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Note on the normalized Laplacian eigenvalues of signed graphs*

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Abstract

The normalized Laplacian of a graph was introduced by F.R.K. Chung and has been studied extensively over the last decade. In this paper, we introduce the notion of the normalized Laplacian of signed graphs and extend some fundamental concepts of the normalized Laplacian from graphs to signed graphs.

1 Introduction

Signed graphs were introduced by Harary [6] in connection with the study of theory of social balance. Since then they have been extensively studied because they come up naturally in many unrelated areas such as topological graph theory, group theory and the classical root systems (see [2, 11]). *A signed graph* $\Gamma = (G, \sigma)$ consists of a simple graph $G = (V, E)$ and a mapping $\sigma : E \to \{+, -\}$. The graph G is called the

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underlying graph of Γ. We will write $V(\Gamma)$ for the vertex set and $E(\Gamma)$ for the edge set of Γ if necessary. For any $u \in V(\Gamma)$, let d_u denote the degree of u.

Let C be a cycle of a signed graph $\Gamma = (G, \sigma)$. The sign of C is denoted by $\text{sgn}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is $+$(respectively, $-$) is called *positive*(respectively, *negative*). A signed graph is called *balanced* if all its cycles are positive. Suppose $\theta : V \to \{+, -\}$ is any sign function. Switching Γ by θ means forming a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ whose underlying graph is the same as G, but whose sign function is defined on an edge $e = uv$ by $\sigma^\theta(e) = \theta(u)\sigma(e)\theta(v)$. Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs with the same underlying graph. Γ_1 and Γ_2 are called *switching equivalent*, written $\Gamma_1 \simeq \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1^\theta$. We denote by $\mathcal{SE}(\Gamma)$ the set of signed graphs switching equivalent to Γ. Switching preserves many signed-graphic invariant, such as the set of positive cycles.

The *standard Laplacian matrix* $L := L(\Gamma) = (L_{uv})$ of a signed graph Γ of order n is the $n \times n$ matrix L defined as follows:

$$L_{uv} = \begin{cases} d_u & \text{if} \quad u = v, \\ -\sigma(uv)1 & \text{if} \quad uv \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Hou et al. [8] described $L(\Gamma)$ by means of its quadratic form:

$$x^T L(\Gamma) x = \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j) x_j)^2, \tag{1}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

Lemma 1 ([8]) *Let $\Gamma = (G, \sigma)$ be a connected signed graph and $L(\Gamma)$ be its Laplacian matrix. Then* $\det L(\Gamma) = 0$ *if and only if Γ is balanced.*

Two matrices M_1 and M_2 of order n are called *signature similar* if there exists a signature matrix, that is, a diagonal matrix $S = \text{diag}(s_1, s_2, \ldots, s_n)$ with diagonal entries $s_i = \pm 1$ such that $M_2 = S M_1 S$.

Lemma 2 ([8]) *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be signed graphs on the same underlying graph G. Then $\Gamma_1 \simeq \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.*

The *normalized Laplacian* of Γ is the $n \times n$ matrix $\mathcal{L} := \mathcal{L}(\Gamma) = (\mathcal{L}_{uv})$ given by

$$\mathcal{L}_{uv} = \begin{cases} 1 & \text{if} \quad u = v \quad \text{and} \quad d_u \neq 0, \\ -\sigma(uv)\frac{1}{\sqrt{d_u d_v}} & \text{if} \quad uv \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Let D denote the diagonal matrix with the (u, u)th entry having value d_u. For a general graph, we have

$$\begin{aligned} \mathcal{L} &= D^{-1/2} L D^{-1/2} \\ &= I - D^{-1/2} A D^{-1/2} \end{aligned} \tag{2}$$

with the convention that $D^{-1}(u, u) = 0$ if $d_u = 0$, where $A = (\sigma(uv)a_{uv})$ is the adjacency matrix of Γ (i.e., $a_{uv} = 1$ if u is adjacent to v, and 0 otherwise). We note that $\mathcal{L}$ can be written as

$$\mathcal{L} = SS^t,$$

where S is the matrix whose rows are indexed by the vertices and whose columns are indexed by the edges of G such that each column corresponding to an edge $e = uv$ has an entry $1/\sqrt{d_u}$ in the row corresponding to u, $-\sigma(uv)1/\sqrt{d_v}$ in the row corresponding to v, and zero elsewhere.

Remark 1 *From Lemma 2, we know that for a signed graph Γ, the signed graphs in the switching equivalent class $\mathcal{SE}(\Gamma)$ share the same Laplacian spectrum. And we shall show they also share the same normalized Laplacian spectrum. For any two signed graphs $\Gamma_1, \Gamma_2 \in \mathcal{SE}(\Gamma)$, by Lemma 2, there exists a signature matrix S such that $L(\Gamma_1) = SL(\Gamma_2)S$. Recalling that the relation between the normalized Laplacian and Laplacian as in Eq. (2), we have*

$$\begin{aligned} \mathcal{L}(\Gamma_1) &= D^{-1/2} L(\Gamma_1) D^{-1/2} \\ &= D^{-1/2} S L(\Gamma_2) S D^{-1/2} \\ &= S D^{-1/2} L(\Gamma_2) D^{-1/2} S \\ &= S \mathcal{L}(\Gamma_2) S. \end{aligned}$$

So $\mathcal{L}(\Gamma_1)$ and $\mathcal{L}(\Gamma_2)$ are signature similar and share the same spectrum.

The normalized Laplacian was introduced by F.R.K. Chung [4] and has been intensively studied in recent years. Grossman [5] has investigated lower bounds for the second least eigenvalue. Chen et al. [3, 10] studied Cauchy interlacing-type properties of the normalized Laplacian. The problem of relating the eigenvalues of the normalized Laplacian for a weighted graph and its subgraph was considered in [1]. Kirkland [9] reconsidered a previously published bound relating some parameters of graphs to the eigenvalues of the normalized Laplacian and provided a corrected version of the bound. In the next section of this paper we first investigate some properties of the normalized Laplacian of signed graphs and then obtain an interlacing-type property of the normalized Laplacian spectra between a signed graph and its underlying graph.

2 Main Results

For a signed graph Γ, the normalized Laplacian $\mathcal{L}(\Gamma)$ is symmetric and positive semi-definite, so its eigenvalues are all real and non-negative and denoted by $0 \leq$

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. We adapt the Courant-Fischer theorem to $\mathcal{L}$ using harmonic eigenfunctions. Recall that

$$\mathcal{L} = D^{-1/2} L D^{-1/2}.$$

We assume that $D^{1/2}$ is invertible, that is, there are no isolated vertices in Γ.

For vectors g and g_j, define the vectors

$$f = D^{-1/2} g, \qquad f_j = D^{1/2} g_j.$$

Note that,

$$g \perp g_{k+1}, g_{k+2}, \ldots, g_n$$

if and only if

$$f \perp f_{k+1}, f_{k+2}, \ldots, f_n.$$

Applying the Courant-Fischer theorem to get the eigenvalue λ_k of $\mathcal{L}$ gives

$$\begin{aligned}
\lambda_k &= \min_{g_{k+1}, g_{k+2}, \ldots, g_n \in \mathbb{R}^n} \max_{\substack{g \neq 0 \\ g \perp g_{k+1}, g_{k+2}, \ldots, g_n}} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} \\
&= \min_{g_{k+1}, g_{k+2}, \ldots, g_n \in \mathbb{R}^n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\langle f, L f \rangle}{\langle D^{1/2} f, D^{1/2} f \rangle} \\
&= \min_{f_{k+1}, f_{k+2}, \ldots, f_n \in \mathbb{R}^n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\langle f, L f \rangle}{\langle D^{1/2} f, D^{1/2} f \rangle} \\
&= \min_{f_{k+1}, f_{k+2}, \ldots, f_n \in \mathbb{R}^n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v} \qquad (3)
\end{aligned}$$

where $\sum_{u \sim v}$ denotes the sum over all unordered pairs $\{u, v\}$ for which u and v are adjacent. The second line and the third line depend on the invertibility of $D^{1/2}$ so that $g \neq 0$ if and only if $f \neq 0$ and minimizing over vectors $g_{k+1}, g_{k+2}, \ldots, g_n$ is equivalent to minimizing over vectors $f_{k+1}, f_{k+2}, \ldots, f_n$. The fourth line depends on the equation (1) so we have $\langle f, Lf \rangle = \sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2$. The vector f can be viewed as a function $f(v)$ on the vertex set. The function $f(v)$ is called a harmonic eigenfunction corresponding to λ_k.

The other half of the Courant-Fischer theorem gives

$$\begin{aligned}
\lambda_k &= \max_{g_1, g_2, \ldots, g_{k-1} \in \mathbb{R}^n} \min_{\substack{g \neq 0 \\ g \perp g_1, g_2, \ldots, g_{k-1}}} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} \\
&= \max_{f_1, f_2, \ldots, f_{k-1} \in \mathbb{R}^n} \min_{\substack{f \neq 0 \\ f \perp f_1, f_2, \ldots, f_{k-1}}} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v}. \qquad (4)
\end{aligned}$$

In particular, we have

$$\lambda_1 = \inf_{f \neq 0} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v}, \qquad (5)$$

$$\lambda_n = \sup_{f \neq 0} \frac{\sum_{u \sim v}(f(u) - \sigma(uv)f(v))^2}{\sum_v f(v)^2 d_v}. \tag{6}$$

Similar to the result in [8], we have the following result by means of which we can compare the spectral radius of signed graphs with that of all-negative graphs. Firstly we need the following preliminary lemma.

Lemma 3 ([8]) *Let $\Gamma = (G, \sigma)$ be a signed graph. Then the following conditions are equivalent:*

(1). $\Gamma = (G, \sigma)$ *is a signed graph such that all odd cycles are negative and all even cycles are positive.*

(2). There exists a partition $V(\Gamma) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is positive and every edge within V_1 or V_2 is negative.

(3). $\Gamma = (G, \sigma) \simeq (G, -)$.

Lemma 4 *Let $\Gamma = (G, \sigma)$ be a connected signed graph of order n. Then*

$$\lambda_n(\sigma) \leq \lambda_n(-)$$

where equality holds if and only if $(G, \sigma) \simeq (G, -)$.

PROOF: From the equation (6), let $f = (f(v) : v \in V(\Gamma))^T$ be a harmonic eigenfunction corresponding to $\lambda_n(\sigma)$. Taking absolute value of $f's$, we get a vector $h = (|f(v)| : v \in V(\Gamma))^T$, and

$$\begin{aligned}\lambda_n(\sigma) &= \frac{\sum_{u \sim v}(f(u) - \sigma(uv)f(v))^2}{\sum_v f(v)^2 d_v} \\ &\leq \frac{\sum_{u \sim v}(|f(u)| + |f(v)|)^2}{\sum_v f(v)^2 d_v} \\ &\leq \lambda_n(-)\end{aligned}$$

If $\lambda_n(\sigma) = \lambda_n(-)$, then $-\sigma(uv)f(u)f(v) = |f(u)||f(v)|$ and h is a harmonic eigenfunction corresponding to $\lambda_n(-)$. Since $\mathcal{L}(-)$ is nonnegative and irreducible, f has no zero entries. Let $S = \{v : f(v) > 0\}$ and $T = \{v : f(v) < 0\}$. From $-\sigma(uv)f(u)f(v) = |f(u)||f(v)|$ it follows that if $\sigma(uv) < 0$ then $f(u)f(v) > 0$, and if $\sigma(uv) > 0$ then $f(u)f(v) < 0$. By Lemma 3, $(G, \sigma) \simeq (G, -)$.

Conversely, if $(G, \sigma) \simeq (G, -)$, by Remark 1, $\mathcal{L}(\sigma)$ and $\mathcal{L}(-)$ share the same spectrum and $\lambda_n(\sigma) = \lambda_n(-)$ particularly. □

Lemma 5 ([11]) *Let G be a graph and T a maximal forest of G. Then each switching equivalent class of signed graphs on the graph G has a unique representative which is $+$ on the edges of T. Indeed, given any prescribed sign function $\sigma_T : T \to \{+, -\}$, each switching class has a single representative which agrees with σ_T on T.*

In a signed graph Γ, let C^- denote the set of all negative cycles. Switching leaves the sign of cycles invariant, so the number of C^-, $|C^-|$, remains unchanged by switching. Let $E^-(\Gamma)$ denote the subset of negative edges in $E(\Gamma)$.

Lemma 6 *For a signed graph Γ, there exists such a $\Gamma' \in \mathcal{SE}(\Gamma)$ such that*

$$|E^-(\Gamma')| \leq |C^-|.$$

PROOF: By Lemma 5, we can select a $\Gamma' \in \mathcal{SE}(\Gamma)$ such that Γ' has sign $+$ on edges of a maximal forest T of Γ'. We know that for each negative edge e in Γ', $T + e$ has one negative cycle. By induction on the value of $|E^-(\Gamma')|$, we can show that the result holds. □

Theorem 1 *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices and m edges. Then*

(i) The spectrum of a signed graph is the union of the spectra of its connected components.

(ii) $\sum_i \lambda_i \leq n$, equality holds if and only if there are no isolated vertices in G.

(iii) For $n \geq 2$ and G is nonempty,

$$0 \leq \lambda_1 < 1$$

where the left equality in the first inequality holds if and only if some connected component of Γ is balanced. When G has no isolated vertices, we have

$$\lambda_n > 1, \qquad \lambda_1 \leq \frac{2|C^-|}{m}.$$

(iv) For all $i \leq n$,

$$\lambda_i \leq 2.$$

with $\lambda_n = 2$ if and only if $\Gamma \simeq (G, -)$.

PROOF: (i) follows from the definition and (ii) follows from considering the trace of $\mathcal{L}$.

(iii) We have that $\det \mathcal{L} = 0$ if and only if $\det L = 0$ by $\mathcal{L} = D^{-1/2} L D^{-1/2}$. By Lemma 1, $\lambda_1 = 0$ if and only if some connected component of Γ is balanced.

By Eq. (5), choosing f such that $f(v) = 1$, for an arbitrary fixed vertex v; $f(w) = 0$, otherwise, we have

$$\begin{aligned}\lambda_1 &= \inf_{f \neq 0} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v} \\ &\leq \frac{\sum_{u \sim v} 1}{d_v} \\ &= 1.\end{aligned}$$

In the same way, by Eq. (6), we can show that $\lambda_n \geq 1$. If $\lambda_1 = 1$, then all eigenvalues of $\mathcal{L}$ are 1 by the result (ii). From the positive semi-definiteness of $\mathcal{L}$ and $n\lambda_1 = \text{trace}(\mathcal{L})$, we obtain immediately that $\mathcal{L} = I$ and so $D^{-1/2}AD^{-1/2} = 0$. This is a contradiction with the fact that G is not empty.

If G has no isolated vertex, i.e., $\sum_i \lambda_i = n$, $\lambda_n = 1$ means eigenvalues of $\mathcal{L}$ all equal 1, that is, $n\lambda_n = \text{trace}(\mathcal{L})$. It is a contradiction by the same way as above.

Choosing $f(v) \equiv 1$ in equation (5) and considering Lemma 6 and Remark 1, we have

$$\begin{aligned}\lambda_1 &= \inf_{f\neq 0} \frac{\sum_{u\sim v}(f(u) - \sigma(uv)f(v))^2}{\sum_v f(v)^2 d_v} \\ &\leq \frac{\sum_{\sigma(uv)=-} 4}{2m} \\ &\leq \frac{2|C^-|}{m}.\end{aligned}$$

(iv) Without loss of generality, suppose that G is connected. In this case, $D^{1/2}$ is invertible and consider the nonnegative matrix

$$M = D^{-1/2}\mathcal{L}(-)D^{1/2}.$$

Since row sums of M all equal 2, $\lambda_n(M) = 2$. Consequently, we have

$$\lambda_n(\sigma) \leq \lambda_n(-) = \lambda_n(M) = 2.$$

So by Lemma 4, $\lambda_n(\sigma) = \lambda_n(-) = 2$ if and only if $\Gamma \simeq (G, -)$.

□

Proposition 1 ([7]) *Let A, B be $n \times n$ Hermitian matrices and suppose that B has rank at most r. Then*

(1). $\lambda_k(A + B) \leq \lambda_{k+r}(A) \leq \lambda_{k+2r}(A + B), k = 1, \ldots, n - 2r$

(2). $\lambda_k(A) \leq \lambda_{k+r}(A + B) \leq \lambda_{k+2r}(A), k = 1, \ldots, n - 2r$

In a signed graph $\Gamma = (G, \sigma)$, $\mathcal{L}(\sigma)$ can be viewed as

$$\mathcal{L}(\sigma) = \mathcal{L}(G) + D^{-1/2}B(\Gamma)D^{-1/2},$$

where $B(\Gamma) = (b_{uv})$ is defined to be a symmetric matrix as

$$b_{uv} = \begin{cases} 2 & \text{if} \quad \sigma(uv) = -, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the rank of $B(\Gamma)$ is r, so is the rank of $D^{-1/2}B(\Gamma)D^{-1/2}$ by the invertibility of $D^{-1/2}$. By Proposition 1, we have

$$\lambda_k(G) \leq \lambda_{k+r}(\sigma) \leq \lambda_{k+2r}(G), k = 1, \ldots, n - 2r \tag{7}$$

From the definition it is easy to see that the rank of $B(\Gamma)$ depends on the edge labelling σ. As a result, the rank of $B(\Gamma)$ may be changed by switching the signed graph. So in case that the rank of $B(\Gamma)$ is too large, the result (7) may be trivial.

For any $\Gamma' \in \mathcal{SE}(\Gamma)$, let $\mathcal{X}(\Gamma')$ denote the family of subsets X of $V(\Gamma')$ such that each negative edge is incident to at least one vertex in X. Now we define the *N-domination number* $\mathcal{N}$ as follows:

$$\mathcal{N} = \min\{|X| : X \in \mathcal{X}(\Gamma'), \Gamma' \in \mathcal{SE}(\Gamma)\}, \tag{8}$$

where some subset X such that $|X| = \mathcal{N}$ is called an *N-dominating set* of Γ'(i.e., the minimum subset of vertices dominating all negative edges with the understanding of switching equivalence).

We can see that for $\mathcal{N}$ there are two properties as follows:

(1). $\mathcal{N} = 0$ if and only if Γ is balanced.

(2). $\mathcal{N} \le |C^-|$, where $|C^-|$ denotes the number of negative cycles in Γ.

Theorem 2 *For a signed graph $\Gamma = (G, \sigma)$ on n vertices without isolated vertex, $\mathcal{N}(\Gamma) = k$ defined as in Eq. (8). The spectrum of $\mathcal{L}(G)$ and $\mathcal{L}(\Gamma)$ are, respectively,*

$$0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n \qquad and \qquad 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

Then

$$\mu_{i-k} \le \lambda_i \le \mu_{i+k}, i = 1, 2, \ldots, n,$$

where $\mu_{1-k} = \mu_{2-k} = \cdots = \mu_0 = 0$ *and* $\mu_{n+1} = \mu_{n+2} = \cdots = \mu_{n+k} = 2.$

Proof: Without loss of generality, suppose that $X = \{1, 2, \ldots, k\}$ is an N-dominating set of Γ. Recall that $\mathcal{L}(\Gamma) = D^{-1/2}LD^{-1/2}$ and no vertex of G is isolated, $D^{-1/2}$ is always invertible. Applying the equation (4) to get the eigenvalue λ_i of $\mathcal{L}(\Gamma)$ gives

$$\begin{aligned}
\lambda_i &= \max_{w_1, w_2, \ldots, w_{i-1} \in \mathbb{R}^n} \min_{\substack{f \neq 0, f \in \mathbb{R}^n \\ f \perp w_1, w_2, \ldots, w_{i-1}}} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v} \\
&= \max_{w_1, w_2, \ldots, w_{i-1} \in \mathbb{R}^n} \min_{\substack{f \neq 0, f \in \mathbb{R}^n \\ f \perp w_1, w_2, \ldots, w_{i-1}}} \frac{\sum_{\sigma(uv)=+} (f(u) - f(v))^2 + \sum_{\sigma(uv)=-} (f(u) + f(v))^2}{\sum_v f(v)^2 d_v} \\
&= \max_{w_1, w_2, \ldots, w_{i-1} \in \mathbb{R}^n} \min_{\substack{f \neq 0, f \in \mathbb{R}^n \\ f \perp w_1, w_2, \ldots, w_{i-1}}} \frac{\sum_{u \sim v} (f(u) - f(v))^2 + 4 \sum_{\sigma(uv)=-} f(u) f(v)}{\sum_v f(v)^2 d_v} \\
&\le \max_{w_1, w_2, \ldots, w_{i-1} \in \mathbb{R}^n} \min_{\substack{f \neq 0, f \in \mathbb{R}^n \\ f \perp w_1, w_2, \ldots, w_{i-1} \\ f \perp e_1, e_2, \ldots, e_k}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
&\le \max_{w_1, w_2, \ldots, w_{i+k-1} \in \mathbb{R}^n} min_{\substack{f \neq 0, f \in \mathbb{R}^n \\ f \perp w_1, w_2, \ldots, w_{i+k-1}}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
&= \mu_{i+k}
\end{aligned}$$

where vectors $e_i (1 \le i \le k)$ in the fourth line denote the n-vector with 1 as the ith element and 0 otherwise. So if $f \perp e_1, e_2, \ldots, e_k$, that is, the values of f in $1, 2, \ldots, k$ all equal 0, then $\sum_{\sigma(uv)=-} f(u)f(v) = 0$.

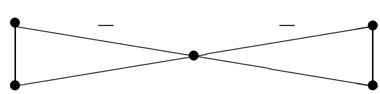
By a similar method, by the equation (3) we have

$$\begin{aligned}
\lambda_i &= \min_{w_{i+1}, w_{i+2}, \ldots, w_n \in \mathbb{R}^n} \max_{\substack{f \ne 0, f \in \mathbb{R}^n \\ f \perp w_{i+1}, w_{i+2}, \ldots, w_n}} \frac{\sum_{u \sim v} (f(u) - \sigma(uv) f(v))^2}{\sum_v f(v)^2 d_v} \\
&= \min_{w_{i+1}, w_{i+2}, \ldots, w_n \in \mathbb{R}^n} \max_{\substack{f \ne 0, f \in \mathbb{R}^n \\ f \perp w_{i+1}, w_{i+2}, \ldots, w_n}} \frac{\sum_{u \sim v} (f(u) - f(v))^2 + 4 \sum_{\sigma(uv)=-} f(u)f(v)}{\sum_v f(v)^2 d_v} \\
&\ge \min_{w_{i+1}, w_{i+2}, \ldots, w_n \in \mathbb{R}^n} \max_{\substack{f \ne 0, f \in \mathbb{R}^n \\ f \perp w_{i+1}, w_{i+2}, \ldots, w_n \\ f \perp e_1, e_2, \ldots, e_k}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
&\ge \min_{w_{i-k+1}, w_{i+2}, \ldots, w_n \in \mathbb{R}^n} \max_{\substack{f \ne 0, f \in \mathbb{R}^n \\ f \perp w_{i-k+1}, w_{i+2}, \ldots, w_n}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \\
&= \mu_{i-k}
\end{aligned}$$

Consequently, $\mu_{i-k} \le \lambda_i \le \mu_{i+k}$. For convenience, we set $\mu_{1-k} = \mu_{2-k} = \cdots = \mu_0 = 0$ and $\mu_{n+1} = \mu_{n+2} = \cdots = \mu_{n+k} = 2$. □

In general, the result in Theorem 2 is always better than the result (7) because $\mathcal{N}(\Gamma)$ is usually not more than the rank of B.

Example 1 *For the signed graph*



Signed graph $\Gamma = (G, \sigma)$.

where the edges unlabelled are positive. It is easy to see that all cycles in Γ *are negative and so* $(G, \sigma) \simeq (G, -)$. *Here,*

$$B(\Gamma) = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we have $r(B(\Gamma)) = 2 > \mathcal{N}(\Gamma) = 1$.

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