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On split and almost CIS-graphs*

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Abstract

A *CIS-graph* is defined as a graph whose every maximal clique and stable set intersect. These graphs have many interesting properties, yet, it seems difficult to obtain an efficient characterization and/or polynomial-time recognition algorithm for CIS-graphs.

An *almost CIS-graph* is defined as a graph that has a unique pair (C, S) of disjoint maximal clique C and stable sets S. We conjecture that almost CIS-graphs are exactly split graphs that have a unique split partition and prove this conjecture for a large hereditary class of graphs that contains, for example, chordal graphs and P_5-free graphs, as well as their complements, etc. We also prove the conjecture in case $|C| = |S| = 2$ and show that the vertex-set $R = V \setminus (C \cup S)$ cannot induce a threshold graph, although we do not prove that $R = \emptyset$, as the conjecture suggests.

1 Introduction

By definition [1], every maximal clique C and every maximal stable set S of a *CIS-graph* G intersect. In this case we also say that G has the *CIS-property*. Otherwise, clearly, there is a disjoint pair (C, S) in G, which is called a *non-CIS-pair*. The above characterization of CIS-graphs is simple but not efficient, since the numbers of maximal cliques and stable sets of a graph can be exponentially large. CIS-graphs were considered by Zang [5], Deng, Li, and Zang [3], and Andrade, Boros, and Gurvich [1]. Some necessary and some sufficient conditions were obtained for the

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CIS-property to hold, yet, it seems difficult to obtain an efficient characterization or polynomial recognition algorithm for CIS-graphs. A similar class of graphs, which might have much simpler structure, was also introduced in [1].

Definition 1. *An* almost CIS-graph *has a unique non-CIS-pair* (C, S)*.*

It seems that almost CIS-graphs are closely related to the following simple and well-known class of graphs. A *split graph* G admits a partition $A \cup B = V(G)$, a *split partition*, such that A is a clique and B is a stable set. Split graphs are exactly $(2K_2, C_4, C_5)$-free graphs according to the result of Foldes and Hammer [4]. A split graph may have several split partitions. For example, Bull (called also A-graph) in Figure 1 has two split partitions, namely $A = \{a, b, e\}$, $B = \{c, d\}$ and $A' = \{a, b\}$, $B' = \{c, d, e\}$.

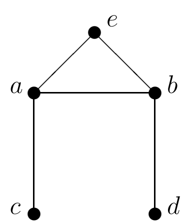

Figure 1: Bull.

If we delete vertex e, then we obtain the split graph P_4, which has a unique split partition $A = \{a, b\}$, $B = \{c, d\}$. The sets A and B are a maximal clique and stable set, respectively, and they are disjoint. It is easy to verify that P_4 is an almost CIS-graph, while Bull is a CIS-graph. The following two claims generalize above simple observations.

Proposition 1. [1] *Every split graph has at most one non-CIS pair.*

For completeness we reproduce a simple prove from [1].

Proof. Let (A, B) be a split partition of a split graph G, where A is a clique and B is a stable set. Obviously, a maximal clique C distinct from A consists of a proper subset of A and one vertex $u \in B$; respectively, a maximal stable set S distinct from B consists of a proper subset of B and one vertex $v \in A$. It is easy to see that $C \cap S = \{u\}$ if u and v are non-adjacent, and $C \cap S = \{v\}$ otherwise. □

In other words, every split graph is either CIS or almost CIS. The next claim shows when the first option takes place.

Proposition 2. *A split graph G has more than one split partition if and only if G is a CIS-graph.*

Proof. Let $A \cup B$ be a split partition of G. By Proposition 1, (A, B) is the only possible non-CIS-pair (C, S) in G. If, indeed, (A, B) is such a pair then G is an almost CIS-graph, by the definition. If not, then either clique A or stable set B is not maximal. In this case G is a CIS-graph. □

Thus, every split graphs with a unique split partition is an almost CIS-graph. It was conjectured in [1] that the inverse claim holds too.

Conjecture 1. *Every almost CIS-graph is a split graph with a unique split partition.*

In other words, every non-split graph has at least two non-CIS pairs. In contrast, by Proposition 1, split graphs have at most one. Somewhat surprisingly, this simple would be characterization of split graphs is not obvious (and, perhaps, even not true). Here we obtain partial results in its support. In particular, we show that it holds for a hereditary class that contains many known extensions of cographs and split graphs, for example, P_5-free graphs, chordal graphs, C_4-free graphs, as well as their complements. We also prove the conjecture in case $|C| = |S| = 2$ and show that the vertex-set $R = V(G) \setminus (C \cup S)$ cannot induce a threshold graph, while the conjecture means that $R = \emptyset$.

2 Partial results

Figure 2 shows a graph H and its complement $\overline{H}$.

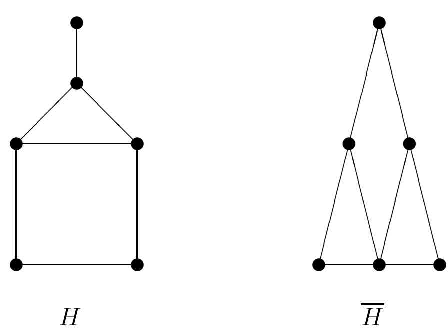

Figure 2: The graph H and its complement $\overline{H}$.

Using H and $\overline{H}$, we define a set $\mathbf{Z}$ of graphs.

Definition 2. *A graph G belongs to $\mathbf{Z}$ if and only if G contains both H and $\overline{H}$ as induced subgraphs, and G is minimal with this property.*

Here minimality of G means that every proper induced subgraph of G does not contain either H or $\overline{H}$. Every graph in $\mathbf{Z}$ has at least 7 vertices and at most 12 vertices. A vertex-minimal member of $\mathbf{Z}$ is shown in Figure 3.

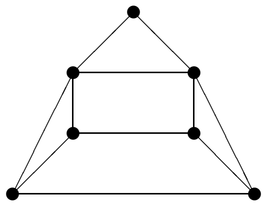

Figure 3: A minimal member of the set **Z**.

We shall use **Z** as the set of minimal forbidden induced subgraphs for a hereditary class.

We use notation $X \sim Y$ (respectively, $X \not\sim Y$) for disjoint subsets $X, Y \subseteq V(G)$ in a graph G to indicate that every vertex of Y is adjacent (respectively, non-adjacent) to every vertex of Y.

Theorem 1. Conjecture 1 *holds for the class* $\mathcal{P}$ *of all* **Z***-free graphs.*

Proof. Let $G \in \mathcal{P}$ be a non-split almost CIS-graph, and let (C, S) be the unique pair consisting of a maximal clique C disjoint from a maximal stable set S. The set $C \cup S$ induces a split graph, therefore there is a vertex $u \in V(G) \setminus (C \cup S)$. We denote:

$$C^- = C \setminus N(u),$$

$$C^+ = C \cap N(u),$$

$$S^- = S \setminus N(u),$$

$$S^+ = S \cap N(u),$$

where $N(u)$ is the neighborhood of u in G. Let L be the set of all vertices $x \notin C \cup S$ that are non-adjacent to the largest number of vertices in S. Assumption (1): $u \in L$ and u is adjacent to the maximum number of vertices in C among all vertices $x \in L$.

The set $C^+ \cup \{u\}$ induces a clique which is disjoint from the maximal stable set S. It implies that there exists a vertex $s_1 \in S$ with $s_1 \sim C^+ \cup \{u\}$ [s_1 is adjacent to all vertices in $C^+ \cup \{u\}$]. Clearly, $s_1 \in S^+$. We subdivide C^-: $C_1^- = \{x \in C^- : x \sim s_1\}$ and $C_2^- = \{x \in C^- : x \not\sim s_1\}$.

Similarly, the set $S^- \cup \{u\}$ induces a stable set which is disjoint from the maximal clique C. It implies that there exists a vertex $c_1 \in C$ with $c_1 \not\sim S^- \cup \{u\}$ [c_1 is non-adjacent to all vertices in $S^- \cup \{u\}$].

Claim 1. $c_1 \sim S^+$.

Proof. Suppose that c_1 is non-adjacent to some vertex $x \in S^+$. Since S a maximal stable set, the vertex c_1 is adjacent to some $y \in S$. Since $c_1 \notin S^- \cup \{u\}$, we have $y \in S^+$, and therefore y is adjacent to u. Consider the stable set S' consisting of c_1

and all its non-neighbors in S. We have $S^- \subseteq S'$ and $x \in S'$. The clique $C' = \{u, y\}$ is disjoint from S', therefore there exists a vertex z such that $z \sim C'$ and $z \not\sim S'$. We have $z \notin C \cup S$. Indeed, z is non-adjacent to $c_1 \in C$ and z is adjacent to $y \in S$. We obtain a contradiction to Assumption (1): z is non-adjacent to a larger number of vertices S than u. Indeed, $z \not\sim S^- \cup \{x\}$ while $u \not\sim S^-$ and $u \sim S^+$. □

Claim 1 shows that $c_1 \sim s_1$ and therefore $c_1 \in C_1^-$, see an illustration in Figure 4.

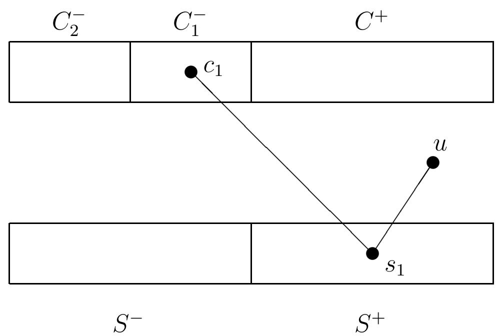

Figure 4: An illustration.

By maximality of C, the vertex s_1 is non-adjacent to some $c_2 \in C$. Clearly, $c_2 \in C_2^-$. Now we subdivide S^-: $S_1^- = \{x \in S^- : x \not\sim c_2\}$ and $S_2^- = \{x \in S^- : x \sim c_2\}$.

The set $C_1^- \cup C^+ \cup \{s_1\}$ induces a clique which is disjoint from the stable set $S_1^- \cup \{u, c_2\}$. Hence there exists a vertex v such that $v \sim C_1^- \cup C^+ \cup \{s_1\}$ and $v \not\sim S_1^- \cup \{u, c_2\}$, see Figure 5.

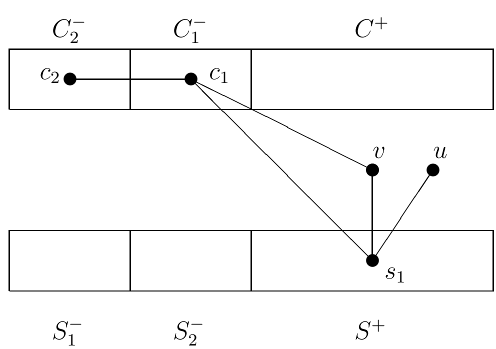

Figure 5: Another illustration.

Claim 2. *The vertex v is adjacent to some vertex in S_2^-.*

Proof. If $v \not\sim S_2^-$ then $v \not\sim S^-$, and therefore $v \in L$. However, the vertex v is adjacent to more vertices in C than u does: $v \sim C^+ \cup C_1^+$ while $u \sim C^+$ and $u \not\sim C^-$. This is a contradiction to Assumption (1). □

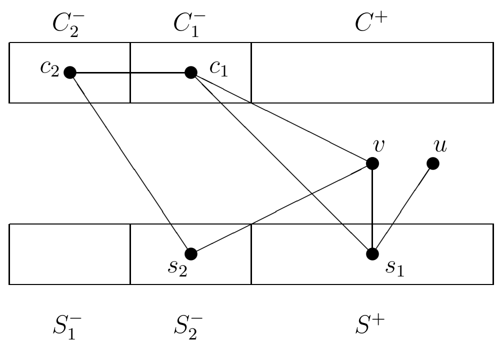

Figure 6: The set $\{u, v, c_1, c_2, s_1, s_2\}$ induces a subgraph isomorphic to H.

According to Claim 2, there exists a vertex $s_2 \in S_2^-$ which is adjacent to v. By the definition of S_2^-, $c_2 \sim s_2$. Also, $s_2 \not\sim \{c_1, u, s_1\}$. Thus, the set $\{u, v, c_1, c_2, s_1, s_2\}$ induces a subgraph isomorphic to H, see Figure 6.

Applying the same argument to $\overline{G}$ proves that G also contains an induced subgraph isomorphic to $\overline{H}$. It implies that G contains an induced subgraph belonging to $\mathbf{Z}$, a contradiction to the assumption $G \in \mathcal{P}$. □

Every graph in $\mathbf{Z}$ contains the 5-path P_5 as an induced subgraph.

Corollary 1. Conjecture 1 *holds for all P_5-free graphs.*

Also, every graph in $\mathbf{Z}$ contains the 4-cycle C_4 as an induced subgraph.

Corollary 2. Conjecture 1 *holds for all C_4-free graphs, and therefore for all chordal graphs.*

The set $\mathbf{Z}$ is self-complementary, that is $G \in \mathbf{Z}$ implies $\overline{G} \in \mathbf{Z}$. Hence the statements complementary to Corollary 1 and to Corollary 2 hold: Conjecture 1 is true for all $\overline{P}_5$-free graphs and for all $\overline{C}_4$-free graphs. Finally, the set $\mathbf{Z}$ is finite, it contains graphs with at most 12 vertices, therefore the class $\mathcal{P}$ is polynomial-time recognizable.

3 Some useful properties

The proof of Theorem 1 contains some interesting properties of non-split almost CIS-graphs. Let G be a non-split graph with a non-CIS pair (C, S), and let $U = V(G) \setminus (C \cup S)$. We define a quasiorder $\leq_C$ on U: $u \leq_C u'$ if and only if $N(u) \cap C \subseteq N(u') \cap C$. An element u is called a *C-maximal vertex* if there is no v with $u <_C v$, where $u <_C v$ if and only if $u \leq_C v$ and $v \not\leq_C u$. Similarly, a quasiorder $\leq_S$ can be defined on U: $u \leq_S u'$ if and only if $N(u') \cap S \subseteq N(u) \cap S$, that is the set of all non-neighbors of u' in S contains the set of all non-neighbors of u in S. *S-Maximal vertices* are are defined similarly to C-maximal vertices. The following observation follows from the definitions.

Property 1. *A non-split almost CIS graph G does not have a vertex which is both C-maximal and S-maximal.*

For example, suppose that $|C| = |S| = 2$ for the unique non-CIS pair (C, S) of G. It is easy to see that $C \cup S$ induces P_4, say $P_4 = (s_1, c_1, c_2, s_2)$, where $C = \{c_1, c_2\}$ and $S = \{s_1, s_2\}$. We apply Property 1 to an arbitrary vertex $u \in U = V(G) \setminus (C \cup S)$. If u is S-maximal then it is non-adjacent to exactly one vertex of S. Note that u cannot be non-adjacent to both vertices of S by maximality of S. According to Property 1, u is not C-maximal, that is u must be non-adjacent to both vertices of C. Thus, the set $C \cup S \cup \{u\}$ induces the graph P_5 shown in Figure 7 [or the symmetric graph, where u is adjacent to s_2 instead of s_1].

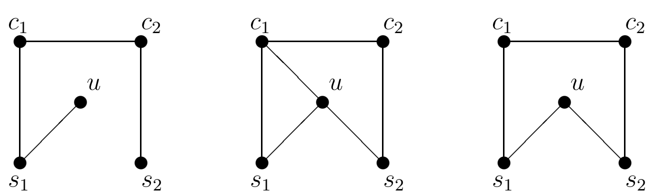

Figure 7: The graphs P_5, $\overline{P}_5$ and C_5.

If u is C-maximal then it is adjacent to exactly one vertex of C. Note that u cannot be adjacent to both vertices of C by maximality of C. According to Property 1, u is not S-maximal, that is u must be adjacent to both vertices of S. Thus, the set $C \cup S \cup \{u\}$ induces the graph $\overline{P}_5$ [or the symmetric graph].

Finally, it is possible that u neither C-maximal nor S-maximal, in which case the set $C \cup S \cup \{u\}$ induces C_5; see the rightmost graph in Figure 7.

A *C-mirror* of a vertex $u \in U$ is a vertex $s \in S$ adjacent to u and such that $N(u) \cap C = N(s) \cap C$. An *$S$-mirror* of a vertex $u \in U$ is a vertex $c \in C$ non-adjacent to u and such that $N(u) \cap S = N(c) \cap S$. The following observation is clear.

Property 2. *Every C-maximal vertex has a C-mirror, and every S-maximal vertex has an S-mirror.*

A *P_4-based graph* has a non-CIS pair (C, S) such that $|C| = |S| = 2$.

Theorem 2. Conjecture 1 *holds for the class of all P_4-based graphs.*

Proof. Consider an arbitrary almost CIS P_4-based graph G with the unique non-CIS pair (C, S). We may assume that G is not a split graph. Let $C \cup S$ induces $P_4 = (s_1, c_1, c_2, s_2)$, where $\{c_1, c_2\} = C$ and $\{s_1, s_2\} = S$. We denote

$X_1 = \{u : u \sim \{c_1, s_1, s_2\}$ and $u \not\sim c_2\}$,

$X_2 = \{u : u \sim \{c_2, s_1, s_2\}$ and $u \not\sim c_1\}$,

$Y_1 = \{u : u \not\sim \{s_2, c_1, c_2\}$ and $u \sim s_1\}$, and

$Y_2 = \{u : u \not\sim \{s_1, c_1, c_2\}$ and $u \sim s_2\}$.

Fact 1. *Every vertex of X_1 is adjacent to every vertex of X_2.*

Proof. Suppose that a vertex $x_1 \in X_1$ is non-adjacent to some vertex $x_2 \in X_2$. An arbitrary maximal stable set that contains both x_1 and x_2 is disjoint from the maximal clique $C = \{c_1, c_2\}$, a contradiction. □

Here is the complementary statement.

Fact 2. *Every vertex of Y_1 is non-adjacent to every vertex of Y_2.*

Now consider a maximal clique C_i in the subgraph induced by X_i, $i = 1, 2$, and a maximal stable set S_i in the subgraph induced by Y_i, $i = 1, 2$. Fact 1 and Fact 2 show that $C_1 \cup C_2$ is a clique in G, and $S_1 \cup S_2$ is a stable set. The set $A = C_1 \cup \{c_1, s_1\}$ is a clique, and the set $B = S_1 \cup S_2 \cup \{c_2\}$ is a stable set.

Fact 3. *There is a vertex $u \in C_1$ which is non-adjacent to all vertices of B.*

Proof. We extend A and B to a maximal clique A' and a maximal stable set B', respectively. Since $(A', B') \neq (C, S)$, there must be a vertex $u \in A' \cap B'$. The proof of Theorem 1 implies that S_1 is non-empty, therefore $u \neq s_1$. Indeed, s_1 is adjacent to S_1, but u is not adjacent to $B \supseteq S_1$. The vertex u is adjacent to both c_1 and s_1, and u is non-adjacent to c_2. It follows that $u \in X_1$. Since $u \sim C_1$, $u \in C_1$ by maximality of C_1. Finally, $u \in B'$ implies that u is non-adjacent to all vertices of $B \subseteq B'$. □

Now, the set $Q = C_1 \cup C_2 \cup \{s_2\}$ is a clique, and the set $T = S_2 \cup \{c_2, s_1\}$ is a stable set.

Fact 4. *There is a vertex $v \in S_2$ which is adjacent to all vertices of Q.*

Proof. We extend Q and T to a maximal clique Q' and a maximal stable set T', respectively. Since $(Q', T') \neq (C, S)$, there must be a vertex $v \in Q' \cap T'$. By Fact 3, the set C_1 is non-empty, therefore $v \neq c_2$. The vertex v is non-adjacent to both c_2 and s_1, and v is adjacent to s_2, hence $v \in Y_2$. Since $v \not\sim S_2$, maximality of S_2 implied that $v \in S_2$. Finally, $v \in Q'$ shows that v is adjacent to all vertices of $Q \subseteq Q'$. $\square$

Fact 3 and Fact 4 produce a contradiction: the vertices u and v are non-adjacent and adjacent simultaneously. $\square$

4 An extension

We can further extend the graph H of Figure 6 using the maximal clique C' containing v and s_2 and the maximal stable set S' containing s_1 and c_2. There must be a vertex $w \in C' \cap S'$. We have $w \notin C \cup S$, since $w \not\sim c_2 \in C$ and $w \sim s_2 \in S$, see Figure 8, where w can be adjacent to the vertices c_1 and/or u.

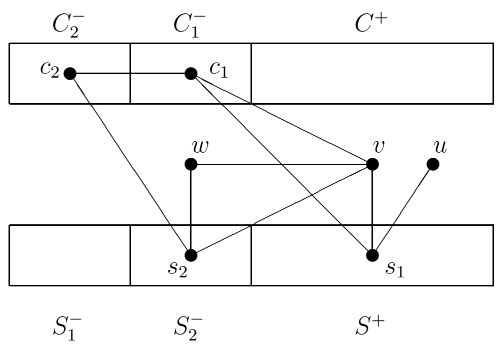

Figure 8: An extension.

The configuration of Figure 8 contains four graphs H_1, H_2, H_3 and H_4 shown in Figure 9:

- H_1: the vertex w is non-adjacent to both u and c_1,
- H_2: the vertex w is non-adjacent to u, and w is adjacent to c_1,
- H_3: the vertex w is adjacent to u, and w is non-adjacent to c_1, and
- H_4: the vertex w is adjacent to both u and c_1.

Now we specify a maximal clique C' and a maximal stable set S' in each H_i, $i = 1, 2, 3, 4$.

- H_1: $C' = \{c_1, s_1, v\}$ and $S' = \{u, w, c_2\}$,
- H_2: $C' = \{c_1, s_1, v\}$ and $S' = \{u, w, c_2\}$,
- H_3: $C' = \{u, s_1\}$ and $S' = \{w, c_1\}$, and
- H_4: $C' = \{v, w, c_1\}$ and $S' = \{s_1, s_2\}$.

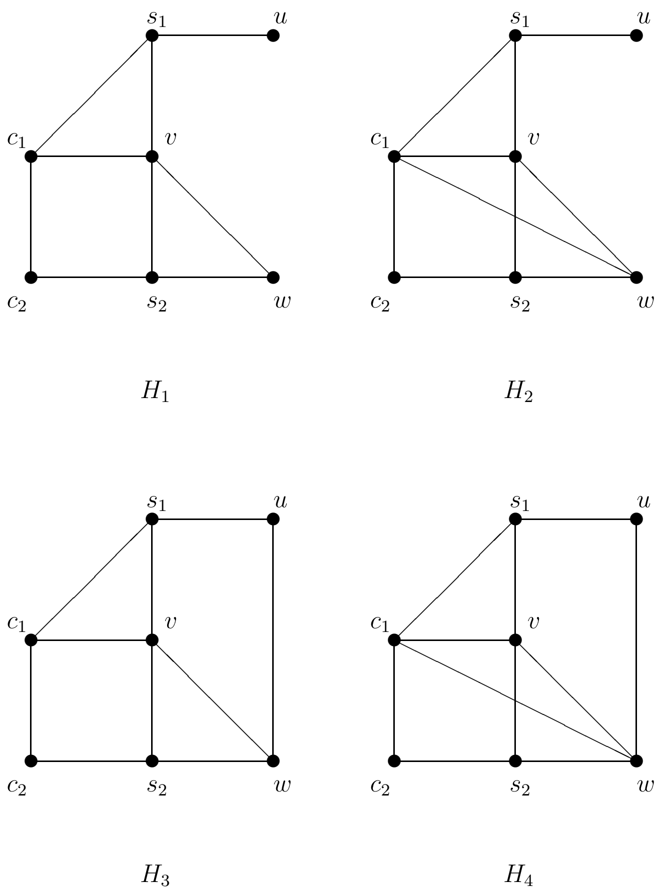

Figure 9: The four variants.

We introduce a new vertex x such that $x \sim C'$ and $x \not\sim C'$. For the graph H_1, $x \notin C \cup S$, since $x \not\sim c_2 \in C$ and $x \sim s_1 \in S$. Depending on adjacency of x to the vertex s_2, we obtain two extensions F_1 and F_2 of H_1; see Figure 10.

The graph H_2 is similar to H_1, and we have two extensions F_3 and F_4 of H_2 shown in Figure 11.

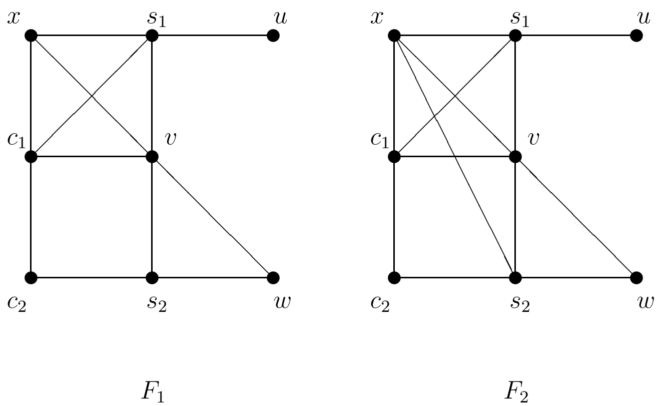

Figure 10: The two extensions F_1 and F_2 of H_1.

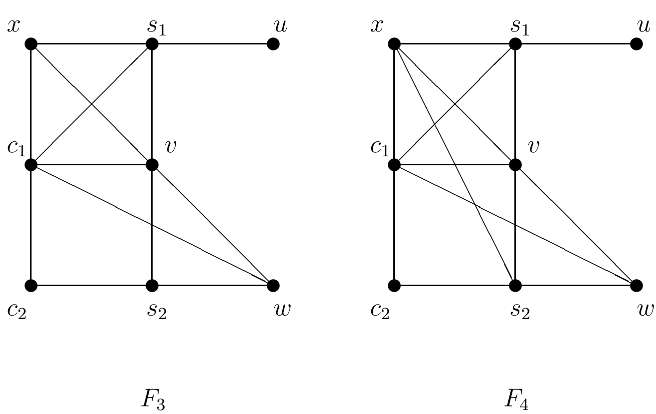

Figure 11: The two extensions F_3 and F_4 of H_2.

In case of H_3, the new vertex x does not belong to $C \cup S$, since $x \not\sim c_1 \in C$ and $x \sim s_1 \in S$. The vertex x may or may not be adjacent to v, c_2, s_2 independently of each other. Thus, we have 8 variants – the graphs F_5, F_6, F_7, F_8 of Figure 12 and the graphs $F_9, F_{10}, F_{11}, F_{12}$ of Figure 13.

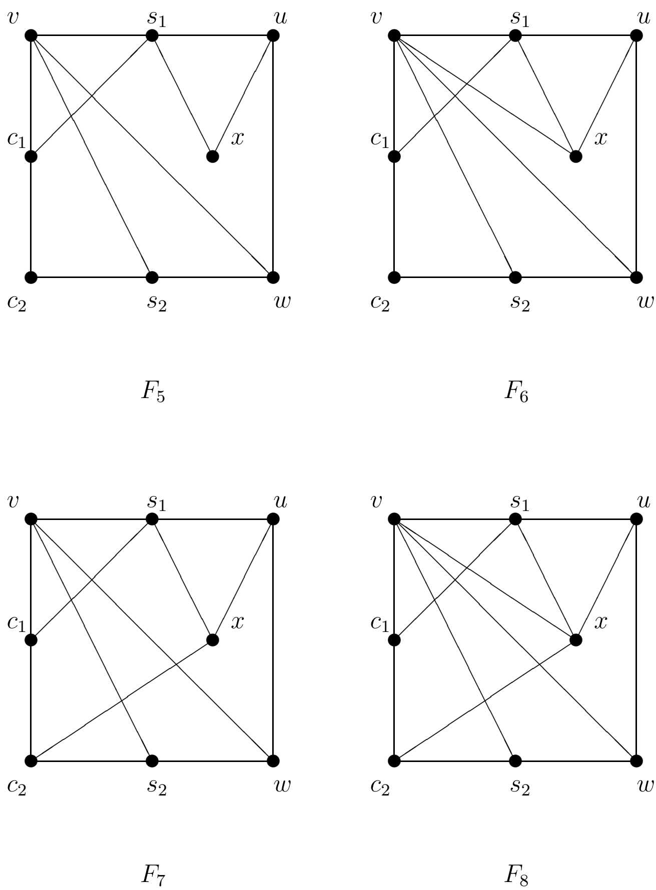

Figure 12: The graphs F_5, F_6, F_7, F_8 (extensions of H_3).

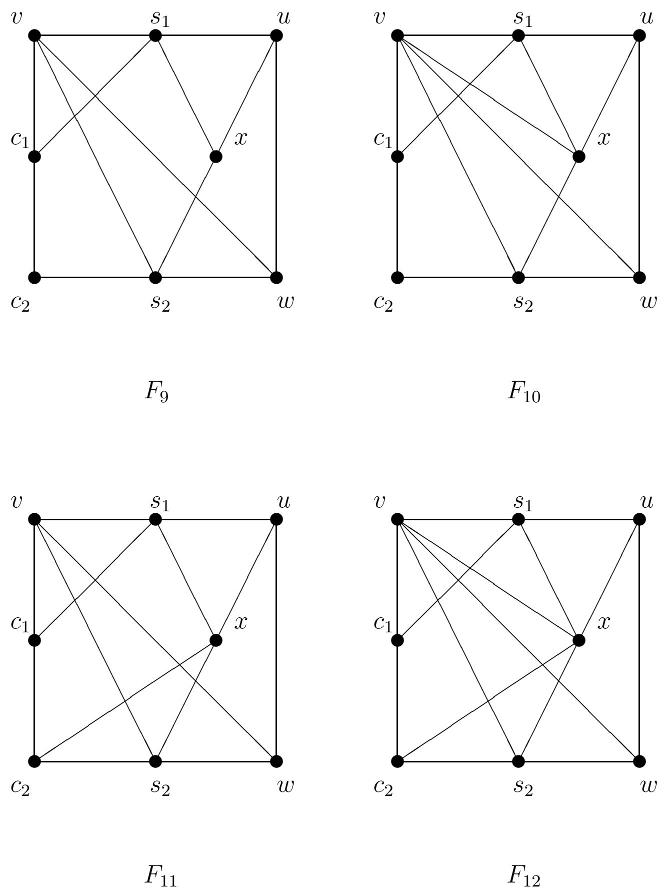

Figure 13: The graphs $F_9, F_{10}, F_{11}, F_{12}$ (extensions of H_3).

In case of H_4, we can extend S' to the original maximal stable set S. As a result, the vertex x will be in S, and therefore we rename it as s_3. Specifying potential edges s_3u and s_3c_2, we obtain the four graphs $F_{13}, F_{14}, F_{15}, F_{16}$ of Figure 14.

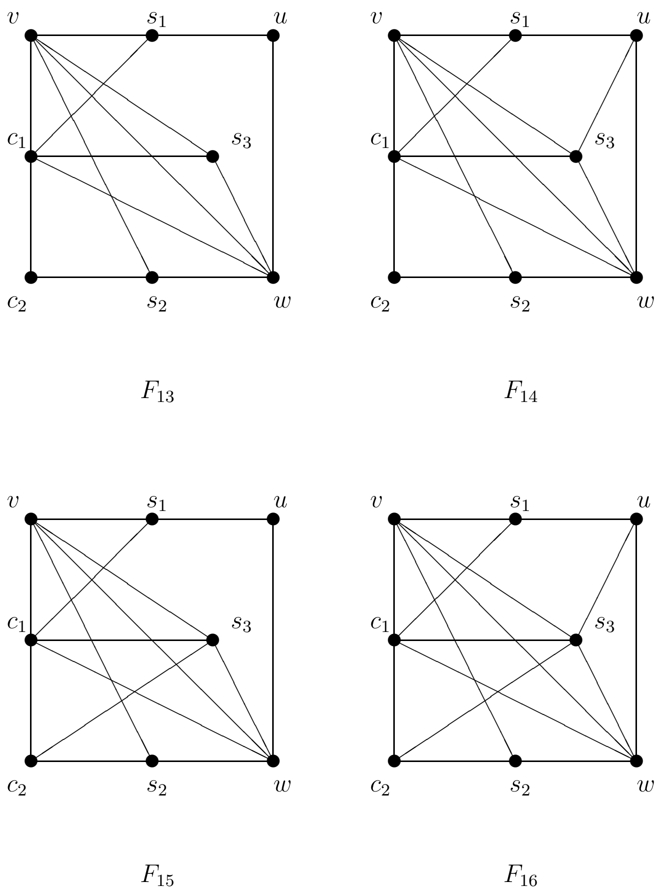

Figure 14: The extensions of H_4.

Now we define a finite set $\mathbf{Z}'$ of graphs. A graph G belongs to $\mathbf{Z}'$ if and only if

- G contains at least one of $F_1, F_2, \ldots, F_{16}$ as induced subgraph, and
- G contains at least one of $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_{16}$ as induced subgraph,

and G is minimal with this property.

Theorem 3. Conjecture 1 *holds for the class* $\mathcal{P}$ *of all* $\mathbf{Z}'$*-free graphs.*

5 Alternating sequences

Let G be an almost CIS-graph with a unique non-CIS pair (C, S). Here we derive some properties of the subgraph induced by the set $R = V(G) \setminus (C \cup S)$. In particular, we show that R cannot induce a threshold graph.

Property 3. *For an arbitrary C-maximal vertex $u \in R$, there exists a vertex $v \in R$ such that*

(i) *v is an S-maximal vertex,*

(ii) *v is non-adjacent to all non-neighbors of u in S,*

(iii) *v is adjacent to u,*

(iv) *v is non-adjacent to an arbitrary C-mirror $s \in S$ of u, and*

(v) *u is non-adjacent to an arbitrary S-mirror $c \in C$ of v.*

Proof. As in the proof of Theorem 1, we denote

$$C^- = C \setminus N(u),$$

$$C^+ = C \cap N(u),$$

$$S^- = S \setminus N(u),$$

$$S^+ = S \cap N(u).$$

Let $s \in S$ be an arbitrary C-mirror of u, see Property 2. Also, there is a vertex $c' \in C^-$ such that $c' \not\sim \{u\} \cup S^-$. The proof of Theorem 1 implies that s and c' are non-adjacent. Indeed, c' is not adjacent to u, and s is a C-mirror of u.

The set $S \cup \{c'\}$ is not stable, therefore c' is adjacent to some $s' \in S$. Clearly, $s' \in S^+$ and $s' \sim u$. Consider the clique $\{u, s'\}$ and the stable set $\{c', s\} \cup S^-$. Their maximal extensions must have a common vertex, say v'. Since $v' \sim s'$ and $v' \not\sim c'$, we have $v' \in R$. The vertex v' is non-adjacent to all non-neighbors of u in S and to the vertex s, therefore R contains a maximal vertex with this property. Thus, we can choose an S-maximal vertex $v'' \not\sim S^- \cup \{s\}$, and both (i) and (ii) hold for v''.

Suppose that v'' is non-adjacent to u. According to Property 2, v'' has an S-mirror $c \in C$. Since $v'' \not\sim s$, we have $c \not\sim s$, and therefore $c \not\sim u$. The set $S \cup \{c\}$ is not stable, therefore c is adjacent to some vertex $s'' \in S$. Clearly, $s'' \in S^+$ and $s'' \sim \{u, v''\}$. Consider the clique $\{u, s''\}$ and the stable set $\{c\} \cup X$, where X consists of all non-neighbors of v'' in S, including s. Their maximal extensions must have a common vertex v. The vertex v has the same set of non-neighbors in S as v'' by S-maximality of v''. In particular, c is an S-mirror for v. Now v satisfies all the properties (i), (ii), (iii), (iv) and (v). $\square$

Here is the complementary result.

Property 4. *For an arbitrary S-maximal vertex $v \in R$, there exists a vertex $u \in R$ such that*

(i) *u is a C-maximal vertex,*

(ii) *u is adjacent to all neighbors of v in C,*

(iii) *u is non-adjacent to v,*

(iv) *u is adjacent to an arbitrary S-mirror $c \in C$ of v, and*

(v) *v is non-adjacent to an arbitrary C-mirror $s \in S$ of u.*

Now we consider the shortest alternating cyclic sequence P of C-maximal vertices $u_i \in R$ and S-maximal vertices $v_j \in R$:

$$P = (u_1, v_1, u_2, v_2, \ldots, u_k, v_k, u_{k+1} = u_1), \tag{1}$$

such that each v_i is constructed for u_i according to Property 3, and each u_{i+1} is constructed for v_i according to Property 4. The sequence P has at least four vertices. Indeed,

$$\{u_1, u_2, \ldots, u_k\} \cap \{v_1, v_2, \ldots, v_k\} = \emptyset,$$

since no vertex is C-maximal and S-maximal simultaneously. In particular, $v_1 \neq u_1$. The vertex u_2 is non-adjacent to v_1, while the vertices u_1 and v_1 are adjacent, therefore $u_2 \neq u_1$. Similarly, $v_2 \neq v_1$. However, it is possible that $u_3 = u_1$, in which case P has length four [$k = 2$].

Theorem 4. *Let G be an almost CIS non-split graph with a non-CIS pair (C, S), and let $R = V(G) \setminus (C \cup S)$. Then G contains a sequence (1) of pairwise distinct vertices $u_i, v_i \in R$, $k \geq 2$, and two sequences*

$$(c_1, c_2, \ldots, c_k), (s_1, s_2, \ldots, s_k) \tag{2}$$

of vertices $c_i \in C$ and $s_j \in S$ such that

(i) *each u_i is a C-maximal vertex,*

(ii) *each v_i is an S-maximal vertex,*

(iii) *each v_i is non-adjacent to all non-neighbors of u_i in S,*

(iv) *each u_i is adjacent to all neighbors of v_{i-1} in C,*

(v) *each u_i is non-adjacent to v_{i-1},*

(vi) *each v_i is adjacent to u_i,*

(vii) *each s_i is a C-mirror of u_i,*

(viii) *each c_i is an S-mirror of v_i,*

(ix) *each v_i is non-adjacent to s_i, and*

(x) *each u_{i+1} is adjacent to c_i.*

Note that the vertices in (2) are not necessarily pairwise distinct. Figure 15 illustrates the minimal case $k = 2$, where $u_1 \not\sim u_2$ and $v_1 \not\sim v_2$. However, the vertices u_1 and u_2 may be adjacent as well as v_1 and v_2.

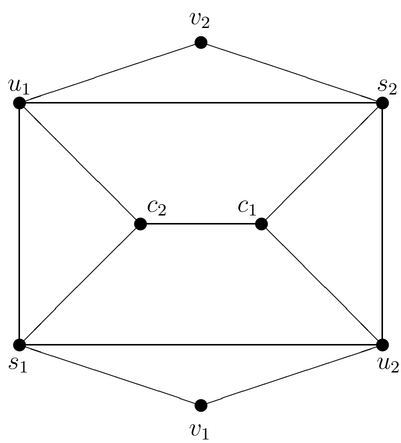

Figure 15: An illustration for $k = 2$.

A graph is *threshold* if it does not contain $2K_2, P_4$ and C_4 as induced subgraphs; see Figure 16.

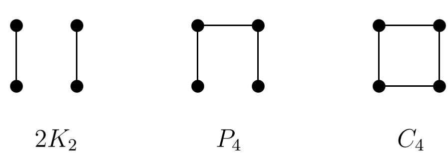

Figure 16: Minimal forbidden induced subgraph for threshold graphs.

Corollary 3. *If G is an almost CIS non-split graph with a non-SIC pair (C, S), then the set $R = V(G) \setminus (C \cup S)$ does not induce a threshold graph.*

Proof. For $k = 2$, the sequence (1) contains four vertices that induces a configuration C shown in see Figure 17, where u_1 and u_2 may or may not be adjacent, as well as v_1 and v_2. Clearly, C contains exactly the graphs $2K_2, P_4$ and C_4.

If $k \geq 3$ then the set $\{u_i, v_i : i = 1, 2, \ldots, k\}$ induce a configuration that contains C. $\square$

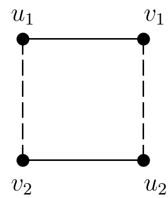

Figure 17: The configuration C.

Acknowledgments

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