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Factorial properties of graphs

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Abstract

We explore an operation on graphs that is analogous to the factorial of a positive integer. We develop formulas for the factorials of complete graphs, odd cycles and complete bipartite graphs, and we prove some properties of the factorial operation. We conclude with a list of open questions.

This paper explores some properties of a factorial operation that associates to each graph G a graph $G!$ which we call the *factorial* of G. In [1] this operation is used to derive some cancellation properties of the direct product of graphs. However, our purpose here is simply to investigate the factorial operation on several classes of graphs and to deduce some of its properties.

Before stating our main definition, we set up notation and recall some relevant constructions. All our graphs are finite, undirected and are allowed to have loops. We thus regard a graph $G = (V(G), E(G))$ as a symmetric relation $E(G)$ on the set $V(G)$. The complete graph on n vertices is denoted K_n, while K_n^* denotes the complete graph with loops at each vertex. We put $V(K_n) = V(K_n^*) = \{0, 1, 2, 3, \ldots, n-1\}$. The complete bipartite graph with partite sets of sizes m and n is denoted $K(m, n)$. The complement of a graph G is denoted $\overline{G}$. The *direct product* of two graphs G and H is the graph $G \times H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are the pairs $(g, h)(g', h')$ with $gg' \in E(G)$ and $hh' \in E(H)$. (See [2] for a standard reference for the direct product.) We denote by $G + H$ the disjoint union of G and H. Given a positive integer n, we let nG denote the graph consisting of n copies of G. An *involution* is an automorphism of G that is its own inverse.

1 The Graph Factorial

The following definition, introduced in [1], is the main topic of this paper.

Definition 1 The **factorial** of a graph G is the graph, denoted $G!$, whose vertices are the permutations of $V(G)$. Permutations λ and μ are adjacent in $G!$ exactly when $gg' \in E(G) \iff \lambda(g)\mu^{-1}(g') \in E(G)$ for all pairs $g, g' \in V(G)$. We denote an edge joining vertices λ and μ as $(\lambda)(\mu)$ in order to avoid confusion with composition.

We remark that $E(G!)$ is indeed a symmetric relation on $V(G!)$ because replacing g and g' in the definition with $\lambda^{-1}(g)$ and $\mu(g')$ gives $\lambda^{-1}(g)\mu(g') \in E(G) \iff gg' \in E(G)$, from which symmetry of G yields $gg' \in E(G) \iff \mu(g)\lambda^{-1}(g') \in E(G)$.

Observe that if λ is an automorphism of G, then $(\lambda)(\lambda^{-1}) \in E(G!)$, but it is not necessarily true that every edge of $G!$ has this form. Notice also that there is a loop at a vertex α of $G!$ precisely when α satisfies $gg' \in E(A) \iff \alpha(g)\alpha^{-1}(g') \in E(G)$ for all pairs $g, g' \in V(G)$. Such an α is called an *anti-automorphism* in [1], and it is proved there that for any bipartite graph B, the condition $G \times B \cong H \times B$ implies $G \cong H$ if and only if every anti-automorphism α of G factors as $\alpha = \lambda\mu$ for some $(\lambda)(\mu) \in E(G!)$. We are not concerned with such applications here. Our purpose is merely to determine factorials of some elementary graphs.

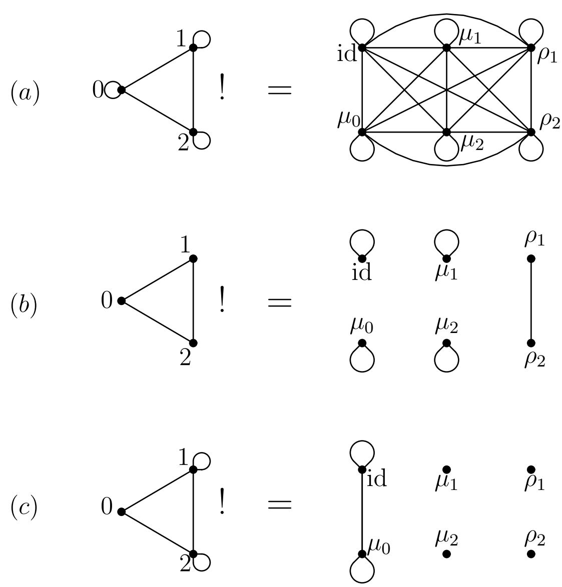

Figure 1: Some Factorials of Graphs

As our first example of a graph factorial, consider $K_n^*!$. From Definition 1 it follows that every pair of permutations of $V(K_n^*)$ must be adjacent in $K_n^*!$, so

$$K_n^*! \cong K_{n!}^*. \tag{1}$$

(Figure 1(a) illustrates the case $n = 3$.) It follows therefore that

$$K_n^*! \;\cong\; K_n^* \times K_{n-1}^* \times K_{n-2}^* \times \cdots \times K_3^* \times K_2^*.$$

Of course we expect no such nice formulas for $G!$ when G is arbitrary, but we do have the following immediate consequence of Definition 1.

$$\overline{G}\,! = G! \tag{2}$$

Figures 1(a)–(c) illustrate factorials for three graphs on the vertex set $\{0, 1, 2\}$. In each case, id is the identity permutation, μ_i is the transposition of the two vertices $\{0, 1, 2\} - \{i\}$, and ρ_1 and ρ_2 are clockwise rotations of $2\pi/3$ and $4\pi/3$, respectively.

The components of $G!$ have a high degree of structure. The following result from [1] is included here for completeness.

Proposition 1 *For any graph G, each nontrivial component of $G!$ is either a K_n^* for some $n \geq 1$, or a complete bipartite graph.*

Proof. We first prove by induction that given any odd walk $(\mu_1)(\mu_2)(\mu_3)\dots(\mu_{2p})$ in $G!$, the pair $(\mu_1)(\mu_{2p})$ is an edge of $G!$. This is trivial if $p = 1$. If $p > 1$, the induction hypothesis guarantees $(\mu_3)(\mu_{2p}) \in E(G!)$, so $(\mu_1)(\mu_2)(\mu_3)(\mu_{2p})$ is a walk in $E(G!)$. Using the fact that the edges of this walk are adjacent in $G!$, we get

$$\begin{aligned} gg' \in E(G) &\iff \mu_1(g)\mu_2^{-1}(g') \in E(G) \\ &\iff \mu_3^{-1}\mu_1(g)\mu_2\mu_2^{-1}(g') \in E(G) \\ &\iff \mu_3\mu_3^{-1}\mu_1(g)\mu_{2p}^{-1}\mu_2\mu_2^{-1}(g') \in E(G) \\ &\iff \mu_1(g)\mu_{2p}^{-1}(g') \in E(G). \end{aligned}$$

Therefore $(\mu_1)(\mu_{2p}) \in E(G!)$.

Now, if C is a component of $G!$ that happens to be bipartite, then there is an odd path between any vertices α, β in opposite partite sets of C. Thus $(\alpha)(\beta) \in E(G!)$, so C is a complete bipartite graph. On the other hand, if C has an odd cycle (possibly just a loop), then there must be an odd walk joining any pair of its vertices, so all pairs of vertices in C are adjacent, and $C \cong K_n^*$. ■

Proposition 1 is evident in Figures 1 (a)–(c), where each nontrivial component is either a K_n^* or a complete bipartite graph. These figures also illustrate what seems to be a general principle: Even though the individual components of $G!$ tend to be highly connected, unless G has a lot of symmetries, $G!$ tends to be relatively sparse. We expect $G!$ to have lots of isolated vertices in general, since it is unlikely that $gg' \in E(G) \iff \alpha(g)\beta^{-1}(g')$ will hold for all pairs g, g'.

2 Factorials of Special Graphs

This section is mainly concerned with computing factorials of special graphs. We first introduce some notation. Let $G \simeq H$ (as opposed to $G \cong H$) mean that the graphs induced on the edges of G and H are isomorphic. In other words, $G \simeq H$ means that G and H are isomorphic once all their isolated vertices are removed. This notational convenience simplifies our discussion because factorials often have many isolated vertices. Thus, for example, if G is the graph in Figure 1(c), we can simply say $G! \simeq K_2^*$, rather than having to invoke the more unwieldly $G! \cong K_2^* + 4K_1$.

We now consider factorials of complete graphs. Let us identify the symmetric group S_n with the permutations of $V(K_n)$, so $\mathrm{Aut}(K_n) = S_n$.

Proposition 2 *The edges of $K_n!$ are precisely the pairs $(\alpha)(\alpha^{-1})$ where $\alpha \in S_n$. Moreover, $K_n! \cong mK_1^* + \frac{1}{2}(n! - m)K_2$, where $m = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!}$.*

Proof. Let $\alpha \in S_n$. It is clear that $(\alpha)(\alpha^{-1}) \in E(K_n!)$ since

$$gg' \in E(K_p) \Longleftrightarrow \alpha(g)(\alpha^{-1})^{-1}(g') \in E(K_n)$$

for all $g, g' \in V(K_n)$. Suppose that there are permutations α and β of $V(K_n)$ such that $(\alpha)(\beta) \in E(K_n!)$ but $\beta \neq \alpha^{-1}$. Then there must exist a vertex g in K_n such that $\alpha(g) \neq \beta^{-1}(g)$. Thus $\alpha(g)\beta^{-1}(g) \in E(K_n)$. Since $(\alpha)(\beta) \in E(K_n!)$ it follows that $gg \in E(K_n)$ and we have a loop in K_n. This is a contradiction, hence $K_n!$ has no edges other than those of form $(\alpha)(\alpha^{-1})$.

The previous paragraph implies that each component of K_n is either a K_1^* or a K_2, where the vertex of each K_1^* is an involution. Therefore, proving $K_n! \cong mK_1^* + \frac{1}{2}(n! - m)K_2$ amounts to showing that S_n has exactly $m = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!}$ involutions. Now, the involutions of S_n are exactly those permutations that are products of disjoint transpositions. For $0 \leq k \leq \lfloor n/2 \rfloor$, the set of involutions that are products of k disjoint transpositions can be counted as follows. Identify a list $(i_1, i_2, \ldots, i_{2k})$ of elements from $\{0, 1, 2, \ldots, n-1\}$ (no repetitions) with the involution $(i_1, i_2)(i_3, i_4) \cdots (i_{2k-1}, i_{2k})$. There are $\frac{n!}{(n-2k)!}$ such lists, but this number over-counts the involutions. We must divide out by the $k!$ ways that these k transpositions can be arranged, and by the 2^k orderings of the elements within the transpositions. Thus S_n has $\frac{n!}{(n-2k)!2^k k!}$ involutions that are products of k disjoint transpositions. The total number of involutions is then $m = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!}$. ■

We now investigate cycles. However, as factorials of even cycles seem to be quite complicated, we will restrict our attention to odd cycles. Let C_{2n+1} denote the cycle on $2n+1$ vertices, and list its vertices as $\{0, 1, 2, \ldots, 2n\}$, where edges join the pairs i and $i+1$, with arithmetic done modulo $2n+1$. We identify the automorphism group $\mathrm{Aut}(C_{2n+1})$ with the dihedral group D_{2n+1}.

Proposition 3 *The edges of $C_{2n+1}!$ are precisely the pairs $(\alpha)(\alpha^{-1})$ where $\alpha \in \mathrm{Aut}(C_{2n+1}) = D_{2n+1}$. Thus, as D_{2n+1} consists of $2n+2$ involutions (including the identity) and $2n$ non-involutions, we have $C_{2n+1}! \simeq (2n+2)K_1^* + nK_2$.*

Proof. Certainly if $\alpha \in \mathrm{Aut}(C_{2n+1})$ it follows immediately from Definition 1 that $(\alpha)(\alpha^{-1})$ is an edge of $C_{2n+1}!$. Conversely, suppose $(\alpha)(\beta) \in E(C_{2n+1}!)$. We just need to show that $\alpha \in \mathrm{Aut}(C_{2n+1})$ and $\beta = \alpha^{-1}$. Consider the closed walk $0\,1\,2\,3 \cdots 2n\,0\,1\,2\,3 \cdots 2n\,0$ that traverses the cycle twice. Since $(\alpha)(\beta) \in E(C_{2n+1}!)$, it follows that

$$\alpha(0)\,\beta^{-1}(1)\,\alpha(2)\,\beta^{-1}(3) \cdots \alpha(2n)\,\beta^{-1}(0)\,\alpha(1)\,\beta^{-1}(2)\,\alpha(3) \cdots \beta^{-1}(2n)\,\alpha(0) \quad (3)$$

is also a (closed) walk in C_{2n+1}. As α and β are injective, all subwalks $\alpha(i)\,\beta^{-1}(i+1)\,\alpha(i+2)$ and $\beta^{-1}(i)\,\alpha(i+1)\,\beta^{-1}(i+2)$ must consist of two distinct (but incident) edges. From this we infer that the walk (3) traverses C_{2n+1} twice. Starting at $\alpha(0)$ and traversing half of this walk brings us to the vertex $\beta^{-1}(0)$, which equals our starting point. Thus $\alpha(0) = \beta^{-1}(0)$. Similarly, starting at $\beta^{-1}(1)$ and traversing half the walk brings us to $\alpha(1)$, which must equal the starting point $\beta^{-1}(1)$, so $\alpha(1) = \beta^{-1}(1)$. Continuing this line of reasoning, we have $\alpha(i) = \beta^{-1}(i)$ for each vertex i, so $\alpha = \beta^{-1}$.

Moreover, as $(\alpha)(\beta) \in E(C_{2n+1}!)$, we have $ij \in E(C_{2n+1})$ if and only if $\alpha(i)\beta^{-1}(j) \in E(C_{2n+1})$, if and only if $\alpha(i)\alpha(j) \in E(C_{2n+1})$. Thus $\alpha \in \mathrm{Aut}(C_{2n+1})$. ■

The next result describes factorials of complete bipartite graphs.

Proposition 4 *The edges of $K(m,n)!$ are precisely the pairs $(\alpha)(\beta)$ of permutations which both restrict to permutations of the two partite sets of $K(m,n)$, or which both interchange the partite sets. Moreover $K(m,n)! \simeq K^*_{m!n!}$ if $m \neq n$, and $K(m,m)! \simeq 2K^*_{m!m!}$.*

Proof. Label the two partite sets of $K(m,n)$ as X and Y. Suppose $(\alpha)(\beta) \in E(K(m,n)!)$. Any even walk $x_0y_0x_1y_1 \ldots x_{p-1}y_{p-1}x_p$ in $K(m,n)$ joining two vertices x_0 and x_p in X gives rise to a corresponding even walk

$$\alpha(x_0)\beta^{-1}(y_0)\alpha(x_1)\beta^{-1}(y_1)\ldots\alpha(x_{p-1})\beta^{-1}(y_{p-1})\alpha(x_p)$$

in $K(m,n)$. Therefore $\alpha(x_0)$ and $\alpha(x_p)$ belong to the same partite set. Consequently, either $\alpha(X) = X$ or $\alpha(X) = Y$. Similarly, either $\beta(X) = X$ or $\beta(X) = Y$. If $\alpha(X) = X$, then we must have $\beta(X) = X$, otherwise any edge xy of $K(m,n)$ leads to an edge $\alpha(x)\beta^{-1}(y)$ that joins vertices in the same partite set. Likewise, if $\alpha(X) = Y$, it follows that $\beta(X) = Y$. Therefore α and β either both preserve or both reverse the partite sets of $K(m,n)$.

Converesly, if α and β are two permutations that either both preserve or both reverse the partite sets of $K(m,n)$, then it is immediate that $xy \in E(K(m,n))$ if and only if $\alpha(x)\beta^{-1}(y) \in E(K(m,n))$, so $(\alpha)(\beta) \in E(K(m,n)!)$.

If $m \neq n$, then there are no permutations that interchange partite sets. By the above paragraph, the edges of $K(m,n)!$ are precisely the pairs of permutations that restrict to permutations of the partite sets. Since there are $m!n!$ such permutations, it follows that $K(m,n)! \simeq K^*_{m!n!}$. Similarly, if $m = n$, then there are $m!m!$ permutations that restrict to permutations on each partite set, and $m!m!$ permutations that interchange partite sets. By the above paragraphs, any two permutations of one type are adjacent in $K(m,m)!$, while two permutations of different types are not adjacent. Thus $K(m,m)! \simeq 2K^*_{m!m!}$. ■

3 A Sum Formula

Our aim in this section is to explore the factorial of the union of two connected graphs. We prove the following proposition, which gives conditions under which $(G + H)!$ is readily expressible in terms of $G!$ and $H!$.

Proposition 5 *Suppose that G and H are connected, nontrivial, non-bipartite graphs. If $G \times K_2 \ncong H \times K_2$, then $(G+H)! \simeq G! \times H!$.*

(We remind the reader that in general $G \times K_2 \cong H \times K_2$ does not imply $G \cong H$. The smallest counterexample is $G = 2K_1^*$ and $H = K_2$.) As a simple example of Proposition 5, let us use it to compute $(K_2^* + K_3)!$. Certainly $K_2^* \times K_2 \ncong K_3 \times K_2$, so Proposition 5 combined with Equation 1 and Proposition 2 give

$$(K_2^* + K_3)! \simeq K_2^*! \times K_3! = K_2^* \times (4K_1^* + K_2) = 4K_2^* + K(2,2).$$

A proof of Proposition 5 follows.

Proof. The proposition will follow from two claims.

Claim 1. *This claim concerns the edges $(\alpha)(\beta) \in E((G+H)!)$. We claim that, given such an edge, then either*

$$\begin{array}{ll} \alpha(V(G)) = V(G), & \alpha(V(H)) = V(H), \\ \beta(V(G)) = V(G), & \beta(V(H)) = V(H), \end{array} \tag{4}$$

or

$$\begin{array}{ll} \alpha(V(G)) = V(H), & \alpha(V(H)) = V(G), \\ \beta(V(G)) = V(H), & \beta(V(H)) = V(G). \end{array} \tag{5}$$

In words, either either α and β both restrict to permutations of $V(G)$ and $V(H)$, or they both interchange the sets $V(G)$ and $V(H)$.

To prove this claim, we consider the cases $\alpha(V(G)) = V(G)$ and $\alpha(V(G)) \neq V(G)$. We will examine these two possibilities and show that the first implies (4) and the second implies (5).

If $\alpha(V(G)) = V(G)$, then by the fact that α is a permutation of $V(G) \cup V(H)$, we have $\alpha(V(H)) = V(H)$. Now if $g \in V(G)$, we must have $\beta(g) \in V(G)$, for if $\beta(g) \in V(H)$ any edge $gg' \in E(G)$ would yield $\beta(g)\alpha^{-1}(g') \in E(G+H)$, and that is impossible since $\beta(g) \in V(H)$ and $\alpha^{-1}(g') \in V(G)$. Thus $\beta(V(G)) = V(G)$, so $\beta(V(H)) = V(H)$ also. Hence we have (4).

Now suppose $\alpha(V(G)) \neq V(G)$, so $\alpha(g_1) \in V(H)$ for some $g_1 \in V(G)$. Let x be an arbitrary vertex of $V(G)$ and take an even walk $g_1g_2g_3g_4 \cdots g_{2p}x$ in G joining g_1 to x. (An even walk is possible since G is a nontrivial non-bipartite graph.) Using the fact that $(\alpha)(\beta)$ is an edge of $(G+H)!$, we obtain the following walk in $G+H$.

$$\alpha(g_1)\beta^{-1}(g_2)\alpha(g_3)g_4 \cdots \beta^{-1}(g_{2p})\alpha(x) \tag{6}$$

Since the initial vertex $\alpha(g_1)$ is in H, the entire walk is in H. Therefore $\alpha(x) \in V(H)$ for any vertex x of G, so

$$\alpha(V(G)) \subseteq V(H).$$

Now, since $g_1 \in V(G)$ and $\alpha(g_1) \in V(H)$, the orbit $\alpha(g_1)$, $\alpha^2(g_1), \alpha^3(g_1) \ldots$ must contain vertices of both G and H, so there is a vertex $h_1 = \alpha^k(g_1) \in V(H)$ for which $\alpha(h_1) \in V(G)$, and repeating the above argument with the roles of G and H reversed gives

$$\alpha(V(H)) \subseteq V(G).$$

By finiteness it follows that $\alpha(V(G)) = V(H)$ and $\alpha(V(H)) = V(G)$.

Finally, note that since the walk (6) is in H, we have $\beta^{-1}(g_2) \in V(H)$ while $g_2 \in V(G)$, so $\beta(V(H)) \neq V(H)$. Repeating the exact same argument we used above under the assumption $\alpha(V(G)) \neq V(G)$, we get $\beta(V(H)) = V(G)$ and $\beta(V(G)) = V(H)$. Therefore we have deduced (5), so the proof of Claim 1 is complete.

Claim 2. *If $\alpha(V(G)) \neq V(G)$, then $G \times K_2 \cong H \times K_2$.* To prove this claim, note that if $\alpha(V(G)) \neq V(G)$, then Claim 1 implies that the conditions (5) hold, so α and β^{-1} restrict to injections $\alpha, \beta^{-1} : V(G) \to V(H)$; and since $(\alpha)(\beta) \in E((G+H)!)$, it must be that $gg' \in E(G) \iff \alpha(g)\beta^{-1}(g') \in E(H)$. Define a map $\Theta : V(G \times K_2) \to V(H \times K_2)$ as

$$\Theta(g, \varepsilon) = \begin{cases} (\alpha(g), \varepsilon) & \text{if } \varepsilon = 0 \\ (\beta^{-1}(g), \varepsilon) & \text{if } \varepsilon = 1. \end{cases}$$

It is immediate that Θ is bijective, and it is an isomorphism because

$$\begin{aligned} (g,0)(g',1) \in E(G \times K_2) &\iff gg' \in E(G) \\ &\iff \alpha(g)\beta^{-1}(g') \in E(H) \\ &\iff \Theta(g,0)\Theta(g',1) \in E(H). \end{aligned}$$

This completes the proof of Claim 2.

Now the proof of Proposition 5 can be completed. Suppose $G \times K_2 \ncong H \times K_2$. We need to show $(G+H)! \simeq G! \times H!$. Since $G \times K_2 \ncong H \times K_2$, claims 1 and 2 imply that conditions (4) hold. Thus the endpoints of any edge $(\alpha)(\beta) \in E((G+H)!)$ restrict to permutations α_G and β_G of $V(G)$, and permutations α_H, β_H of $V(H)$. Let $E[(G+H)!]$ denote the subgraph of $(G+H)!$ induced on the edges of $(G+H)!$. (That is, $E[(G+H)!]$ is $(G+H)!$ with isolated vertices removed.) Similarly, $E[G! \times H!]$ stands for $G! \times H!$ with isolated vertices removed. Consider the map

$$\begin{aligned} \Upsilon : V(E[(G+H)!]) &\to V(E[G! \times H!]) \\ \alpha &\mapsto (\alpha_G, \alpha_H). \end{aligned}$$

This is a homomorphism as follows: If $(\alpha)(\beta) \in E((G+H)!)$, it is straightforward that $gg' \in E(G) \Leftrightarrow \alpha_G(g)\beta_G^{-1}(g') \in E(G)$, so $(\alpha_G)(\beta_G) \in E(G!)$; and likewise $(\alpha_H)(\beta_H) \in E(H!)$. Thus $\Upsilon(\alpha)\Upsilon(\beta) = (\alpha_G, \alpha_H)(\beta_G, \beta_H) \in E(G! \times H!)$. Conversely, given an edge $(\alpha, \alpha')(\beta, \beta')$ of $G! \times H!$, let $\alpha \cup \alpha'$ (resp. $\beta \cup \beta'$) be the permutation of $V(G+H)$ that acts as α (resp. β) on $V(G)$ and α' (resp. β') on $V(H)$. It is straightforward that $(\alpha \cup \alpha')(\beta \cup \beta') \in E((G+H)!)$ and $\Upsilon(\alpha \cup \alpha')\Upsilon(\beta \cup \beta') = (\alpha, \alpha')(\beta, \beta')$. We have shown $(G+H)! \simeq G! \times H!$. ∎

Proposition 5 does not tell us the structure of $(G+H)!$ in the case $G \times K_2 \cong H \times K_2$. Indeed under these conditions it is not generally true that $(G+H)! \simeq G! \times H!$. For example, let $G = H = K_3^*$. Then by Equations (2), (1) and Proposition 4 we have

$$(G+H)! = \overline{K(3,3)}! = K(3,3)! \simeq 2K_{36}^* \ncong K_{36}^* \simeq G! \times H!.$$

This leads us to conjecture that Proposition 5 can be extended to state that if $G \times K_2 \cong H \times K_2$, then $(G+H)! = 2(G! \times H!)$. However, there are some intriguing subtleties that have not yet been resolved.

We remark in passing that the proof of Proposition 5 easily adapts to prove that if $G_1, G_2, \ldots, G_n$ are connected, nontrivial, non-bipartite graphs for which $G_i \times K_2 \not\cong G_j \times K_2$ for all $1 \leq i < j \leq n$, then $(G_1 + G_2 + \cdots + G_n)! \simeq G_1! \times G_2! \times \cdots \times G_n!$.

4 Open Questions

The ideas presented here lead to many unanswered questions. For example, Proposition 1 states that the nontrivial components of a factorial graph are either K_n^*'s or $K(m,n)$'s. Is the converse true? Is any graph with $k!$ vertices and whose nontrivial components are K_n^*'s and $K(m,n)$'s (and which has at least one K_n^*) a factorial of some other graph? The authors have found no examples of graphs whose factorials contain a K_n^* where n is odd. (Other than $n = 1$.) They have found no example of a factorial that has a $K(m,n)$ component with $m \neq n$. Is there some deeper structure that governs the types of components that a factorial can have?

Computing factorials of special graphs is an entertaining problem. In particular, finding the factorials of even cycles appears to be somewhat subtle. There are many special graphs (complete n-partite graphs, hypercubes, etc.) whose factorials have not been investigated.

Finally the results of [1] show that factorials are related to the problem of direct-product cancellation of graphs. Moreover, our sum formula $(G + H)! \simeq G! \times H!$ is structurally similar to the exponentiation formula $K^{(G+H)} \cong K^G \times K^H$, and exponentiation is also linked with cancellation in [3]. This is a connection that may deserve further investigation.

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