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New skew-Hadamard matrices via computational algebra

I. S. Kotsireas

Wilfrid Laurier University
Department of Physics and Computer Science
75 University Avenue West
Waterloo, ON N2L 3C5
Canada
ikotsire@wlu.ca

C. Koukouvinos

National Technical University of Athens
Department of Mathematics
Zografou 15773, Athens
Greece
ckoukouv@math.ntua.gr

Abstract

In this paper we formalize three constructions for skew-Hadamard matrices from a Computational Algebra point of view. These constructions are the classical 4 Williamson array construction, an 8 Williamson array construction and a construction based on $OD(16; 1, 1, 2, 2, 2, 2, 2, 2, 2)$, a 9-variable full orthogonal design of order 16. Using our Computational Algebra formalism in conjunction with supercomputing, we perform exhaustive and partial searches for all three constructions. The computational results of these searches indicate a fundamental difference among these constructions, namely that the 8 Williamson array construction and the full orthogonal design construction exhibit exponential growths in the number of solutions. Subsequently, we analyze the computational results of these searches to locate inequivalent skew-Hadamard matrices, using the profile criterion. We show how to use the doubling construction to construct inequivalent skew-Hadamard matrices of order $2n$ from skew-Hadamard matrices of order n. Combining our computational results and the doubling construction we establish constructively 30 new lower bounds for the numbers of inequivalent skew-Hadamard matrices of orders 60, 68, 72, 76, 80, 84, 88, 92, 100, 104, 108, 112, 116, 120, 136, 144, 152, 160, 168, 176, 184, 200, 208, 216, 224, 232, 240, 288, 352, 416. All

the inequivalent skew-Hadamard matrices constructed in this paper are available on the web page `http://www.cargo.wlu.ca/skew-Hadamard`.

1 Introduction

A $(1,-1)$ matrix H of order n is called *Hadamard matrix* if $HH^T = H^TH = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n. A $(1,-1)$ matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric. A Hadamard matrix is normalized if all entries in its first row and column are equal to 1. A skew-Hadamard matrix H of order n can always be put in the skew-normal form

$$H = \begin{pmatrix} 1 & e^T \\ -e & C+I \end{pmatrix},$$

where e^T is the $1 \times (n-1)$ vector of ones, i.e. $e^T = (1,\ldots,1)$, and C is a skew-symmetric $(0,1,-1)$ matrix. A necessary and sufficient condition for a Hadamard matrix H of order n to be a skew-Hadamard matrix, is that $H + H^T = 2I_n$.

Two Hadamard matrices are said to be *equivalent* if one can be transformed into the other by a series of row or column permutations and negations. It is well known that if n is the order of a Hadamard matrix then n is necessarily $1, 2$ or a multiple of 4. However, it still remains open if a Hadamard matrix of order n exists for every $n \equiv 0 \ (mod \ \ 4)$. Since a Hadamard matrix of order 428 is found recently (see [21]), the smallest order for which a Hadamard matrix is not yet known is $n = 668$.

Skew-Hadamard matrices are of great interest (see [33]) because of their applications in constructing orthogonal designs, D-optimal weighing designs for $n \equiv 3 \ (mod\ 4)$ (see for example [28]), and edge designs (see for example [11, 18, 19]).

The comprehensive survey article [27], discusses the existence and the equivalence of skew-Hadamard matrices, presents some known construction methods of skew-Hadamard matrices and the known results and also some new inequivalent skew-Hadamard matrices of order 52.

1.1 The 4 Williamson array skew-Hadamard construction

Theorem 1 ([39]) *Let A, B, C and D be square matrices of order n. Further, let A be skew-type and circulant and B, C, D be back-circulant matrices whose first rows satisfy the following equations:*

$$\left.\begin{array}{ll} a_{1,j} = -a_{1,n+2-j} & \\ b_{1,j} = b_{1,n+2-j} & \\ c_{1,j} = c_{1,n+2-j} & 2 \le j \le n \\ d_{1,j} = d_{1,n+2-j} & \\ a_{11} = b_{11} = c_{11} = d_{11} = +1, & \end{array}\right\} \tag{1}$$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ and every element is $+1$ or -1. If

$$AA^T + BB^T + CC^T + DD^T = 4nI_n, \tag{2}$$

then

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix} \tag{3}$$

is a skew-Hadamard matrix of order $4n$. □

1.1.1 Diophantine constraints

Hadamard matrices of the Williamson type, from the 4 Williamson array are subject to some Diophantine constraints that can be derived by multiplying (2) from the right with the column vector $\mathbf{e} = (1, 1, \ldots, 1)^t$ and from the left with the row vector $\mathbf{e}^t$, see [16]. Thus we obtain the representation of $4n$ as a sum of four squares

$$4n = a^2 + b^2 + c^2 + d^2, \tag{4}$$

where a (resp. b, c, d) is the sum of the elements of each row (and column) of the matrix A (resp. B, C, D). Since A is skew-type, we have $a = a_{11} = 1$. Therefore we are looking for representations of $4n$ as a sum of four odd squares, where at least one of the squares is equal to 1.

1.1.2 Construction with Polynomials

In the rest of this paper, we take n to be an odd positive integer and we set $m = \frac{n-1}{2}$. Following Williamson's analysis (see [39]) let T and R be the two $n \times n$ matrices

$$T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 \\ 1 & \ldots & 0 & 0 & 0 \end{bmatrix}.$$

The matrix T has the property $T^n = I_n$. Then the matrices A, B, C, D can be defined by the following polynomials:

$$\begin{array}{ccccccccc} A & = & I_n & + & a_1 T & + & \cdots & + & a_{n-1}T^{n-1} \\ B & = & I_n & + & (b_1 T & + & \cdots & + & b_{n-1}T^{n-1})R \\ C & = & I_n & + & (c_1 T & + & \cdots & + & c_{n-1}T^{n-1})R \\ D & = & I_n & + & (d_1 T & + & \cdots & + & d_{n-1}T^{n-1})R \end{array} \tag{5}$$

where the coefficients a_i satisfy the skew-symmetry conditions

$$a_i = -a_{n-i}, \ i = 1, \ldots, m$$

and the coefficients b_i, c_i, d_i satisfy the symmetry conditions

$$b_i = b_{n-i}, c_i = c_{n-i}, d_i = d_{n-i,} \ i = 1, \ldots, m.$$

In this notation, the equation (4) must be satisfied with

$a = 1, b = \mid 1 + 2(b_1 + \ldots + b_m) \mid, c = \mid 1 + 2(c_1 + \ldots + c_m) \mid, d = \mid 1 + 2(d_1 + \ldots + d_m) \mid .$

For illustration, using the 4 Williamson array for $n = 15$, $(m = 7)$ equation (4) becomes

$$4 \cdot 15 = 60 = a^2 + b^2 + c^2 + d^2 \tag{6}$$

which has the solutions $(a = 1,\ b = 1,\ c = 3,\ d = 7)$ and $(a = 1,\ b = 3,\ c = 5,\ d = 5)$, with $b = \mid 1 + 2(b_1 + \ldots + b_7) \mid$, $c = \mid 1 + 2(c_1 + \ldots + c_7) \mid$, $d = \mid 1 + 2(d_1 + \ldots + d_7) \mid$.

1.2 An 8 Williamson array skew-Hadamard construction

Consider the 8×8 Williamson array

$$W = \begin{pmatrix} A & -B & -C & -D & -E & -F & -G & -H \\ B & A & -D & C & -F & E & H & -G \\ C & D & A & -B & -G & -H & E & F \\ D & -C & B & A & -H & G & -F & E \\ E & F & G & H & A & -B & -C & -D \\ F & -E & H & -G & B & A & D & -C \\ G & -H & -E & F & C & -D & A & B \\ H & G & -F & -E & D & C & -B & A \end{pmatrix}$$

which specifies the left matrix representation of an octonion over the set of real numbers and has the property

$$WW^T = (A^2 + B^2 + C^2 + D^2 + E^2 + F^2 + G^2 + H^2) \times I_8.$$

when A, B, C, D, E, F, G, H as viewed as numbers. See [24] and references therein, for computational results using this 8 Williamson array and for details on the derivation of the array.

By analogy with the 4 Williamson array construction defined in the previous paragraph, the matrices A, B, C, D, E, F, G, H can be defined by the following polynomials:

$$\begin{array}{lllll} A = I_n + a_1 T + \cdots + a_{n-1} T^{n-1}, & B = I_n + (b_1 T + \cdots + b_{n-1} T^{n-1}) R \\ C = I_n + (c_1 T + \cdots + c_{n-1} T^{n-1}) R, & D = I_n + (d_1 T + \cdots + d_{n-1} T^{n-1}) R \\ E = I_n + (e_1 T + \cdots + e_{n-1} T^{n-1}) R, & F = I_n + (f_1 T + \cdots + f_{n-1} T^{n-1}) R \\ G = I_n + (g_1 T + \cdots + g_{n-1} T^{n-1}) R, & H = I_n + (h_1 T + \cdots + h_{n-1} T^{n-1}) R \end{array} \tag{7}$$

where the coefficients a_i satisfy the skew-symmetry conditions

$$a_i = -a_{n-i},\ \ i = 1, \ldots, m$$

and the coefficients $b_i, c_i, d_i, e_i, f_i, g_i, h_i$ satisfy the symmetry conditions

$$b_i = b_{n-i}, c_i = c_{n-i}, d_i = d_{n-i}, e_i = e_{n-i}, f_i = f_{n-i}, g_i = g_{n-i}, h_i = h_{n-i},\ i = 1, \ldots, m.$$

By analogy with the 4 Williamson array construction defined in the previous paragraph, Hadamard matrices of the Williamson type from the 8 Williamson array are subject to the Diophantine constraints

$$8n = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2, \tag{8}$$

where a (resp. b, c, d, e, f, g, h) is the sum of the elements of each row (and column) of the matrix A (resp. B, C, D, E, F, G, H). Since A is skew-type, we have $a = a_{11} = 1$. Therefore we are looking for representations of $8n$ as a sum of eight odd squares, where at least one of the squares is equal to 1.

1.3 An OD16 array skew-Hadamard construction

Consider the 9-variable full orthogonal design of order 16 $OD(16; 1, 1, 2, 2, 2, 2, 2, 2, 2)$

$$OD_{16} = \begin{bmatrix}
A & B & C & D & E & F & G & H & I & B & C & D & E & F & G & H \\
-B & A & -D & C & -F & E & H & -G & -B & I & D & -C & F & -E & -H & G \\
-C & D & A & -B & -G & -H & E & F & -C & -D & I & B & G & H & -E & -F \\
-D & -C & B & A & -H & G & -F & E & -D & C & -B & I & H & -G & F & -E \\
-E & F & G & H & A & -B & -C & -D & -E & -F & -G & -H & I & B & C & D \\
-F & -E & H & -G & B & A & D & -C & -F & E & -H & G & -B & I & -D & C \\
-G & -H & -E & F & C & -D & A & B & -G & H & E & -F & -C & D & I & -B \\
-H & G & -F & -E & D & C & -B & A & -H & -G & F & E & -D & -C & B & I \\
-I & B & C & D & E & F & G & H & A & -B & -C & -D & -E & -F & -G & -H \\
-B & -I & D & -C & F & -E & -H & G & B & A & D & -C & F & -E & -H & G \\
-C & -D & -I & B & G & H & -E & -F & C & -D & A & B & G & H & -E & -F \\
-D & C & -B & -I & H & -G & F & -E & D & C & -B & A & H & -G & F & -E \\
-E & -F & -G & -H & -I & B & C & D & E & -F & -G & -H & A & B & C & D \\
-F & E & -H & G & -B & -I & -D & C & F & E & -H & G & -B & A & -D & C \\
-G & H & E & -F & -C & D & -I & -B & G & H & E & -F & -C & D & A & -B \\
-H & -G & F & E & -D & -C & B & -I & H & -G & F & E & -D & -C & B & A
\end{bmatrix} \tag{9}$$

which has the property that

$$OD_{16}OD_{16}^T = (A^2 + 2B^2 + 2C^2 + 2D^2 + 2E^2 + 2F^2 + 2G^2 + 2H^2 + I^2) \times I_{16}$$

when A, B, C, D, E, F, G, H, I as seen as numbers.

By analogy with the 8 Williamson array construction defined in the previous paragraph, the matrices A, B, C, D, E, F, G, H, I can be defined by the polynomials (7) and the additional polynomial

$$I = I_n + (i_1 T + \cdots + i_{n-1} T^{n-1}) R \tag{10}$$

where the coefficients a_j satisfy the skew-symmetry conditions

$$a_j = -a_{n-j}, \ j = 1, \ldots, m,$$

and the coefficients $b_j, c_j, d_j, e_j, f_j, g_j, h_j, i_j$ satisfy the same symmetry conditions as before and also $i_j = i_{n-j}$, $j = 1, \ldots, m$. The corresponding Diophantine constraints are

$$16n = a^2 + 2b^2 + 2c^2 + 2d^2 + 2e^2 + 2f^2 + 2g^2 + 2h^2 + i^2. \tag{11}$$

See [25] and references therein, for computational results using the OD16 array and for details on the derivation of the array.

1.4 The Doubling construction

This construction method was first given in [30] where it was used for the construction of a skew-Hadamard matrix of order 184 using the skew-Hadamard matrix of order 92 which was also constructed in the same paper.

Theorem 2 *Suppose that $H_n = S + I_n$ is a skew-Hadamard matrix of order n. Then*

$$H_{2n} = \begin{pmatrix} S + I_n & S + I_n \\ S - I_n & -S + I_n \end{pmatrix}$$

is a skew-Hadamard matrix of order $2n$.

1.5 Existence results for skew-Hadamard matrices

The problem of existence of skew-Hadamard matrices has been studied extensively, see [3, 4, 5, 7, 8, 9, 13, 15, 16, 17, 20, 26, 32, 34, 36, 37, 38, 40, 41, 42]. However, there are a lot of orders for which it is still unknown whether skew-Hadamard matrices exist. Recently, a skew-Hadamard matrix of order 236 has been constructed in [12], and of orders 188 and 388 in [10]. The smallest order for which a skew-Hadamard matrix is not known, is the order $276 = 4 \cdot 69$. The current status on known results and open problems on the existence of skew-Hadamard matrices of order $2^t m$, m odd, $m < 500$, is described in [27].

J. Seberry Wallis [31] conjectured that skew-Hadamard matrices exist for all orders divisible by 4.

1.6 Inequivalent skew-Hadamard matrices

The problem of establishing lower bounds for the numbers of inequivalent skew-Hadamard matrices of a given order has also received a lot of attention in the literature. Let N_n denote the number of inequivalent skew-Hadamard matrices for a given order n. We summarize the known results for N_n in the table below, taken from [27].

n	2	4	8	12	16	20	24	28	32	36	40	44	48	52
N_n	1	1	1	1	2	1	16	54	≥ 6	≥ 18	≥ 22	≥ 59	≥ 1	≥ 561

(12)

Inequivalent skew-Hadamard matrices for orders 2 to 52.

In Table (12), we denote by n the order of the skew-Hadamard matrix and by N_n the number of known inequivalent skew-Hadamard matrices of order n.

For $n = 4, 8, 12, 20, 24, 28, 32, 44, 48$ the conditions of Paley's theorem ([29]) are satisfied and thus we can construct a skew-Hadamard matrix of order n. N. Ito has determined that for general skew-Hadamard matrices, there is a unique matrix of each order less than 16, two of order 16, and 16 of order 24. H. Kimura has found 49 of order 28 and 6 of order 32, see [35, p.492]. Later Baartmans, Lin and Wallis [1] improved the lower bound found by Kimura for order 28, showing that there are exactly 54 inequivalent skew-Hadamard matrices of order 28. 22 skew-Hadamard matrices of order 40 were constructed in a resent paper (see [14]). Results for orders $36, 44$ have been recently given in [19, 18], respectively. A skew-Hadamard matrix of order 52 was first found in [2]. New results for $n = 52$ are presented in [27].

2 Computational Algebra approach

From the point of view of Computational Algebra we see that we need $4(\frac{n-1}{2}) = 2n-2$ coefficients to describe the skew-Hadamard 4 Williamson array construction using the polynomials (5). Similarly, we need $8(\frac{n-1}{2}) = 4n-4$ coefficients to describe the skew-Hadamard 8 Williamson array construction using the polynomials (7). Finally, we need $9(\frac{n-1}{2})$ coefficients to describe the orthogonal design construction using the polynomials (10).

For illustration, using the 4 Williamson array for $n = 15$ we obtain the following 7 equations in the 28 variables $a_1, \ldots, a_7$, $b_1, \ldots, b_7$, $c_1, \ldots, c_7$, $d_1, \ldots, d_7$:

```
a1a2+a2a3+a3a4+a4a5+a5a6+a6a7+b1b2+b2b3+b3b4+b4b5+b5b6+b6b7+
c1c2+c2c3+c3c4+c4c5+c5c6+c6c7+d1d2+d2d3+d3d4+d4d5+d5d6+d6d7+
b1+c1+d1+1 = 0

a1a3+a2a4+a3a5+a4a6+a5a7-a6a7+b1b3+b2b4+b3b5+b4b6+b5b7+b6b7+
c1c3+c2c4+c3c5+c4c6+c5c7+c6c7+d1d3+d2d4+d3d5+d4d6+d5d7+d6d7+
b2+c2+d2+1 = 0

-a1a2+a1a4+a2a5+a3a6+a4a7-a5a7+b1b2+b1b4+b2b5+b3b6+b4b7+b5b7+
c1c2+c1c4+c2c5+c3c6+c4c7+c5c7+d1d2+d1d4+d2d5+d3d6+d4d7+d5d7+
b3+c3+d3+1 = 0

-a1a3+a1a5+a2a6+a3a7-a4a7-a5a6+b1b3+b1b5+b2b6+b3b7+b4b7+b5b6+
c1c3+c1c5+c2c6+c3c7+c4c7+c5c6+d1d3+d1d5+d2d6+d3d7+d4d7+d5d6+
b4+c4+d4+1 = 0

-a1a4+a1a6-a2a3+a2a7-a3a7-a4a6+b1b4+b1b6+b2b3+b2b7+b3b7+b4b6+
c1c4+c1c6+c2c3+c2c7+c3c7+c4c6+d1d4+d1d6+d2d3+d2d7+d3d7+d4d6+
b5+c5+d5+1 = 0

-a1a5+a1a7-a2a4-a2a7-a3a6-a4a5+b1b5+b1b7+b2b4+b2b7+b3b6+b4b5+
c1c5+c1c7+c2c4+c2c7+c3c6+c4c5+d1d5+d1d7+d2d4+d2d7+d3d6+d4d5+
b6+c6+d6+1 = 0

-a1a6-a1a7-a2a5-a2a6-a3a4-a3a5+b1b6+b1b7+b2b5+b2b6+b3b4+b3b5+
c1c6+c1c7+c2c5+c2c6+c3c4+c3c5+d1d6+d1d7+d2d5+d2d6+d3d4+d3d5+
b7+c7+d7+1 = 0
```

where the 28 variables satisfy the additional 28 equations

$$a_1^2 = 1,\ \ldots,\ d_7^2 = 1.$$

We remark that each equation is of total degree 2 but also contains three monomials of degree 1, namely the variables b_i, c_i, d_i, $i = 1, \ldots, 7$ linearly. Here is a solution of the equations for $n = 15$: ($-$ stands for -1, $+$ stands for $+1$)

$$\underbrace{-\,-\,-\,-\,-\,+\,-}_{\texttt{a1 a2 a3 a4 a5 a6 a7}}\ \underbrace{-\,+\,-\,-\,+\,+\,-}_{\texttt{b1 b2 b3 b4 b5 b6 b7}}\ \underbrace{+\,-\,+\,+\,+\,+\,-}_{\texttt{c1 c2 c3 c4 c5 c6 c7}}\ \underbrace{+\,+\,-\,-\,+\,-\,+}_{\texttt{d1 d2 d3 d4 d5 d6 d7}}.$$

The above solution belongs to the first class of solutions of the associated Diophantine constraint (6) because it satisfies the identities

$$b = \mid 1 + 2(b_1 + \ldots + b_7) \mid = 1, c = \mid 1 + 2(c_1 + \ldots + c_7) \mid = 7, d = \mid 1 + 2(d_1 + \ldots + d_7) \mid = 3.$$

We run several C programs automatically generated with Maple, to perform exhaustive searches for all odd values of n up to $n = 25$ and partial searches for $n = 27, 29$. The searches for all odd values of n until $n = 23$ were performed with serial C programs. The exhaustive search for $n = 25$ was performed using a serial C program that was broken into 16 C programs using the bash shell Linux utility. The individual C programs were run simultaneously on a SHARCnet high-performance cluster at the University of Western Ontario, London ON, Canada. The partial searches for $n = 27, 29$ were performed on SHARCnet high-performance clusters at the University of Western Ontario, London ON, Canada, McMaster University, Hamilton ON, Canada and on a WestGrid high-performance cluster at the University of Calgary, Calgary AB, Canada.

The results of our searches are summarized below:

n	3	5	7	9	11	13	15	17	19	21	23	25	27	29
# sol's	6	12	66	36	180	432	528	192	768	720	792	1080	≥ 810	≥ 222

(13)

Exhaustive and partial searches for the 4 Williamson array

n	3	5	7	9	11	13	15
# solutions	6	2,100	56,070	1,179,360	$\geq 5,592,430$	$\geq 841,214$	$\geq 91,648$

(14)

Exhaustive and partial searches for the 8 Williamson array

n	3	5	7	9	11	13
# solutions	42	2,100	48,510	982,800	23,362,500	$\geq 3,722,034$

(15)

Exhaustive and partial searches for the OD16 orthogonal design

3 New inequivalent skew-Hadamard matrices from the 4 and 8 Williamson array and the OD16 array

In this section we describe how many inequivalent skew-Hadamard matrices of the twenty-two orders 40, 44, 48, 52, 56, 60, 68, 72, 76, 80, 84, 88, 92, 100, 104, 108, 112, 116, 120, 144, 176, 208 we constructed using the results of the exhaustive and partial searches from the 4 and the 8 Williamson arrays and the OD16 array (see tables (13), (14) and (15)).

k	11	13	15	17	19	21	23	25	27	29
n	44	52	60	68	76	84	92	100	108	116
N_n	≥ 5	≥ 11	≥ 22	≥ 4	≥ 16	≥ 20	≥ 12	≥ 18	≥ 26	≥ 10

Table 1: Inequivalent skew-Hadamard matrices from the 4 Williamson array

k	5	7	9	11	13	15
n	40	56	72	88	104	120
N_n	≥ 6	≥ 75	≥ 990	≥ 3673	≥ 2661	≥ 5738

Table 2: Inequivalent skew-Hadamard matrices from the 8 Williamson array

Let N_k denote the number of inequivalent skew-Hadamard matrices of order k. We summarize our results for N_k, in the following three tables.

All the inequivalent skew-Hadamard matrices described in this paragraph, can be downloaded from the web page `http://www.cargo.wlu.ca/skew-Hadamard`.

We used the following Magma function to check whether the matrices are skew-Hadamard (and Hadamard).

```
IsSkewHadamard := function(H) if IsHadamard(H) eq true
        then
          n := NumberOfRows(H);
          In2 := DiagonalMatrix([2 : i in [1..n]]);
          if H+Transpose(H) eq In2
            then return(true);
            else return("this is not a skew-Hadamard matrix");
          end if;
        else return("this is not a Hadamard matrix");
end if; end function;
```

We used the profile criterion (implemented in Magma's `HadamardInvariant` command) to prove that the matrices are inequivalent.

4 New inequivalent skew-Hadamard matrices from the doubling construction

In this section we describe how many inequivalent skew-Hadamard matrices of the twenty-two orders 80, 88, 96, 104, 112, 120, 136, 144, 152, 160, 168, 176, 184, 200, 208, 216, 224, 232, 240, 288, 352, 416 we constructed, by using the doubling method

k	3	5	7	9	11	13
n	48	80	112	144	176	208
N_n	≥ 1	≥ 6	≥ 62	≥ 366	≥ 1586	≥ 1143

Table 3: Inequivalent skew-Hadamard matrices from the OD16 array

on the matrices constructed in the previous section. The doubling construction is particularly easy to implement, given an initial set of skew-Hadamard matrices.

Remark: For some of the orders obtained by doubling, there are already a number of inequivalent skew-Hadamard matrices from the 4 and 8 Williamson arrays. In the present paper there are six such orders, namely, 88, 104, 120, 144, 176 and 208.

Remark: When the doubling construction is used on a set of k inequivalent skew-Hadamard matrices of a fixed order n, we can obtain at most k inequivalent skew-Hadamard matrices of order $2n$. In the case of the 4 Williamson array, the number of inequivalent skew-Hadamard matrices of order $2n$ is usually less than k (in most cases it's about $\frac{k}{2}$ as has been verified experimentally). However, in the case of the 8 Williamson array, the number of inequivalent skew-Hadamard matrices of order $2n$ is always exactly equal to k (as has been verified experimentally). This experimental observation may be linked to the exponential behavior of the solution sets for the 8 Williamson array.

We use the notations and tools of the previous paragraph. Below is a summary of the results on N_n for the twenty-two orders by doubling. The inequalities for the six orders 88, 104, 120, 144, 176 and 208 below, are to be understood as the numbers of inequivalent matrices for these orders, coming from doubling. These inequalities must also be interpreted taking into account the previous (much larger) bounds for these three orders.

- Results from doubling 4 Williamson array matrices

$$N_{88} \geq 3, \quad N_{104} \geq 6, \quad N_{120} \geq 11, \quad N_{136} \geq 2, \quad N_{152} \geq 8,$$

$$N_{168} \geq 10, \quad N_{184} \geq 6, \quad N_{200} \geq 9, \quad N_{216} \geq 13, \quad N_{232} \geq 5.$$

- Results from doubling 8 Williamson array matrices

$$N_{80} \geq 6, \quad N_{112} \geq 75, \quad N_{144} \geq 990,$$

$$N_{176} \geq 3673, \quad N_{208} \geq 2661, \quad N_{240} \geq 5738.$$

- Results from doubling OD16 array matrices

$$N_{96} \geq 1, \quad N_{160} \geq 6, \quad N_{224} \geq 62, \quad N_{288} \geq 366, \quad N_{352} \geq 1586, \quad N_{416} \geq 1143.$$

5 Structure of the ideal for skew-Hadamard matrices

The study of the entire solution sets (varieties) of the polynomial systems of equations (ideals) arising in the construction of skew-Hadamard matrices for the 4 and 8 Williamson arrays reveals some similarities and differences with the ideals associated to other constructions for Hadamard matrices, see for example [22], [23], [24].
Specifically, if we look at the sequence of the numbers of solutions for all values of the parameter, we observe the two properties:

- *non-lacunarity*, i.e. there are solutions for all values of the parameters we examined so far. This property holds in the case of Hadamard matrices with two circulant cores see [23], but doesn't hold in the case of Hadamard matrices, from the 4 Williamson array, see [6].

- *non-monotonicity*, i.e. the sequence is neither (strictly) increasing nor (strictly) decreasing. A monotonicity property (strictly increasing) holds for Hadamard matrices with two circulant cores see [23]. Hadamard matrices, from the 4 Williamson array, exhibit non-monotonicity, see [24].

In addition, if we look at the two sequences of the numbers of solutions for skew-Hadamard matrices from the 4 Williamson array and Hadamard matrices from the 4 Williamson array, we see that they exhibit a similar rate of growth.
An important difference of the solution sets associated with the 4 and the 8 Williamson arrays and the OD16 array, is that in the cases of the 8 Williamson and the OD16 array skew-Hadamard construction, we observe an exponential growth in the number of solutions, as a function of the order of the block matrices.
An important difference of the solution sets associated with skew-Hadamard matrices, as opposed to the solution sets associated to non-skew Hadamard matrices, is that there is a phenomenon of break of symmetry. To illustrate this phenomenon, we study the case $n = 5$ for skew-Hadamard matrices from the 4 Williamson array.

The equations for $n = 5$ are:

```
-a1*a2+b1*b2+c1*c2+d1*d2+b2+c2+d2+1 = 0
 a1*a2+b1*b2+c1*c2+d1*d2+b1+c1+d1+1 = 0
```

There are exactly 12 solutions (with ± 1 values) of these equations:

```
1 -1 -1 -1 -1 -1 -1 -1 1 2 -1 -1 -1 -1 -1 1 -1 -1 3 -1 -1 -1 1 -1
-1 -1 -1 4 -1 1 -1 -1 -1 -1 1 -1 5 -1 1 -1 -1 1 -1 -1 -1 6 -1 1 1
-1 -1 -1 -1 -1 7 1 -1 -1 -1 -1 -1 1 -1 8 1 -1 -1 -1 1 -1 -1 -1 9 1
-1 1 -1 -1 -1 -1 -1 10 1 1 -1 -1 -1 -1 -1 1 11 1 1 -1 -1 -1 1 -1
-1 12 1 1 -1 1 -1 -1 -1 -1
```

where each solution is given in the format `solution # a1 a2 b1 b2 c1 c2 d1 d2`.
Using the bash shell script

```
awk '{print $2 $3 $4 $5 $6 $7 $8 $9}' $1 > $1.awked sed -f
sedChanges $1.awked > $1.seded rm $1.awked
```

where `sedChanges` is the SED script

```
s/-1/0/g s/$/,/g $s/,$/]:/ 1s/^/l:=[/
```

we can compute the ranks of these 12 solutions in the boolean cube of $2^8 = 256$ elements, via their 12 binary representations

```
00000001 00000100 00010000 01000010 01001000 01100000 10000010
10001000 10100000 11000001 11000100 11010000
```

Translating these binary numbers in the decimal system, we obtain the ranks of the 12 solutions as follows:

$$[1, 4, 16, 66, 72, 96, 130, 136, 160, 193, 196, 208].$$

Now we see that the solutions are not situated symmetrically in the interval $[1, 256]$, which is the interval representation of the boolean cube of $2^8 = 256$ elements.

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7 Conclusion

In this paper we use Computational Algebra methods to establish constructively 30 new lower bounds for the numbers of inequivalent skew-Hadamard matrices of orders 60, 68, 72, 76, 80, 84, 88, 92, 100, 104, 108, 112, 116, 120, 136, 144, 152, 160, 168, 176, 184, 200, 208, 216, 224, 232, 240, 288, 352, 416. All the inequivalent skew-Hadamard matrices constructed in this paper are available in the web page `http://www.cargo.wlu.ca/skew-Hadamard` .

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