AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **42** (2008), Pages 301–307

Parameters of independence and domination by monochromatic paths in graphs

Iwona Włoch

Rzeszów University of Technology
Faculty of Mathematics and Applied Physics
ul. W.Pola 2,35-959 Rzeszów
Poland
iwloch@prz.edu.pl

Abstract

In this paper we generalize parameters of independence and domination for graphs. The behavior of parameters for independence and domination by monochromatic paths is studied when an edge or a vertex is deleted. Then we give main results concerning μ-feasible sequences and interpolation.

1 Introduction

Let G be a finite graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. By a *path* from a vertex x_1 to a vertex x_n, $n \geq 2$, we mean a sequence of vertices $x_1, \ldots, x_n$ and edges $x_i x_{i+1} \in E(G)$, for $i = 1, \ldots, n-1$ and we denote it by $x_1 \ldots x_n$. If G_1 and G_2 are two graphs then their *union*, denoted by $G_1 \cup G_2$, has $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* $G_1 + G_2$ of graphs G_1 and G_2 is defined as the disjoint union of G_1 and G_2 with additional edges linking each vertex of G_1 with each vertex of G_2. Let G be a graph on $V(G) = \{x_1, \ldots, x_n\}$, $n \geq 1$, and let $\mathcal{H} = (H_i)_{i \in \mathcal{I}}$ where $\mathcal{I} = \{i; x_i \in V(G)\}$ is a sequence of vertex disjoint graphs on $V(H_i) = \{y_1^i, \ldots, y_{p_i}^i\}$, $p_i \geq 1$. The *corona* of G and $\mathcal{H}$ is the graph $G \circ \mathcal{H}$ such that $V(G \circ \mathcal{H}) = V(G) \cup \bigcup_{i=1}^{n} V(H_i)$ and

$$E(G \circ \mathcal{H}) = E(G) \cup \bigcup_{i=1}^{n} E(H_i) \cup \bigcup_{i=1}^{n} \{x_i y_t^i \; ; \; t = 1, \ldots, p_i\}.$$

A graph G is said to be *edge-m-coloured* if its edges are coloured with m colours. A path is called *monochromatic* if all its edges are coloured alike. A subset $S \subset V(G)$ is *independent by monochromatic paths* of the edge-coloured graph G if for any two different vertices $x, y \in S$ there does not exist a monochromatic path between them. In addition a subset containing only one vertex and the empty set also are independent by monochromatic paths sets of G. For convenience we will write imp-set of G instead of independent by monochromatic paths set of G. For a proper edge-colouring of the graph G an imp-set of G is an independent set of G in the classical sense. The lower and upper independence by monochromatic paths numbers $i_{mp}(G)$ and $\alpha_{mp}(G)$ of G are respectively the minimum and maximum cardinalities of maximal imp-set of vertices of G. A subset $Q \subseteq V(G)$ is *dominating by monochromatic paths* of the edge-coloured graph G if for each $x \in (V(G) \setminus Q)$ there exists a monochromatic path from $x \ldots y$, for some $y \in Q$. We will write a dmp-set of G instead of a dominating by monochromatic paths set of G. For a proper edge colouring of the graph G dmp-set of G is a dominating set of G in the classical sense. The lower and upper domination by monochromatic paths numbers $\gamma_{mp}(G)$ and $\Gamma_{mp}(G)$ of G are respectively the minimum and maximum cardinalities of minimal dmp-set of vertices of G. Parameters $\gamma_{mp}(G)$ and $\alpha_{mp}(G)$ will be called as the domination by monochromatic paths and independence by monochromatic paths numbers, respectively.

The concept of independence by monochromatic paths and domination by monochromatic paths generalizes independence and domination in the classical sense and was studied for instance in [1], [2], [6], [7].

In this paper we present general properties of imp-sets and dmp-sets in graphs. Next we study the behavior of parameters $\alpha_{mp}(G)$, $i_{mp}(G)$, $\gamma_{mp}(G)$ and $\Gamma_{mp}(G)$ when an edge or a vertex are deleted. We give the characterization of μ-feasible sequences for the parameter μ being monochromatic parameters of independence and domination. We also studied interpolation properties for these parameters.

2 General properties of imp-sets and dmp-sets in edge-coloured graphs

Theorem 1 *For an arbitrary edge-coloured graph G and a subset $S \subset V(G)$ the following conditions are equivalent:*

(1) S is a maximal imp-set of G

(2) S is an imp-set of G and a dmp-set of G

(3) S is both a maximal imp-set and a minimal dmp-set of G.

Proof: Let S be a maximal imp-set of G. We shall show that S is a dmp-set of G. Assume on contrary that S is not a dmp-set of G. This means that there exists a vertex $x \in (V(G) \setminus S)$ such that for every $y \in S$ there does not exist a monochromatic path from $x \ldots y$ in G. Hence $S \cup \{x\}$ is an imp-set of G, a contradiction with maximality of the set S.

Let now S be an imp-set and a dmp-set of G. Then S is a maximal imp-set of G, for otherwise S would not be a dmp-set of G. We will prove that S is a minimal dmp-set of G. Assume on contrary that S is not a minimal dmp-set of G. Hence there exists $x \in S$ such that $S' = S \setminus \{x\}$ is a dmp-set of G. So there is a monochromatic path from x to S, a contradiction that S is an imp-set of G. Thus S is a minimal dmp-set of G.

This suffices to complete the proof of the theorem. □

The next result is an immediate consequence of Theorem 1.

Corollary 1 *For an arbitrary edge-coloured graph G we have $\gamma_{mp}(G) \leq \alpha_{mp}(G)$.*

Theorem 2 *For any vertex x of an edge-coloured graph G*
(1) $\gamma_{mp}(G) - 1 \leq \gamma_{mp}(G - x)$
(2) $\alpha_{mp}(G) - 1 \leq \alpha_{mp}(G - x)$
(3) $i_{mp}(G) - 1 \leq i_{mp}(G - x)$.

PROOF: (1). If Q is a minimum dmp-set of $G-x$, then $\gamma_{mp}(G-x) = |Q|$. Moreover the set $Q \cup \{x\}$ dominates by monochromatic paths G and therefore $\gamma_{mp}(G) \leq |Q \cup \{x\}| = \gamma_{mp}(G - x) + 1$.

(2). Let S be a maximum imp-set of G. Then $|S| = \alpha_{mp}(G)$. We distinguish two possible casses:

(2.1). $x \notin S$.

Then it is clear that $\alpha_{mp}(G - x) \geq |S| = \alpha_{mp}(G)$ hence $\alpha_{mp}(G - x) > \alpha_{mp}(G) - 1$.

(2.2). $x \in S$

In this case $S \setminus \{x\}$ is an imp-set of $G - x$ and this implies that $\alpha_{mp}(G - x) \geq |S \setminus \{x\}| = \alpha_{mp}(G) - 1$.

So, from the above possibilities we obtain that $\alpha_{mp}(G - x) \geq \alpha_{mp}(G) - 1$.

(3). Let S be a minimum maximal imp-set of $G - x$. If there does not exist in G a monochromatic path $x \ldots y$, for all $y \in S$ then $S \cup \{x\}$ is a maximal imp-set of G and consequently $i_{mp}(G) \leq |S \cup \{x\}| = i_{mp}(G - x) + 1$. If there is $y \in S$ and a monochromatic path between $x \ldots y$ in G then the set S is a maximal imp-set of G, so $i_{mp}(G) \leq i_{mp}(G - x) < i_{mp}(G - x) + 1$. Hence the above possibilities give that $i_{mp}(G) \leq i_{mp}(G - x) + 1$.

This completes the proof. □

Theorem 3 *For every edge-coloured graph G and every edge e of G*
(1) $\gamma_{mp}(G) \leq \gamma_{mp}(G - e) \leq \gamma_{mp}(G) + 1$
(2) $\alpha_{mp}(G) \leq \alpha_{mp}(G - e) \leq \alpha_{mp}(G) + 1$

PROOF: (1). Let $e = xy$ be an arbitrary edge of the edge-coloured graph G. Assume that Q is a minimum dmp-set of $G - e$. Then Q is a dmp-set of G and $\gamma_{mp}(G) \leq |Q| = \gamma_{mp}(G - e)$. If Q is a minimum dmp-set of G then at least one of the sets Q, $Q \cup \{x\}$, $Q \cup \{y\}$ is a dmp-set in $G - e$ and hence $\gamma_{mp}(G - e) \leq |Q| + 1 = \gamma_{mp}(G) + 1$.

(2). Since every imp-set of G is an imp-set of $G - e$, so $\alpha_{mp}(G) \leq \alpha_{mp}(G - e)$. To prove the inequality $\alpha_{mp}(G - e) \leq \alpha_{mp}(G) + 1$ assume that S is a maximum imp-set of $G - e$. If S is also an imp-set of G then $\alpha_{mp}(G - e) = |S| \leq \alpha_{mp}(G) < \alpha_{mp}(G) + 1$. Hence assume that S is not an imp-set of G. This means that there are vertices

$u, v \in S$ such that there is no a monochromatic path $u \ldots v$ in $G - e$ whereas there is a monochromatic path $u \ldots v$ in G. Evidently this monochromatic path contains the edge $e = xy$. Let H be a maximal monochromatic subgraph of the graph G containing the edge e. Evidently $u, v \in V(H)$. Moreover in the graph $G - e$ we have $z \notin S$ for every vertex $z \in (V(H) \setminus \{u, v\})$, otherwise there exists a monochromatic path from z to S, a contradiction the fact that S is an imp-set of $G - e$. The definition of an imp-set implies that exactly one vertex from monochromatic subgraph can belong to an imp-set of G. Because $S \cap V(H) = \{u, v\}$, so it is obvious that $S \setminus \{u\}$ and $S \setminus \{v\}$ are imp-sets of G. Therefore $\alpha_{mp}(G) \geq |S \setminus \{u\}| = |S \setminus \{v\}| = \alpha_{mp}(G - e) - 1$.

This completes the proof. □

3 Independence and domination feasible sequences

Let μ be an integer-valued graphical invariant. A sequence $(a_0, a_1, \ldots, a_n)$ of positive integers is a μ-feasible sequence if there exists a graph G with distinguished vertices $x_1, x_2, \ldots, x_n$ such that $\mu(G) = a_0$ and $\mu(G - x_1 - x_2 - \ldots - x_i) = a_i$, for $i = 1, 2, \ldots, n$. Hence μ-feasible sequences describe the possible behaviors of the invariant μ in successive vertex-deleted subgraphs. In [4], [5], μ-feasible sequences were studied for μ being the lower (upper) independence number i (α) and the lower domination number γ.

The following is immediate from Theorem 2.

Theorem 4 *Let $\mu = (a_0, a_1, \ldots, a_n)$ be a sequence of positive integers. If*

(1) μ is an α_{mp}-feasible sequence

(2) μ is a γ_{mp}-feasible sequence

(3) μ is an i_{mp}-feasible sequence

then $a_j \geq a_{j-1} - 1$ for $j = 1, \ldots, n$.

Theorem 5 *Let $\mu = (a_0, a_1, \ldots, a_n)$ be a sequence of positive integers. If $a_{j-1} \geq a_j \geq a_{j-1} - 1$ for $j = 1, \ldots, n$ then μ is an α_{mp}-feasible sequence and μ is a Γ_{mp}-feasible sequence.*

PROOF: Assume that $\mu = (a_0, a_1, \ldots, a_n)$ is a sequence of positive integers with $a_{j-1} \geq a_j \geq a_{j-1} - 1$, for $j = 1, 2, \ldots, n$. We shall show that there exists an edge-coloured graph G with distinguished vertices $x_1, \ldots, x_n$ such that $\alpha_{mp}(G) = \Gamma_{mp}(G) = a_0$ and $\alpha_{mp}(G - x_1 - \ldots - x_n) = \Gamma_{mp}(G - x_1 - \ldots - x_n) = a_i$ for $i = 1, \ldots, n$. To prove it we apply the construction of the graph G from [4] for edge-coloured graph. Let K_{n+1} be an edge-coloured complete graph with the vertex set $\{v_1, \ldots, v_{n+1}\}$ and G_{a_0} is a graph with $V(G_0) = \{y_1, \ldots, y_{a_0}\}$ and $E(G_0) = \emptyset$. We define the edge-coloured graph G as the join $G = K_{n+1} + G_{a_0}$, where every two added edges between $V(K_{n+1})$ and $V(G_{a_0})$ have different colours which are not used for colouring of the graph K_{n+1}. The definition of the graph $K_{n+1} + G_{a_0}$ imply that

$\alpha_{mp}(K_{n+1}+G_{a_0}) = \Gamma_{mp}(K_{n+1}+G_{a_0}) = a_0$. In the graph $K_{n+1}+G_{a_0}$ we choose vertices $x_1, \ldots, x_n$ in the following way $x_i = \begin{cases} v_{i-k} & \text{if} \quad a_i = a_{i-1} \quad \text{and} \quad a_0 - a_i = k \\ y_k & \text{if} \quad a_i = a_{i-1} - 1 \quad \text{and} \quad a_0 - a_i = k, \end{cases}$

for $i = 1, \ldots, n$. It is clear that if $a_i = a_0 - k$ for some nonnegative k, then the graph $(K_{n+1}+G_{a_0}) - x_1 - \ldots - x_i$ is obtained from $K_{n+1}+G_{a_0}$ by the removal of k vertices belonging to the subgraph G_{a_0} and $i-k$ vertices belonging to the subgraph K_{n+1}. Hence $(K_{n+1}+G_{a_0}) - x_1 - \ldots - x_i$ is isomorphic to $K_{n+1-(i-k)} + G_{a_0-k}$ and it is clear that $\alpha_{mp}((K_{n+1}+G_{a_0}) - x_1 - \ldots - x_i) = \Gamma_{mp}((K_{n+1}+G_{a_0}) - x_1 - \ldots - x_n) = a_0 - k = a_i$. Consequently μ is an α_{mp}-feasible sequence and μ is a Γ_{mp}-feasible sequence.

Thus the theorem is proved. □

Theorem 6 *Let $\mu = (a_0, a_1, \ldots, a_n)$ be a sequence of positive integers with $a_0 = n$. If $a_j \geq a_{j-1}$ for $j = 1, \ldots, n$ then μ is an α_{mp}-feasible sequence and μ is a Γ_{mp}-feasible sequence.*

Proof: Assume that $\mu = (a_0, a_1, \ldots, a_n)$ is a sequence of positive integers with $a_0 = n$ and $a_j \geq a_{j-1}$ for $j = 1, \ldots, n$. We shall prove that there exists an edge-coloured graph G with distinguished vertices $x_1, \ldots, x_n$ such that $\alpha_{mp}(G) = \Gamma_{mp}(G) = a_0 = n$ and $\alpha_{mp}(G - x_1 - \ldots - x_i) = \Gamma_{mp}(G - x_1 - \ldots - x_i) = a_i$, for $i = 1, \ldots, n$. Let K_n be an edge-coloured graph with the vertex set $V(K_n) = \{v_1, \ldots, v_n\}$ and $\mathcal{H}$ be a family of graphs $H_1, \ldots, H_n$ with $V(H_i) = \{y_1^i, \ldots, y_{p_i}^i\}$, and $E(H_i) = \emptyset$, for $i = 1, \ldots, n$, where numbers p_i are defined as follows: $p_1 = a_1 - (n-1)$ and $p_i = a_i - a_{i-1} + 1$ for $i = 2, \ldots, n$. From the assumption it is obvious that $p_i \geq 1$ for every $i = 1, \ldots, n$. We define the graph G as the corona $K_n \circ \mathcal{H}$, where for every $i = 1, \ldots, n$ all edges $x_i y_t^i$, $t = 1, \ldots, p_i$ are coloured alike by a colour not used in the colouring of the graph K_n. Firstly we shall show that $\alpha_{mp}(K_n \circ \mathcal{H}) = n$. Evidently the definition of $K_n \circ \mathcal{H}$ implies that at most one vertex from each H_i, $i = 1, \ldots, n$ can belong to the imp-set of $K_n \circ \mathcal{H}$. Hence the maximum imp-set S of $K_n \circ \mathcal{H}$ has the form $S = \bigcup_{i=1}^{n} S_i$ where S_i is a 1-element set of $V(H_i)$ containing an arbitrary vertex from H_i. This immediately gives that $\alpha_{mp}(K_n \circ \mathcal{H}) = n$. Similarly $\Gamma_{mp}(K_n \circ \mathcal{H}) = n$. In the graph $K_n \circ \mathcal{H}$ we choose vertices $x_1, \ldots, x_n$ as $x_i = v_i$, for $i = 1, \ldots, n$. It is clear that $(K_n \circ \mathcal{H}) - x_1 - \ldots - x_i$ is a disconnected graph isomorphic to the union of $(K_{n-i} \circ (\mathcal{H} \setminus \bigcup_{p=1}^{i} H_p)) \cup \bigcup_{p=1}^{i} H_p$. Therefore $\alpha_{mp}((K_n \circ \mathcal{H}) - x_1 - \ldots - x_i) = \Gamma_{mp}((K_n \circ \mathcal{H}) - x_1 - \ldots - x_i) = p_1 + p_2 + \ldots + p_i + n - i$. Because $p_1 = a_1 - (n-1)$ and $p_i = a_i - a_{i-1} + 1$ then by simple calculations we obtain that $\alpha_{mp}((K_n \circ \mathcal{H}) - x_1 - \ldots - x_i) = \Gamma_{mp}((K_n \circ \mathcal{H}) - x_1 - \ldots - x_i) = a_i$. Thus μ is an α_{mp}-feasible sequence and μ is a Γ_{mp}-feasible sequence, which completes the proof. □

4 Interpolation properties of independence and domination parameters

For a connected graph G let $\mathcal{T}(G)$ be the set of all spanning trees of G. An integer - valued graph function π is interpolate over a connected graph G if the set $\pi(\mathcal{T}(G)) = \{\pi(T); T \in \mathcal{T}(G)\}$ listed in increasing order is a set of consecutive integers. A function π interpolates over a family $\mathcal{F}$ of graphs if π interpolates over each graph of the family $\mathcal{F}$. Finally we say that π is an interpolating function if π interpolates over each connected graph. In [3] a sufficient condition was given for an integer-valued graph function to be an interpolating function.

Theorem 7 *[3] An integer-valued graph function π is an interpolating function if one of the following conditions is satisfied.*

(1) For every graph G and every edge e of G we have $\pi(G) \leq \pi(G-e) \leq \pi(G)+1$.

(2) For every graph G and every edge e of G we have $\pi(G)-1 \leq \pi(G-e) \leq \pi(G)$.

The following results are obvious using Theorem 3 and Theorem 7

Theorem 8 *The monochromatic domination number γ_{mp} is an interpolating function.*

Theorem 9 *The monochromatic independence number α_{mp} is an interpolating function.*

The monochromatic lower independence number i_{mp} is not an interpolating function. This follows by simple counter example shown on Fig.1., in which the edge-coloured graph G has only three nonisomorphic spanning trees T_1, T_2, T_3 such $i_{mp}(T_1) = 2$ and $i_{mp}(T_2) = i_{mp}(T_3) = 4$.

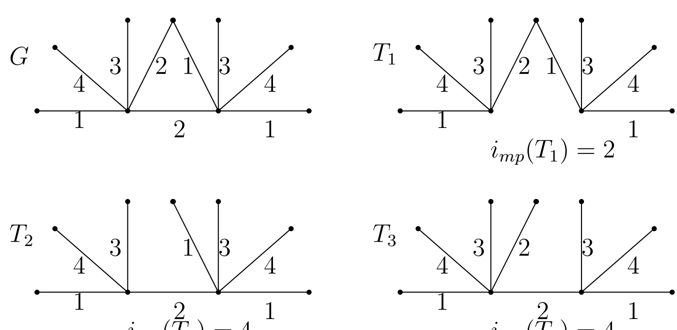

Fig.1.

References

[1] H. Galeana-Sanchez, Kernels in edge coloured digraphs, *Discrete Math.* 184 (1998), 87–99.

[2] H. Galeana-Sanchez and L.A.J. Ramirez, Monochromatic kernel-perfectness of special classes of digraphs, *Discussiones Math. Graph Theory* 27(3) (2007), 389–401.

[3] F. Harary and H.J. Plantholt, Classification of interpolation theorems for spanning trees and others families of spanning subgraphs, *J. Graph Theory* 13 (1989), 703–712.

[4] J. Topp, Sequences of graphical invariants, *Networks* 25 (1995), 1–5.

[5] J. Topp, *Domination, independence and irredundence in graphs*, Dissertationes Mathematicae, Warszawa 1995.

[6] A. Włoch and I. Włoch, Monochromatic Fibonacci numbers of graphs, *Ars Combin.* 82 (2007), 125–132.

[7] I. Włoch, Some operations of graphs that preserve the property of well-covered by mnochromatic paths, *Australas. J. Combin.* 40 (2008), 229–236.

(Received 20 Jan 2008; revised 15 June 2008)