

Antimagic labeling of the union of two stars

DAFIK

*Department of Mathematics
Education Universitas Jember
Indonesia
d.dafik@gmail.com*

MIRKA MILLER JOE RYAN

*School of Electrical Engineering and Computer Science
The University of Newcastle
NSW 2308
Australia
mirka.miller@newcastle.edu.au joe.ryan@newcastle.edu.au*

MARTIN BAČA

*Department of Applied Mathematics
Technical University, Košice
Slovak Republic
martin.baca@tuke.sk*

Abstract

Let G be a graph of order p and size q . An (a, d) -edge-antimagic total labeling of G is a one-to-one map f taking the vertices and edges onto $1, 2, \dots, p + q$ so that the edge-weights $w(u, v) = f(u) + f(v) + f(uv)$, $w \in E(G)$, form an arithmetic progression, starting from a and having common difference d . Moreover, such a labeling is called *super* (a, d) -edge-antimagic total if $f(V(G)) = \{1, 2, \dots, p\}$. This paper considers such labelings applied to a disjoint union of two stars $K_{1,m}$ and $K_{1,n}$.

1 Preliminaries

As a standard notation, assume that $G = G(V, E)$ is a finite, simple and undirected graph with p vertices and q edges. We follow the notation and terminology of [15] and [16].

A labeling of a graph is any map that carries some set of graph elements to the positive integers. An (a, d) -edge-antimagic vertex labeling on G is a one-to-one map f from $V(G)$ onto the integers $1, 2, \dots, p$, with the property that the set of all the edge-weights, $\{w(uv) = f(u) + f(v) : uv \in E(G)\}$, is $\{a, a+d, a+2d, \dots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

An (a, d) -edge-antimagic total labeling of G is defined as a one-to-one map f from $V(G) \cup E(G)$ onto the set $\{1, 2, \dots, p+q\}$, so that the edge-weights $w(uv) = f(u) + f(v) + f(uv)$, $uv \in E(G)$, form an arithmetic progression $a, a+d, a+2d, \dots, a+(q-1)d$, for two integers $a > 0$ and $d \geq 0$.

For brevity's sake, we often refer to an edge-antimagic vertex labeling or an edge-antimagic total labeling as an *EAV labeling* or an *EAT labeling*, respectively.

An (a, d) -EAT labeling f is called *super* if it has the property that the vertex labels are the integers $1, 2, \dots, p$, that is, the smallest possible labels, and $f(E(G)) = \{p+1, p+2, \dots, p+q\}$. A graph G is called (a, d) -EAT or *super* (a, d) -EAT if there exists an (a, d) -EAT or a super (a, d) -EAT labeling of G , respectively.

The definition of (a, d) -EAT labeling was introduced by Simanjuntak, Bertault and Miller in [11]. This labeling and super (a, d) -EAT labeling are natural extensions of the concept of *magic valuation*, defined by Kotzig and Rosa [9] (see also [1],[6],[14]), and the concept of *super edge-magic labeling*, defined by Enomoto, Lladó, Nakamigawa and Ringel in [5]; MacDougall and Wallis [10] refer to this labeling as *strongly edge-magic*. Different kinds of antimagic graphs were studied by Bodendiek and Walther [3] and [4], and Hartsfield and Ringel [7].

Ivančo and Lučkaničová [8] described some constructions of super edge-magic (super $(a, 0)$ -edge-antimagic total) labeling for disconnected graphs, namely $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. Super (a, d) -EAT labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana, Ismailmuza, Baskoro and Assiyatun in [12].

In this paper we investigate the existence of super (a, d) -EAT labeling for the disjoint union of two stars $K_{1,m}$ and $K_{1,n}$.

2 A few known lemmas

In this section, we recall three known lemmas that will be useful in the next section. The following lemma appeared in [2] and provides an upper bound for feasible values of d .

Lemma 1 [2] *If a (p, q) -graph is super (a, d) -EAT then $d \leq \frac{2p+q-5}{q-1}$.*

Lemma 2 [6] *A (p, q) -graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p + q + s$, where $s = \min(S)$ and $S = \{a - (p + 1), a - (p + 2), \dots, a - (p + q)\}$.*

The previous lemma provides a necessary and sufficient condition for a graph to be super edge-magic. In our terminology the previous lemma states that a (p, q) -graph G is super $(a, 0)$ -EAT if and only if there exists an $(a - p - q, 1)$ -EAV labeling.

The following lemma was proved in [13].

Lemma 3 [13] *Let \mathfrak{A} be a sequence $\mathfrak{A} = \{c, c + 1, c + 2, \dots, c + k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{A})$ of the elements of \mathfrak{A} , such that $\mathfrak{A} + \Pi(\mathfrak{A}) = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \dots, 2c + \frac{3k}{2} - 1, 2c + \frac{3k}{2}\}$.*

3 Disjoint union of two stars

It is proved in [13] that the star $K_{1,n}$ has a super (a, d) -EAT labeling if and only if either (i) $d \in \{0, 1, 2\}$ and $n \geq 1$, or (ii) $d = 3$ and $1 \leq n \leq 2$. Now, we will study super edge-antimagicness of the disjoint union of two stars, denoted by $K_{1,m} \cup K_{1,n}$. The disjoint union of $K_{1,m}$ and $K_{1,n}$ is the disconnected graph with vertex set $V(K_{1,m} \cup K_{1,n}) = \{x_{1,j} : j = 0, 1, \dots, m\} \cup \{x_{2,i} : i = 0, 1, \dots, n\}$ and edge set $E(K_{1,m} \cup K_{1,n}) = \{x_{1,0}x_{1,j} : j = 1, 2, \dots, m\} \cup \{x_{2,0}x_{2,i} : i = 1, 2, \dots, n\}$.

If the graph $K_{1,m} \cup K_{1,n}$ is super (a, d) -EAT then, according to Lemma 1, for $p = m + n + 2$ and $q = m + n$, we have $d \leq 3 + \frac{2}{m+n-1}$. We can see that:

- (i) If $m \geq 2$ and $n \geq 2$ then there is no super (a, d) -EAT labeling of $K_{1,m} \cup K_{1,n}$ with $d > 3$.
- (ii) If $m + n = 3$ then there is no super (a, d) -EAT labeling of $K_{1,m} \cup K_{1,n}$ with $d > 4$.
- (iii) If $m + n = 2$ then there is no super (a, d) -EAT labeling of $K_{1,m} \cup K_{1,n}$ with $d > 5$.

If $m + n = 2$ then we have the graph $K_{1,1} \cup K_{1,1}$. Assume that $K_{1,1} \cup K_{1,1}$ has a super (a, d) -EAT labeling. This means that $\sum_{k=1}^6 k = 2a + d$.

For $d = 0, 2$ and 4 the value a is not an integer therefore for the graph $K_{1,1} \cup K_{1,1}$ there is no super (a, d) -EAT labeling.

For $d = 1, 3$ and 5 the requested super (a, d) -EAT labeling f_1 is described by the following table.

d	$f_1(x_{1,0})$	$f_1(x_{2,0})$	$f_1(x_{1,1})$	$f_1(x_{2,1})$	$f_1(x_{1,0}x_{1,1})$	$f_1(x_{2,0}x_{2,1})$
1	2	1	3	4	5	6
3	1	2	3	4	5	6
5	2	4	1	3	5	6

Assume that $K_{1,m} \cup K_{1,n}$, for $m + n = 3$, has a super (a, d) -EAT labeling $f_2 : V(K_{1,2} \cup K_{1,1}) \cup E(K_{1,2} \cup K_{1,1}) \rightarrow \{1, 2, \dots, 8\}$, and $W = \{w(uv) : uv \in E(K_{1,2} \cup K_{1,1})\} = \{a, a + d, a + 2d\}$ is the set of edge-weights. In the computation of the edge-weights

of $K_{1,2} \cup K_{1,1}$, the label of a vertex of degree two is used twice, but the labels of the remaining vertices are used once each, and also the labels of edges are used once each. The sum of all vertex and edge labels, used to calculate the edge-weights, is equal to the sum of the edge-weights. If s_1 is the label of the vertex of degree two then

$$s_1 + \sum_{u \in V(K_{1,2} \cup K_{1,1})} f_2(u) + \sum_{uv \in E(K_{1,2} \cup K_{1,1})} f_2(uv) = \sum_{uv \in E(K_{1,2} \cup K_{1,1})} w(uv)$$

and

$$a = 12 - d + \frac{s_1}{3}.$$

Since a must be an integer then, for s_1 , we have only one possible value, namely, $s_1 = 3$.

For $d = 0, 1, 2$ and 3 the requested super (a, d) -EAT labeling f_2 can be done by the following two tables.

d	$f_2(x_{1,0})$	$f_2(x_{1,1})$	$f_2(x_{1,2})$	$f_2(x_{2,0})$	$f_2(x_{2,1})$
0	3	2	4	1	5
1	3	4	2	1	5
2	3	2	1	4	5
3	3	2	1	4	5

d	$f_2(x_{1,0}x_{1,1})$	$f_2(x_{1,0}x_{1,2})$	$f_2(x_{2,0}x_{2,1})$
0	8	6	7
1	6	7	8
2	8	7	6
3	8	6	7

For $d = 4$ the smallest value of edge-weight $a = 9$ can be obtained only from the triple $(1, 2, 6)$, where 1 and 2 are values of adjacent vertices of degree one and 6 is the value of the edge. The remaining vertices of degree one must be labeled by the values 4 and 5. Thus, we have the triples $(3, 4, 7)$ and $(3, 5, 8)$ or $(3, 4, 8)$ and $(3, 5, 7)$. This contradicts the fact that $K_{1,2} \cup K_{1,1}$ has a super $(9, 4)$ -EAT labeling.

□

Theorem 1 *Let m, n be integers satisfying the condition $m \geq n \geq 2$. The graph $K_{1,m} \cup K_{1,n}$ has an $(a, 1)$ -EAV labeling if and only if m is a multiple of $n + 1$.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq n \geq 2$, has an $(a, 1)$ -EAV labeling $f_3 : V(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, m + n + 2\}$ and that $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a + 1, a + 2, \dots, a + m + n - 1\}$ is the set of the edge-weights. The sum of the elements of W is

$$\sum_{uv \in E(K_{1,m} \cup K_{1,n})} w(uv) = (m+n)a + \frac{(m+n)(m+n-1)}{2}.$$

In the computation of the edge-weights of $K_{1,m} \cup K_{1,n}$, the label of the central vertex, $f_3(x_{1,0})$ or $f_3(x_{2,0})$, is used m or n times, respectively, and the labels of the remaining vertices are used once each. Let $s_1 = f_3(x_{1,0})$ and $s_2 = f_3(x_{2,0})$. The sum of all vertex labels, used to calculate the edge-weights, is equal to

$$\begin{aligned} & (m-1)f_3(x_{1,0}) + (n-1)f_3(x_{2,0}) + \sum_{k=1}^{m+n+2} k = \\ & (m-1)s_1 + (n-1)s_2 + \frac{(m+n+3)(m+n+2)}{2}. \end{aligned}$$

The sum of the vertex labels, used to obtain the edge-weights, is naturally equal to the sum of all the edge-weights. Thus,

$$(m+n)a = 3(m+n+1) + (m-1)s_1 + (n-1)s_2. \quad (1)$$

Since only one endpoint of any edge belongs to $\{x_{1,0}, x_{2,0}\}$, then $s_1 + s_2 \notin \{a, a+1, a+2, \dots, a+m+n-1\}$. Without loss of generality, we may assume that $s_1 + s_2 < a$. If $s_1 + s_2 > a + m + n - 1$ then we consider $(a', 1)$ -EAV labeling g , given by $g(v) = m + n + 3 - f_3(v)$, for all $v \in V(K_{1,m} \cup K_{1,n})$.

If $1 \notin \{s_1, s_2\}$ then $a > s_1 + s_2 > \min_{1 \leq j \leq m} f_3(x_{1,j}) + s_2 \geq 1 + s_2 \geq a$ or $a > s_1 + s_2 > s_1 + \min_{1 \leq i \leq n} f_3(x_{2,i}) \geq s_1 + 1 \geq a$, a contradiction.

Suppose $s_1 = 2$ and $s_2 = 1$. Then, from (1), it follows that

$$(m+n)(a-4) = m,$$

which implies that m is a multiple of $m+n$, a contradiction.

Suppose $s_1 > 2$ and $s_2 = 1$. We can say that $a = s_1 + 2$ because if $\min_{1 \leq i \leq n} f_3(x_{2,i}) = 2$ then $\min_{1 \leq i \leq n} f_3(x_{2,i}) + s_2 < s_1 + s_2 < a$; thus the vertex labeled by 2 must belong to $K_{1,m}$. From (1), it follows that

$$\begin{aligned} (m+n)(s_1+2) &= 3(m+n+1) + (m-1)s_1 + (n-1) \text{ and} \\ (s_1-2)(n+1) &= m, \end{aligned}$$

which means that m is a multiple of $n+1$.

We assume that $m = t(n+1)$, and consider the vertex labeling f_3 , described by Ivančo and Lučkaničová in [8].

$$f_3(x_{1,j}) = \begin{cases} 2+t, & \text{if } j=0 \\ \lceil \frac{j}{t} \rceil + j, & \text{if } 1 \leq j \leq m, \end{cases}$$

$$f_3(x_{2,i}) = \begin{cases} 1, & \text{if } i = 0 \\ 1 + (i + 1)(t + 1), & \text{if } 1 \leq i \leq n. \end{cases}$$

The vertex labeling f_3 is a bijective function from $K_{1,m} \cup K_{1,n}$ onto the set $\{1, 2, \dots, m + n + 2\}$. The edge-weights of $K_{1,m} \cup K_{1,n}$, under the labeling f_3 , constitute the sets

$$W_{f_3}^1 = \{w_{f_3}^1(x_{1,0}x_{1,j}) : \text{if } 1 \leq j \leq m\} = \{2 + t + \lceil \frac{j}{t} \rceil + j : \text{if } 1 \leq j \leq m\},$$

$$W_{f_3}^2 = \{w_{f_3}^2(x_{2,0}x_{2,i}) : \text{if } 1 \leq i \leq n\} = \{2 + (i + 1)(t + 1) : \text{if } 1 \leq i \leq n\}.$$

Hence the set $\bigcup_{k=1}^2 W_{f_3}^k = \{t + 4, t + 5, \dots, m + n + t + 3\}$ consists of consecutive integers. Thus f_3 is a $(t + 4, 1)$ -EAV labeling. \square

With respect to Lemma 2, the $(t + 4, 1)$ -EAV labeling f_3 extends to a super $(a, 0)$ -EAT labeling, where for $p = m + n + 2$ and $q = m + n$, the value $a = 2m + 2n + t + 6$. Thus we have the following theorem, which was proved by Ivančo and Lučkaničová in [8].

Theorem 2 [8] *Let m, n be integers satisfying the condition $m \geq n \geq 2$. The graph $K_{1,m} \cup K_{1,n}$ has a super edge-magic labeling if and only if m is a multiple of $n + 1$.*

Furthermore, we obtain the following theorem.

Theorem 3 *If $m \geq n \geq 2$ and m is a multiple of $n + 1$ then the graph $K_{1,m} \cup K_{1,n}$ has a super $(a, 2)$ -EAT labeling.*

Proof. We assume that $m \geq n \geq 2$ and m is a multiple of $n + 1$. Let $m = t(n + 1)$. Using the $(t + 4, 1)$ -EAV labeling f_3 from Theorem 1, we define a total labeling $f_4 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, 2m + 2n + 2\}$ as follows.

$$f_4(v) = f_3(v), \text{ for every vertex } v \in V(K_{1,m} \cup K_{1,n}),$$

$$f_4(x_{1,0}x_{1,j}) = m + n + 1 + \left\lceil \frac{j}{t} \right\rceil + j, \quad \text{for } 1 \leq j \leq m,$$

$$f_4(x_{2,0}x_{2,i}) = m + n + 2 + i(t + 1), \quad \text{for } 1 \leq i \leq n.$$

The edge-weights of $K_{1,m} \cup K_{1,n}$, under the total labeling f_4 , constitute the sets

$$W_{f_4}^1 = \{w_{f_4}^1(x_{1,0}x_{1,j}) = w_{f_3}^1(x_{1,0}x_{1,j}) + f_4(x_{1,0}x_{1,j}) : \text{if } 1 \leq j \leq m\} =$$

$$\{m + n + t + 3 + 2\lceil \frac{j}{t} \rceil + 2j : \text{if } 1 \leq j \leq m\},$$

$$W_{f_4}^2 = \{w_{f_4}^2(x_{2,0}x_{2,i}) = w_{f_3}^2(x_{2,0}x_{2,i}) + f_4(x_{2,0}x_{2,i}) : \text{if } 1 \leq i \leq n\} =$$

$$\{m + n + 4 + (2i + 1)(t + 1) : \text{if } 1 \leq i \leq n\}.$$

Hence the set $\bigcup_{k=1}^2 W_{f_4}^k = \{m + n + t + 7, m + n + t + 9, \dots, 3m + 3n + t + 5\}$ consists of an arithmetic sequence, with the first term $m + n + t + 7$ and the common difference $d = 2$. Thus f_4 is a super $(m + n + t + 7, 2)$ -EAT labeling. \square

We are not able to give an answer as to whether or not there exists a super $(a, 2)$ -EAT labeling of $K_{1,m} \cup K_{1,n}$ for other values of m and n . Therefore, we propose the following open problem.

Open Problem 1 *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq n \geq 2$, if m is not a multiple of $n + 1$ determine whether there is a super $(a, 2)$ -EAT labeling.*

By using $(t + 4, 1)$ -EAV labeling f_3 , with respect to Lemma 3, we can claim

Theorem 4 *If $m + n$ is odd, $m \geq n \geq 2$, and m is a multiple of $n + 1$ then the graph $K_{1,m} \cup K_{1,n}$ has a super $(a, 1)$ -EAT labeling.*

Proof. From Theorem 1, the graph $K_{1,m} \cup K_{1,n}$ has $(t + 4, 1)$ -EAV labeling. Let a set $\mathfrak{A} = \{c, c + 1, c + 2, \dots, c + k\}$ be the set of the edge weights of the vertex labeling f_3 , for $c = t + 4$ and $k = m + n - 1$. In light of Lemma 3, there exists a permutation $\Pi(\mathfrak{A})$ of the elements of \mathfrak{A} such that $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + m + n + 3] = \{c + \frac{3m+3n+5}{2}, c + \frac{3m+3n+5}{2} + 1, \dots, c + \frac{5m+5n+3}{2}\}$. If $[\Pi(\mathfrak{A}) - c + m + n + 3]$ is an edge labeling of $K_{1,m} \cup K_{1,n}$ then $\mathfrak{A} + [\Pi(\mathfrak{A}) - c + m + n + 3]$ gives the set of the edge weights of $K_{1,m} \cup K_{1,n}$, which implies that the total labeling is super $(a, 1)$ -EAT, where $a = c + \frac{3m+3n+5}{2} = \frac{3(m+n)+2t+13}{2}$. This concludes the proof. \square

Theorem 5 *If $m = n \geq 2$ then the graph $K_{1,m} \cup K_{1,n}$ has a $(4, 2)$ -EAV labeling.*

Proof. Let $m = n \geq 2$. Consider the bijection $f_5 : V(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, m + n + 2\}$, where

$$f_5(x_{1,j}) = \begin{cases} 1, & \text{if } j = 0 \\ 2j + 1, & \text{if } 1 \leq j \leq m, \end{cases}$$

$$f_5(x_{2,i}) = \begin{cases} m + n + 2, & \text{if } i = 0 \\ 2i, & \text{if } 1 \leq i \leq n. \end{cases}$$

We observe that the edge-weights of $K_{1,m} \cup K_{1,n}$, under the vertex labeling f_5 , constitute the sets

$$W_{f_5}^1 = \{w_{f_5}^1(x_{1,0}x_{1,j}) : \text{if } 1 \leq j \leq m\} = \{2j + 2 : \text{if } 1 \leq j \leq m\},$$

$$W_{f_5}^2 = \{w_{f_5}^2(x_{2,0}x_{2,i}) : \text{if } 1 \leq i \leq n\} = \{m + n + 2 + 2i : \text{if } 1 \leq i \leq n\}.$$

Hence, the elements of the set $\bigcup_{k=1}^2 W_{f_5}^k = \{4, 6, \dots, m + 3n + 2\}$ can be arranged to form an arithmetic sequence, with the first term 4 and the common difference $d = 2$. Thus f_5 is a $(4, 2)$ -EAV labeling. \square

Theorem 6 *If $m = n \geq 2$ then the graph $K_{1,m} \cup K_{1,n}$ has a super $(2m + 2n + 6, 1)$ -EAT and super $(m + n + 7, 3)$ -EAT labelings.*

Proof. Let $m = n \geq 2$. From Theorem 5, it follows that the graph $K_{1,m} \cup K_{1,n}$ has a $(4, 2)$ -EAV labeling. We will distinguish two cases, according to whether $d = 1$ or $d = 3$.

Case 1. $d = 1$

Define $f_6 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, 2m + 2n + 2\}$ to be the bijective function such that

$$f_6(v) = f_5(v), \text{ for all vertices } v \in V(K_{1,m} \cup K_{1,n}),$$

$$f_6(x_{1,0}x_{1,j}) = 2m + 2n + 3 - j, \text{ for } 1 \leq j \leq m,$$

$$f_6(x_{2,0}x_{2,i}) = m + 2n + 3 - i, \text{ for } 1 \leq i \leq n.$$

By direct computation, it is not difficult to verify that the edge-weights constitute the arithmetic progression $2m + 2n + 6, 2m + 2n + 7, \dots, 3m + 3n + 5$. Thus f_6 is a super $(2m + 2n + 6, 1)$ -EAT labeling.

Case 2. $d = 3$

Consider the labeling $f_7 : V(K_{1,m} \cup K_{1,n}) \cup E(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, 2m + 2n + 2\}$, such that

$$f_7(v) = f_5(v), \text{ for all vertices } v \in V(K_{1,m} \cup K_{1,n}),$$

$$f_7(x_{1,0}x_{1,j}) = m + n + 2 + j, \text{ for } 1 \leq j \leq m,$$

$$f_7(x_{2,0}x_{2,i}) = 2m + n + 2 + i, \text{ for } 1 \leq i \leq n.$$

We can see that the labeling f_7 uses each integer from the set $\{1, 2, \dots, 2m + 2n + 2\}$ exactly once and the set of edge-weights consists of an arithmetic sequence with the first value $m + n + 7$ and the common difference $d = 3$. Thus f_7 is a super $(m + n + 7, 3)$ -EAT labeling. \square

In the case when $m + n \geq 4$ is even and $m \neq n$, we do not know of any super $(a, 1)$ -EAT labeling for $K_{1,m} \cup K_{1,n}$. Therefore, we propose the following

Open Problem 2 *For the graph $K_{1,m} \cup K_{1,n}$, $m + n \geq 4$ even and $m \neq n$, determine if there exists a super $(a, 1)$ -EAT labeling.*

From Theorem 6, we have that for $m = n \geq 2$ the graph $K_{1,m} \cup K_{1,n}$ has super $(m + n + 7, 3)$ -EAT labeling but for $m > n \geq 2$ we do not know if such a labeling exists or not. Therefore, we propose another open problem.

Open Problem 3 *For the graph $K_{1,m} \cup K_{1,n}$, $m > n \geq 2$, determine if there exists a super $(a, 3)$ -EAT labeling.*

Concluding this paper, let us prove the following theorem.

Theorem 7 *For the graph $K_{1,m} \cup K_{1,n}$, $m \geq n \geq 2$, there is no $(a, 3)$ -EAV labeling.*

Proof. Assume that $K_{1,m} \cup K_{1,n}$, $m \geq n \geq 2$, has an $(a, 3)$ -EAV labeling $f : V(K_{1,m} \cup K_{1,n}) \rightarrow \{1, 2, \dots, m + n + 1, m + n + 2\}$ and $W = \{w(uv) : uv \in E(K_{1,m} \cup K_{1,n})\} = \{a, a + 3, a + 6, \dots, a + (m + n - 1)3\}$ is the set of edge-weights. The minimum possible edge weight is at least 3. It follows that $a \geq 3$. The maximum possible edge weight is no more than $(p - 1) + p = 2m + 2n + 3$.

Consequently, $a + 3(m + n - 1) \leq 2m + 2n + 3$ and $3 \leq 2 + \frac{2}{m+n-1}$, which is impossible when $m + n \geq 4$. \square

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