(G) = k$, $\deg v = d$, and $N(v) = \{v_1, v_2, \ldots, v_d\}$. Let $c : V(G) \to \mathbb{N}_k$ be a metric k-coloring of G and let $V_1, V_2, \ldots, V_k$ be the resulting color classes. First, we assume that $V_i \not\subseteq N[v]$ for all i with $1 \le i \le k$. Now let $c' : V(G - v) \to \mathbb{N}_{k+d}$ be the coloring defined by

$$c'(x) = \begin{cases} c(x) & \text{if } x \notin N(v) \\ k+i & \text{if } x = v_i \ \ (1 \le i \le d), \end{cases}$$

where $V'_1, V'_2, \ldots, V'_{k+d}$ are the color classes resulting from c'. Therefore, $V'_i \subseteq V_i$ for $1 \le i \le k$ and $|V'_{k+i}| = 1$ for $1 \le i \le d$. We show that c' is a metric $(k+d)$-coloring of $G - v$. For $u \in V(G - v)$, observe that $d_G(u, w) \le d_{G-v}(u, w)$ for each $w \in V(G - v) - \{u\}$. Hence for every i with $1 \le i \le k$ for which $V'_i \ne \emptyset$,

$$d_G(u, V_i) \le d_{G-v}(u, V'_i).$$

Let x and y be two adjacent vertices in $G - v$. Then x and y are adjacent in G and $\text{code}_c(x) \ne \text{code}_c(y)$. We show that $\text{code}_{c'}(x) \ne \text{code}_{c'}(y)$. By Observation 1.1, we may assume that $c'(x) = c'(y)$. Hence $x, y \in V(G - v) - N(v) = V(G) - N[v]$. Suppose, without loss of generality, that $x, y \in V'_1$ and so $c'(x) = c'(y) = 1$. Since $c(x) = c(y) = 1$ as well, we may write

$$\begin{aligned} \text{code}_c(x) &= (0, a_2, \cdots, a_k) \\ \text{code}_{c'}(x) &= (0, a'_2, \cdots, a'_k, a'_{k+1}, \cdots, a'_{k+d}) \\ \text{code}_c(y) &= (0, b_2, \cdots, b_k) \\ \text{code}_{c'}(y) &= (0, b'_2, \cdots, b'_k, b'_{k+1}, \cdots, b'_{k+d}). \end{aligned}$$

If $d_{G-v}(x, v_j) \ne d_{G-v}(y, v_j)$ for some j with $1 \le j \le d$, then $a'_{k+j} \ne b'_{k+j}$ and so $\text{code}_{c'}(x) \ne \text{code}_{c'}(y)$, as desired. Hence we may assume that $d_{G-v}(x, v_i) = d_{G-v}(y, v_i)$ for all i with $1 \le i \le d$. We claim that

$$d_G(x, u) = d_G(y, u) \ \text{ for every } u \in N[v]. \tag{3}$$

We consider two cases.

Case 1. $u = v$. We may assume, without loss of generality, that

$$a = d_G(x, v) \le d_G(y, v). \tag{4}$$

Let $P : x = w_0, w_1, \ldots, w_a = v$ be an $x - v$ path of length a in G. Then $d_{G-v}(x, w_{a-1}) = a - 1$. Since $w_{a-1} \in N(v)$, it follows that $d_{G-v}(y, w_{a-1}) = a - 1$. Let P' be a $y - w_{a-1}$ path of length $a - 1$ in $G - v$. Then P' followed $v = w_a$ is a $y - v$ walk of length a in G and so $d_G(y, v) \le a$. It then follows by (4) that $d_G(x, v) = d_G(y, v) = a$.

Case 2. $u \in N(v)$. Then $d_{G-v}(x, u) = d_{G-v}(y, u)$. Assume, to the contrary, that $d_G(x, u) \ne d_G(y, u)$, say $b = d_G(x, u) < d_G(y, u)$. Since $b = d_G(x, u) < d_G(y, u) \le d_{G-v}(y, u) = d_{G-v}(x, u)$, every $x - u$ path of length b in G contains v, say $Q : x = x_0, x_1, \ldots, x_{b-1} = v, u$ is an $x - u$ geodesic in G. Hence $d_G(x, v) = d_G(y, v) = b - 1$

by Case 1. Since a $y - v$ path of length $b-1$ in G followed by u is a $y-u$ walk in G of length b, it follows that $b = d_G(x,u) < d_G(y,u) \le b$, which is impossible.

Therefore, as claimed, $d_G(x,u) = d_G(y,u)$ for every $u \in N[v]$. Since c is a metric coloring of G, it follows that $a_i \ne b_i$ for some i with $2 \le i \le k$, say $a_2 < b_2$. Let $z \in V(G-v) - N(v) = V(G) - N[v]$ such that $c(z) = c'(z) = 2$ and

$$d_G(x,z) = d_G(x,V_2) = a_2.$$

Since $d_G(y,V_2) = b_2 > a_2$, it follows that $b_2 = a_2 + 1$ and by (3) no $x - z$ path of length a_2 can contain a vertex in $N[v]$, implying that $d_{G-v}(x,z) = d_G(x,z) = a_2$. Then

$$a'_2 \le d_{G-v}(x,z) = a_2 < b_2 = d_G(y,V_2) \le d_{G-v}(y,V'_2) = b'_2.$$

Hence $a'_2 \ne b'_2$ and so $\mathrm{code}_{c'}(x) \ne \mathrm{code}_{c'}(y)$. Therefore, c' is a metric coloring of $G-v$ and we obtain $\mu(G-v) \le k+d = \mu(G) + \deg v$.

If it should occur that $V_i \subseteq N[v]$ for one or more integers i with $1 \le i \le k$, then we need not recolor the vertices belonging to each such set V_i. Suppose that there are $j^*(\ge 1)$ such sets, say $V_i \subseteq N[v]$ for $1 \le i \le j^*$ and $V_i \not\subseteq N[v]$ for $j^*+1 \le i \le d$ (if $j^* < d$). Then the coloring c'' of $G-v$ defined by

$$c''(x) = \begin{cases} k+i & \text{if } j^* < d \text{ and } x = v_{j^*+i}\ (1 \le i \le d-j^*) \\ c(x) & \text{otherwise} \end{cases}$$

is a metric $(k+d-j^*)$-coloring. Therefore, $\mu(G-v) \le k+d-j^* < \mu(G) + \deg v$. ■

We now show that the upper bound for $\mu(G-v)$ in Theorem 4.1 is sharp. For a given positive integer d and an integer $k \ge d+2$, let $H = K_{2k-2}$, where $V(H) = U \cup W$ with $U = \{u_1, u_2, \ldots, u_{k-1}\}$ and $W = \{w_1, w_2, \ldots, w_{k-1}\}$. The graph G is constructed from H by adding two new vertices v_1 and v_2 and joining v_1 to $w_1, w_2, \ldots, w_d$ and joining v_2 to $w_{d+1}, w_{d+2}, \ldots, w_{k-1}$. Figure 3 shows the graph G for $d = 3$ and $k = 5$. Hence $\deg v_1 = d$ and $\deg v_2 = k-1-d \ge 1$. We show that $\mu(G) = k$ and $\mu(G-v_1) = k+d$.

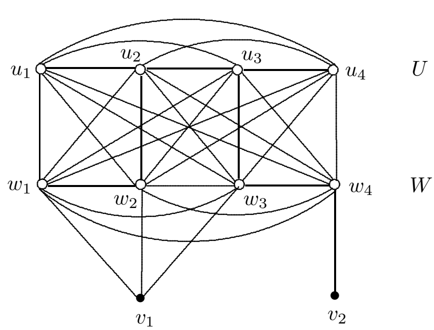


Figure 3: A graph G with $\mu(G) = 5$ and $\mu(G-v_1) = 8$

The k-coloring $c_1 : V(G) \to \mathbb{N}_k$ defined by

$$c_1(x) = \begin{cases} i & \text{if } x \in \{u_i, w_i\} \text{ for } 1 \le i \le k-1 \\ k & \text{if } x \in \{v_1, v_2\} \end{cases}$$

is a metric coloring and so $\mu(G) \le k$. Furthermore, the $(k+d)$-coloring $c_2 : V(G - v_1) \to \mathbb{N}_{k+d}$ given by

$$c_2(x) = \begin{cases} i & \text{if } x = u_i \text{ for } 1 \le i \le k-1 \\ k+i-1 & \text{if } x = w_i \text{ for } 1 \le i \le d \\ i & \text{if } x = w_i \text{ for } d+1 \le i \le k-1 \\ k+d & \text{if } x = v_2 \end{cases}$$

is also a metric coloring and so $\mu(G - v_1) \le k + d$. It remains to verify that $\mu(G) \ge k$ and $\mu(G - v_1) \ge k + d$. By Observation 1.2, the vertices of U must be assigned distinct colors in any metric coloring of G. Hence $\mu(G) \ge |U| = k - 1$. If $\mu(G) = k - 1$, then some vertex of W is assigned the same color as a vertex in U by a metric $(k-1)$-coloring c of G, say $c(u_1) = c(w_1) = 1$. However then, $\text{code}_c(u_1) = \text{code}_c(w_1)$, which is impossible. Thus $\mu(G) = k$. Similarly, the vertices of $X = U \cup \{w_1, w_2, \ldots, w_d\}$ must be assigned distinct colors in any metric coloring of $G - v_1$ and so $\mu(G - v_1) \ge |X| = k + d - 1$. If $\mu(G - v_1) = k + d - 1$, then some vertex in $\{w_{d+1}, w_{d+2}, \ldots, w_{k-1}\}$ is assigned the same color as a vertex in X by a metric $(k + d - 1)$-coloring of G. However then, these two vertices have the same metric color code, which is impossible. Hence $\mu(G - v_1) = \mu(G) + \deg v_1$.

For a nontrivial connected graph G and a vertex v of G, it can occur that $\mu(G - v) = \mu(G)$ or $\mu(G - v) = \mu(G) - 1$. For example, for the graph G of Figure 1 and the vertex v of degree 7 in G, we have $\mu(G - v) = \mu(G) = 3$. If $G = K_{s,t} + uv$, where $s + t \ge 3$ and u and v are nonadjacent vertices in $K_{s,t}$, then $\mu(G) = 3$ and $\mu(G-v) = 2$. Thus $\mu(G-v) = \mu(G)-1$ in this case. In fact, the deletion of a vertex from a graph can decrease the metric chromatic number by 2. We now illustrate this fact. Consider the graph G of Figure 4 and the vertex v of G, where v is adjacent to all other vertices of G.

Since $G - v$ is not bipartite and there is a metric 3-coloring of $G - v$ (as shown in Figure 4), it follows that $\mu(G - v) = 3$. It remains to show that $\mu(G) = 5$. A metric (and proper) 5-coloring of G is shown in Figure 4 and so $\mu(G) \le 5$. Assume, to the contrary, that there exists a metric 4-coloring $c : V(G) \to \mathbb{N}_4$. Suppose, without loss of generality, that $c(v) = 4$ and so $\text{code}(v) = (1, 1, 1, 0)$. Let $X = \{u_2, u_3, w_2, w_3\}$. By Observation 1.2,

$$c(u_i) \ne c(w_i) \quad \text{for } 1 \le i \le 4 \tag{5}$$

and so $|c(X)| \ge 2$. On the other hand, $|c(X)| \ne 4$; for otherwise, if $c(X) = \{1, 2, 3, 4\}$, then the vertex in X colored 4 has the color code $(1, 1, 1, 0) = \text{code}(v)$, a contradiction. Thus $2 \le |c(X)| \le 3$ and there are two vertices in X colored the same. By (5) no three vertices in X can be assigned the same color by c. We consider two cases, according to the number of vertices in X that are colored 4.

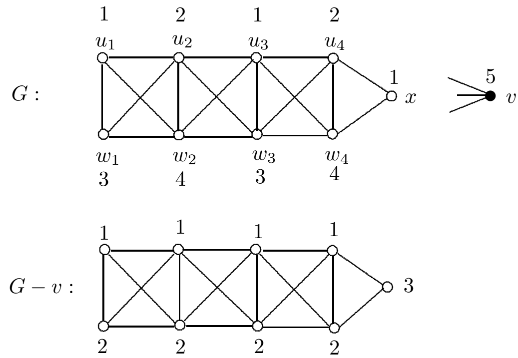


Figure 4: A graph G with $\mu(G) = 5$ and $\mu(G - v) = 3$

Case 1. *At most one vertex in* X *is assigned the color* 4. Assume, without loss of generality, that $c(u_2) = c(u_3) = 1$. First, suppose that $c(w_2) = 2$. Then $c(w_3) \in \{2, 3, 4\}$. If $c(w_3) = 2$, then $\{\text{code}(u_2), \text{code}(u_3)\} = \{(0, 1, 1, 1), (0, 1, 2, 1)\}$. Suppose first that $\text{code}(u_2) = (0, 1, 1, 1)$. Thus $c(u_1) = 3$ or $c(w_1) = 3$, say $c(u_1) = 3$. However then, regardless of the color of w_1, $\text{code}(w_1) \in \{\text{code}(u_1), \text{code}(u_2), \text{code}(w_2), \text{code}(v)\}$, which is impossible. A contradiction arises in a similar way if $\text{code}(u_3) = (0, 1, 1, 1)$. If $c(w_3) = 3$, then $\text{code}(u_2) = \text{code}(u_3) = (0, 1, 1, 1)$, a contradiction. If $c(w_3) = 4$, then $3 \notin \{c(u_4), c(w_4)\}$, for otherwise, $\text{code}(w_3) = (1, 1, 1, 0) = \text{code}(v)$, a contradiction. Hence $\text{code}(u_3) = (0, 1, 2, 1)$ and $\text{code}(u_2) = (0, 1, 1, 1)$. Thus $c(u_1) = 3$ or $c(w_1) = 3$, say the former. However then, regardless of the color of w_1, $\text{code}(w_1) \in \{\text{code}(u_1), \text{code}(u_2), \text{code}(w_2), \text{code}(v)\}$, which is impossible. A similar argument produces a contradiction if $c(w_3) = 2$.

Case 2. *Exactly two vertices in* X *are colored* 4, *say* $c(u_2) = c(u_3) = 4$. Suppose, without loss of generality, that $c(w_2) = 1$. Then $c(w_3) \in \{1, 2, 3\}$. If $c(w_3) = 1$, then since $\text{code}(v) = (1, 1, 1, 0)$, it follows that

$$\{\text{code}(u_2), \text{code}(u_3)\} \subseteq \{(1, 1, 2, 0), (1, 2, 1, 0), (1, 2, 2, 0)\}.$$

Thus, we may assume that $\text{code}(u_2) = (1, 2, a_3, 0)$, where $a_3 \in \{1, 2\}$. This implies that $2 \notin \{c(u_1), c(w_1)\}$. If $1 \in \{c(u_1), c(w_1)\}$, say $c(u_1) = 1$, then $\text{code}(u_1) = \text{code}(w_2)$, regardless of the color assigned to w_1. Similarly, if $4 \in \{c(u_1), c(w_1)\}$, say $c(u_1) = 4$, then $\text{code}(u_1) = \text{code}(u_2)$, regardless of the color assigned to w_1. However then, $c(u_1) = c(w_1) = 3$, which is also impossible. If $c(w_3) = 2$ or $c(w_3) = 3$, say the former, then

$$\{\text{code}(u_2), \text{code}(u_3), \text{code}(v)\} \subseteq \{(1, 1, 1, 0), (1, 1, 2, 0)\},$$

which is a contradiction.

Therefore, as claimed, $\mu(G) = 5$. Hence $\mu(G - v) = \mu(G) - 2$ for the graph G of Figure 4 and the vertex v in G. Whether there exists a connected graph G and a vertex v of G for which $\mu(G - v) < \mu(G) - 2$ is not known.

We have seen in (1) that if G is a nontrivial connected graph with $\mu(G) = a$ and $\chi(G) = b$, then $2 \le a \le b$. We now investigate those pairs a, b of integers with $2 \le a \le b$ that are realizable as the metric chromatic number and chromatic number, respectively, of some connected graph. In the proof given for the following theorem, the notation $[\alpha, \beta]$ denotes the closed interval from α to β, that is, $[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \le x \le \beta\}$.

Theorem 4.2 *For each pair a, b of integers with $2 \le a \le b \le 2^{a-1}$, there exists a connected graph G with $\mu(G) = a$ and $\chi(G) = b$.*

Proof. If $a = b$, then $\mu(K_b) = \chi(K_b) = b$. Hence we may assume that $3 \le a < b \le 2^{a-1}$. Consider the function $f_a : [0, a-2] \to [a, 2^{a-1}]$ defined by $f_a(x) = 2^x(a-x)$. Note that f_a is strictly increasing on $[0, a-2]$. Consequently, there exists an integer $p \in [1, a-2]$ such that

$$a = f_a(0) \le f_a(p-1) < b \le f_a(p) \le f_a(a-2) = 2^{a-1}.$$

Also, observe that

$$p \le 2^{p-1} < \frac{f_a(p-1)}{a-p} < \left\lceil \frac{b}{a-p} \right\rceil \le 2^p.$$

Let $q = \left\lceil \frac{b}{a-p} \right\rceil$. Then $p + 1 \le q \le 2^p$ and

$$b = (q-1)(a-p) + r, \tag{6}$$

where $1 \le r \le a - p$.

Let $H = K_b$ where $V(H)$ can be partitioned into q subsets $X_1, X_2, \ldots, X_q$ such that $|X_i| = a - p$ for $1 \le i \le q-1$ and $|X_q| = r$. Write $X_i = \{x_{i,j} : 1 \le j \le |X_i|\}$ for $1 \le i \le q$. Also, let $S_1, S_2, \ldots, S_{2^p}$ be the 2^p subsets of $\mathbb{N}_p$, where $|S_1| \le |S_2| \le \cdots \le |S_{2^p}|$. (Hence $S_1 = \emptyset$, $|S_i| = 1$ for $2 \le i \le p+1$, and $S_{2^p} = \mathbb{N}_p$.) Let Y be a set of $\sum_{i=1}^{p} i\binom{p}{i}$ vertices disjoint from $V(H)$ such that Y can be partitioned into $2^p - 1$ subsets $Y_2, Y_3, \ldots, Y_{2^p}$ for which $|Y_i| = |S_i|$ for $2 \le i \le 2^p$. A graph G is constructed from H by

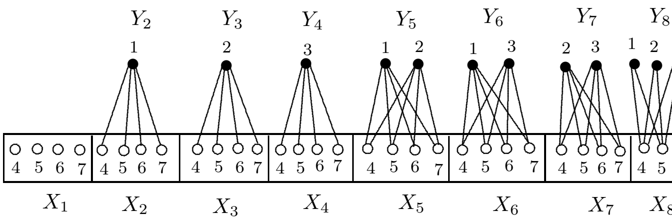
(i) adding the vertices in $\cup_{i=2}^{q} Y_i$ to H and

(ii) joining each vertex in Y_i to every vertex in X_i for $2 \le i \le q$.

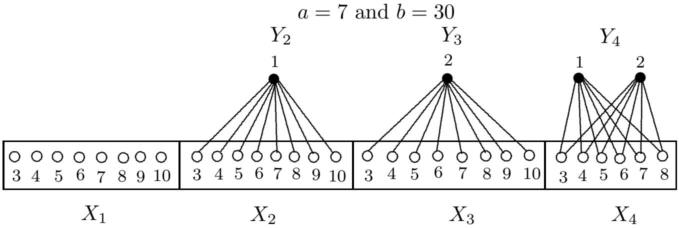
Figure 5 shows the graphs G in the case when $(a, b) = (7, 30)$ and $(a, b) = (10, 30)$.

Since $\chi(G) = b$, it remains to show $\mu(G) = a$. The coloring $c_1 : V(G) \to \mathbb{N}_a$ defined by

$$\begin{aligned} c_1(x_{i,j}) &= p + j \quad \text{for } 1 \le i \le q-1 \text{ and } 1 \le j \le a-p \\ c_1(x_{q,j}) &= p + j \quad \text{for } 1 \le j \le r \\ c_1(Y_i) &= S_i \quad \text{for } 2 \le i \le q \end{aligned}$$



$a = 7$ and $b = 30$



$a = 10$ and $b = 30$

Figure 5: Graphs in the proof of Theorem 4.2 for $a \in \{7, 10\}$ and $b = 30$

is a metric a-coloring of G and so $\mu(G) \leq a$. Assume, to the contrary, that $\mu(G) < a$. Then there exists a metric $(a-1)$-coloring of G. Permuting the colors $1, 2, \ldots, a-1$, if necessary, we obtain an $(a-1)$-coloring $c : V(G) \to \mathbb{N}_{a-1}$ of G such that $c(X) = \mathbb{N}_\ell$, where then $\ell \leq a-1$ and $X = V(H)$. By Observation 1.2, the $a-p$ vertices in X_1 must be colored differently by c. Therefore, $a - p \leq \ell \leq a-1$. Observe that the first ℓ coordinates of the code of each vertex in X are all 1 except one coordinate is 0. Furthermore, each of the remaining $a-1-\ell$ coordinates is either 1 or 2. Therefore,

$$b \leq 2^{a-1-\ell} \cdot \ell = 2^{a-1} \left(2^{-\ell} \cdot \ell\right).$$

The function $g(x) = 2^{-x} \cdot x$ defined on $\mathbb{R}$ is decreasing on $\left(\frac{1}{\ln 2}, \infty\right)$ and $\frac{1}{\ln 2} < 2 \leq a-p$, implying that

$$b \leq 2^{a-1} \left(2^{-\ell} \cdot \ell\right) \leq 2^{a-1} \left[2^{-(a-p)} \cdot (a-p)\right] = 2^{p-1}(a-p).$$

However then,

$$2^{p-1}(a-p+1) = f_a(p-1) < b \leq 2^{p-1}(a-p),$$

which is a contradiction. Therefore, we conclude that $\mu(G) = a$. ■

It is not known whether there is a graph G with $\mu(G) = a$ and $\chi(G) = b$ where $a \geq 3$ and $b > 2^{a-1}$. However, if such a graph G exists, then it follows by Proposition 3.1 that $\omega(G) < b$. In particular, it is not known if there is a 5-chromatic graph whose metric chromatic number is 3.

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