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On the connectivity of extremal Ramsey graphs

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Abstract

An (r, b)-graph is a graph that contains no clique of size r and no independent set of size b. The set of extremal Ramsey graphs $ERG(r, b)$ consists of all (r, b)-graphs with $R(r, b) - 1$ vertices, where $R(r, b)$ is the classical Ramsey number. We show that any $G \in ERG(r, b)$ is $r - 1$ vertex connected and $2r - 4$ edge connected for $r, b \geq 3$.

1 Introduction

Let $R(r, b)$ be the classical Ramsey number. An (r, b)-*graph* is a graph that contains no clique of size r and no independent set of size b. Let $ERG(r, b)$ (to abbreviate *extremal Ramsey graphs*) consist of all (r, b)-graphs with $R(r, b) - 1$ vertices. In this paper we identify $G \in ERG(r, b)$ with a red-blue coloring of the complete graph and we denote the graphs induced by the color classes as G_{red} and G_{blue}.

In his talk at the 2005 British Combinatorial Conference, David Penman established various properties of extremal Ramsey graphs and conjectured that for $k \geq 3$ every graph $G \in ERG(k, k)$ is connected. Here we will observe that this conjecture is true. In fact, its validity follows fairly straightforwardly from a result of Xiaodong, Zheng, and Radziszowski [7] (see Section 2). For the sake of completeness we present a short self-contained proof, which follows by adopting the methods in [7]. (More

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recently, Shane Malik and David Penman [4] reported to the authors that they had independently proved this conjecture.)

Theorem 1 *Let $r \geq 3$, $b \geq 2$, and $G \in ERG(r, b)$. The red graph G_{red} is connected.*

Proof. Assume that $b \geq 3$, for otherwise $G_{\text{red}} = K_{r-1}$ is clearly connected.

Suppose on the contrary that $V(G) = V_1 \cup V_2$ is a proper partition of the vertices with only blue edges across. Pick $x_i \in V_i, i = 1, 2$. Create a new graph G' by adding a new vertex x and coloring the incident edges as follows. Color xx_1 and xx_2 blue. Color xy with $y \in V_i - x_i$ by the color of $x_i y$. Finally, recolor $x_1 x_2$ red.

Since we have strictly increased the number of vertices, G' must contain either a red K_r or a blue K_b. First suppose there is a red K_r on a set R. R must contain x. Indeed, the only difference between $G' - x$ and G is the color of the edge $x_1 x_2$ and this solitary red edge joining V_1 and V_2 cannot be in a red K_r. Since xx_1 and xx_2 are blue, $x_1, x_2 \notin R$ and $R - x$ lies inside one part, say V_1. But then $R - x + x_1$ is a red K_r in G, a contradiction. Next suppose we have a blue K_b on a set B. We must have $x \in B$. At least one of x_1, x_2 is not in B since $x_1 x_2$ is red now. Next suppose $x_1 \notin B$. But then $R - x + x_1$ is a blue K_b in G (because $x_1 x_2$ is blue in the original graph G.) This contradiction proves the theorem. ∎

In the remainder of the paper, we explore how connected the red graph of $G \in ERG(r, b)$ must be. We show that for $r, b \geq 3$, the (vertex) connectivity $\kappa(G_{\text{red}}) \geq r - 1$ and the edge connectivity $\lambda(G_{\text{red}}) \geq 2r - 4$. There is no doubt that these bounds are very far from best possible, which may be even exponential in $\min(r, b)$. However, it is not clear how to get any essential improvement.

2 Vertex Connectivity

We will use the following result of Xiaodong, Zheng, and Radziszowski [7, Theorem 3] which builds upon the ideas from Burr, Erdős, Faudree, and Schelp [1].

Theorem 2 (Xiaodong et al. [7]) *If $2 \leq p \leq q$ and $3 \leq r$, then*

$$R(r, p + q - 1) \geq R(r, p) + R(r, q) + \begin{cases} r - 3, & \text{if } p = 2, \\ r - 2, & \text{if } p \geq 3. \end{cases}$$

In particular, the case $p = 2$ and $q = b - 1$ gives the original result of Burr et al. [1, Theorem 1] (see also [7, Theorem 1] for a small correction, namely that (1) is not true for $b = 2$): for any $r \geq 2$ and $b \geq 3$,

$$R(r, b) \geq R(r, b - 1) + 2r - 3. \tag{1}$$

It follows that for any $G \in ERG(r, b)$, with $b \geq 3$, the minimum red degree

$$\delta(G_{\text{red}}) \geq 2r - 4. \tag{2}$$

Indeed, take any vertex x and let $d_{G_{\mathrm{red}}}(x)$ denote its red degree. The blue neighborhood of x has $R(r,b)-2-d_{G_{\mathrm{red}}}(x)$ vertices and induces an $(r,b-1)$-graph. Hence,

$$R(r,b)-2-d_{G_{\mathrm{red}}}(x) \le R(r,b-1)-1,$$

and $d_{G_{\mathrm{red}}}(x) \ge 2r-4$ by (1), as required.

Theorem 3 *Let $r,b \ge 3$ and $G \in ERG(r,b)$. The (vertex) connectivity of the red graph G_{red} satisfies $\kappa(G_{\mathrm{red}}) \ge r-1$.*

Proof. Suppose on the contrary that we can find a partition $V(G) = V_1 \cup V_2 \cup X$ such that all edges between V_1 and V_2 are blue and $0 < |X| \le r-2$ (X is nonempty by Theorem 1). Let $p-1$ and $q-1$ be the sizes of a maximum blue clique inside V_1 and V_2 respectively. Assume without loss of generality that $p \le q$.

If $p = 2$, then V_1 spans a red clique, so $|V_1| \le r-1$. The red degree of each vertex of V_1 is at most $|V_1|-1+|X| \le 2r-4$. By (2), all these inequalities are equalities; in particular, $|V_1| = r-1$, $|X| = r-2$, and all edges between V_1 and X are red. But then V_1 plus any vertex from $X \ne \emptyset$ spans a red K_r, a contradiction.

So assume $p \ge 3$. We have $(p-1)+(q-1) \le b-1$, and

$$R(r,p)-1+R(r,q)-1 \ge |V_1|+|V_2| = |V(G)|-|X| = R(r,b)-1-|X|.$$

It follows from Theorem 2 that $|X| \ge r-1$, a contradiction. ∎

Corollary 4 *Let $r \ge b \ge 3$ and $G \in ERG(r,b)$. The red graph G_{red} is Hamiltonian.*

Proof. The Chvátal-Erdős condition [2] states that a graph of order at least 3 has a Hamiltonian cycle if it is k-connected and does not contain a set of $k+1$ independent points. By Theorem 3, this condition is satisfied for G_{red} when $r \ge b \ge 3$. ∎

3 Edge Connectivity

Theorem 5 *Let $r,b \ge 3$ and $G \in ERG(r,b)$. The edge connectivity of the red graph G_{red} satisfies*

$$\lambda(G_{\mathrm{red}}) \ge \min\{\delta(G_{\mathrm{red}}), \kappa(G_{\mathrm{red}})+r-3\}. \tag{3}$$

Note that we have the trivial upper bound $\lambda(G_{\mathrm{red}}) \le \delta(G_{\mathrm{red}})$. Our lower bounds for $\delta(G_{\mathrm{red}})$ and $\kappa(G_{\mathrm{red}})$ imply $\lambda(G_{\mathrm{red}}) \ge 2r-4$.

Proof. For $r=3$, the statement $\lambda(G_{\mathrm{red}}) \ge \min\{\delta(G_{\mathrm{red}}), \kappa(G_{\mathrm{red}})\} = \kappa(G_{\mathrm{red}})$ is simply the trivial lower bound.

Consider $r \ge 4$. Suppose that the claim is not true, that is, we can find a proper partition $V(G) = V_1 \cup V_2$ such that we have at most $k = \min\{\delta(G_{\mathrm{red}})-1, \kappa(G_{\mathrm{red}})+r-4\}$ red edges across. Call these red edges F. Let $v_i = |V_i|$. We claim

$v_i \geq \delta(G_{\rm red})+1$, $i=1,2$. Indeed, there is a vertex in V_i whose F-degree is at most $\lfloor \frac{k}{v_i} \rfloor \leq \lfloor \frac{\delta(G_{\rm red})-1}{v_i} \rfloor$. We have $v_i - 1 + \lfloor \frac{\delta(G_{\rm red})-1}{v_i} \rfloor \geq \delta(G_{\rm red})$, which routinely implies that $v_i \geq \delta(G_{\rm red})+1$. Take $x_i \in V_i$ not incident to any edge of F, which is possible since $v_i \geq \delta(G_{\rm red})+1 > |F|$.

Let X be a minimal set of vertices that cover the edge set F. Clearly X is a cutset of vertices for $G_{\rm red}$ ($x_i \notin X$ so $V_i - X \neq \emptyset$ for $i=1,2$) and therefore $|X| \geq \kappa(G_{\rm red})$. By Kőnig's theorem for bipartite graphs, there is a matching $M \subset F$ of size $|X|$. Hence $|F|-|M| \leq k - \kappa(G_{\rm red}) \leq r-4$. As F consists of a matching along with no more than $r-4$ additional edges, there is no red K_{r-1} that intersects both V_1 and V_2.

Construct G' from G as in the proof of Theorem 1. Namely, add a new vertex x. Color xx_1 and xx_2 blue. For any vertex $y \in V_i \setminus \{x_i\}$, color xy with the color of x_iy, $i=1,2$. Finally, recolor x_1x_2 red. We have strictly increased the number of vertices and $G \in ERG(r,b)$ so we must have a large monochromatic clique in G'. First, suppose we have a red K_r in G', say on a set R. This set R must contain x because the only red edge added to G, namely x_1x_2, cannot appear in a red K_r. But then $x_1, x_2 \notin R$. Moreover, $R - x$ lies entirely inside either V_1 or V_2. Suppose that $R - x \subset V_1$. But then $R - x + x_1$ is a red r-clique in G, a contradiction.

Next suppose that we have a blue $K_b \subset G'$, on a set B. We have $x \in B$ and at least one of x_1, x_2 does not belong to B. Suppose $x_1 \notin B$. But then $B - x + x_1$ spans a blue K_b in G (note that all edges between x_1 and V_2 are blue in G by the definition of x_1), a contradiction. ∎

4 Some Small Extremal Ramsey Graphs

In conclusion, we compare our lower bounds on connectivity with the actual connectivities for some small extremal Ramsey graphs.

(r,b)	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$	$(4,4)$
$R(r,b)$	6	9	14	18	23	18
$\lvert ERG(r,b)\rvert$	1	3	1	7	191	1
$\kappa(G_{\rm red})$ lower bound	2	2	2	2	2	3
$\kappa(G_{\rm red})$ actual	2	2	4	4	4,5,6	8
$\lambda(G_{\rm red})$ lower bound	2	2	2	2	2	4
$\lambda(G_{\rm red})$ actual	2	2	4	4	4,5,6	8
$\kappa(G_{\rm blue})$ lower bound	2	3	4	5	6	3
$\kappa(G_{\rm blue})$ actual	2	4	8	11	15	8
$\lambda(G_{\rm blue})$ lower bound	2	4	6	8	10	4
$\lambda(G_{\rm blue})$ actual	2	4	8	11	15	8

Table 1: Lower bounds for connectivity from Theorems 3 and 5 and the actual connectivity for small extremal Ramsey graphs

Let us explain how this table is constructed. Currently, there are 6 pairs (r,b)

with $3 \leq r \leq b$ such that the graphs in $ERG(r, b)$ have been completely enumerated; see Brendan McKay's combinatorial data website [3]. Specifically, the extremal Ramsey graph catalog for $r = 3$ and $4 \leq b \leq 7$ and $r = b = 4$ (in addition to the pair $(r, b) = (3, 3)$) was obtained from [3]. This data (in the graph6 format) was processed by McKay's `showg` executable and subsequently checked and analyzed by the Combinatorica package for Mathematica.

For every such pair we checked all extremal (r, b)-graphs and wrote down the observed edge and vertex connectivities into the table rows marked 'actual.' As it turns out, the pair $(r, b) = (3, 7)$ is the only pair from our list where different extremal (r, b)-graphs may have different connectivities. Of the 191 graphs in $ERG(3, 7)$, 3 are red 4-connected, 178 are red 5-connected and 10 are red 6-connected (while the edge connectivity happens to coincide with the vertex connectivity).

For the ease of reference, we have also included our lower bounds, $\kappa(G_{\text{red}}) \geq r-1$ and $\lambda(G_{\text{red}}) \geq 2r - 4$ of Theorem 3 and the comment immediately after Theorem 5, respectively. As expected, there is certainly room for improvement in the lower bounds for connectivity, even for these small graphs.

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