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Decycling sets in certain cartesian product graphs with one factor complete

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Abstract

A *decycling set* in a graph G is a set D of vertices such that $G - D$ is acyclic. The *decycling number* of G, $\phi(G)$, is the cardinality of a smallest decycling set in G. We obtain sharp bounds on the value of the cartesian product $\phi(G \Box K_r)$ when $r \geq 3$ and prove that when G belongs to one of several well-known families of graphs, including bipartite graphs and graphs of maximum degree 3, then $\phi(G \Box K_3) = n + \phi(G)$ and $\phi(G \Box K_r) = n(r-2)$ for $r \geq 4$, where n is the order of G. We prove also that every cubic graph $G \neq K_4$ contains an independent decycling set.

1 Introduction

A *decycling set* in a graph G, also known in the literature as a *vertex feedback set*, is a set D of vertices such that $G - D$ is acyclic. The *decycling number* of G, denoted by $\phi(G)$, is the cardinality of a smallest decycling set in G. We call a decycling set of minimum size a *ϕ-set* for G. The corresponding problem of finding the minimum number of edges that must be deleted from a graph G of order n having m edges and c components is known as the *cycle rank* of G and is easily shown to be $m - n + c$ (see for example [17], p.46). In contrast, it has been shown by Karp [6] that the decision problem of finding $\phi(G)$ for an arbitrary graph G is NP-Complete. The problem remains difficult even when restricted to some well-known families of graphs, for example, bipartite graphs or planar graphs. On the other hand, it has been shown to be polynomial for graphs of maximum degree 3 in [9] and [16], grids in [5], permutation graphs in [7], interval and comparability graphs in [8], and “snakes” (graphs consisting of a finite sequence of chordless cycles, each having just one edge in common with the preceding cycle and one with the following cycle) in [3].

Other results on the decycling number can be found in [2]. The decycling number of cubic graphs is treated in [10],[13]; of regular graphs in general in [14], [15] and of random regular graphs in [4]. Hypercubes are treated in [5], [11]; and the cartesian product of two cycles in [12].

The *cartesian product* $G := G_1 \Box G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in G if and only if either (i) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or (ii) $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. In [5] (Theorem 1.8), the decycling number of the cartesian product of a graph G with K_2 is considered and sharp bounds for $\phi(G\Box K_2)$ are obtained for an arbitrary graph G in terms of $\phi(G)$ and $\alpha(G)$, where $\alpha(G)$ denotes the covering number of G.

Theorem 1.1 (Beineke, Vandell) *For any graph G,*

$$2\phi(G) \leq \phi(G\Box K_2) \leq \phi(G) + \alpha(G).$$

It is easily seen that for all $n \geq 2$, $\phi(K_n) = n - 2$ and $\phi(K_n \Box K_2) = 2n - 4$, so that the lower bound is achieved when $G = K_n$. The authors prove in [5] that the upper bound is achieved by $G = P_n$, the path of order n.

In this paper we consider the problem of finding bounds on $\phi(H)$ when H is the cartesian product of a graph G of order n with a complete graph K_r, where $r \geq 3$. In Section 2, we obtain a result analogous to Theorem 1.1 that

$$\max\{3\phi(G), n + \phi(G)\} \leq \phi(G\Box K_3) \leq n + 2\phi(G),$$

and further, that when G satisfies certain conditions, the upper bound in the second inequality can be reduced to $n + \phi(G)$. In Section 3, we obtain bounds for $\phi(G\Box K_r)$ when $r \geq 4$ and show that if G satisfies a slightly stronger set of conditions, then $\phi(G\Box K_r) = n(r-2)$. These results enable us to prove that when G belongs to one of several elementary families of graphs, including bipartite graphs and graphs with maximum degree 3, then $\phi(G\Box K_3) = n+\phi(G)$ and $\phi(G\Box K_r) = n(r-2)$ when $r \geq 4$. This implies in particular that the problem of determining $\phi(G\Box K_r)$ when G has maximum degree 3 is polynomial for $r \geq 3$. We also show that every cubic graph $G \neq K_4$ contains an independent decycling set.

All graphs considered in this paper are simple. We use the following notation. For $X \subseteq V(G)$, $\langle X\rangle$ denotes the subgraph of G induced by X. The number of components of G is denoted by $c(G)$ and the maximum vertex degree by $\Delta(G)$. Additionally, when we regard the graph $G\Box K_r$ as the graph K_r in which each vertex is replaced by a copy of G, we label the vertices of K_r as $1, 2, \ldots, r$, the copy of G replacing vertex i as G_i and the copy of vertex $v \in V(G)$ in G_i as v_i, $i = 1, 2, \ldots, r$. Similarly, for any subgraph $H \subseteq G$ or for any set $S \subseteq V(G)$, H_i denotes the copy of H in G_i and $S_i := S \cap V(G_i)$, $i = 1, 2, \ldots, r$.

2 Decycling $G\Box K_3$

Let G be a graph of order $n \geq 2$. We can visualise the graph $G\Box K_r$ in two different ways: either as the graph G in which each vertex $v \in V(G)$ is replaced by a copy of K_r, or as K_r in which each vertex is replaced by a copy of G. In this latter case, we label the vertices of K_r as $1, 2, \ldots, r$, the copy of G replacing vertex i as G_i and the copy of vertex $v \in V(G)$ in G_i as v_i, $i = 1, 2, \ldots, r$. Similarly, for any set $S \subseteq V(G)$, S_i denotes the set of copies of the vertices in S in G_i, $i = 1, 2, \ldots, r$. In finding a sharp lower bound for $\phi(G\Box K_3)$, note that a graph G may contain a set of vertices T that are no help in decycling G, in the sense that $\phi(G-T) = \phi(G)$. For example, if G is a graph with minimum vertex degree 1 and T is the set of leaves in G, then $\phi(G-T) = \phi(G)$. Again, if G' is a graph obtained from G by successive subdivisions of its edges, then we can set $T := V(G') \setminus V(G)$ and $\phi(G') = \phi(G)$.

Lemma 2.1 *Let G be a graph of order $n \geq 2$. Let T be a maximum set of vertices in G such that $\phi(G-T) = \phi(G)$. Then*

$$\phi(G\Box K_3) \geq \max\{n + \phi(G), 3\phi(G) + |T|\}.$$

Proof. Let D be a ϕ-set for $G\Box K_3$. In order to decycle the copy of K_3 at each vertex of G, D contains at least one copy of each vertex of G. Furthermore, for each cycle C in G, D contains at least two copies of some vertex of C. For, suppose otherwise. Then G contains a cycle C such that only one copy of each of its vertices occurs in D. Consider any edge of C, say xy. Then for some i, $1 \leq i \leq 3$, D contains neither x_i nor y_i. But then the edge x_iy_i occurs in the graph $G\Box K_3 - D$. Thus $G\Box K_3 - D$ contains a copy of each edge of C and hence contains a cycle, a contradiction. It follows that each vertex of some decycling set for G occurs twice in D. Hence $|D| \geq |V(G)| + \phi(G)$.

However, D contains a decycling set for G in each copy of G. But no ϕ-set for G contains any vertex of T. Since each vertex of G occurs at least once in D, we also have $|D| \geq 3\phi(G) + |T|$, and the result follows. ■

Although we are mostly concerned in the remainder of this section with graphs which achieve the first of the lower bounds established in Lemma 2.1, the second bound is higher in some cases. For example, suppose G is the graph obtained by appending p leaves to arbitrarily chosen vertices of the complete graph K_m, where $m \geq 5$. Then $|T| = p$, and the second bound gives an improvement of $m - 4$ over the first bound.

Lemma 2.2 *Let G be a graph of order $n \geq 2$ that admits a partition (X, Y) of $V(G)$ such that $\langle X\rangle$ and $\langle Y\rangle$ are acyclic. Let D be a decycling set for G. If*

(i) *$D \cap K$ contains a vertex cover of K for each non-trivial component K of $\langle X\rangle$ or $\langle Y\rangle$; and*

(ii) *for each cycle C in G, there is a component K of $\langle X\rangle$ or $\langle Y\rangle$ such that a maximal path in $\langle V(C) \cap V(K)\rangle$ has an endvertex in D,* *then*

$$\phi(G\Box K_3) \leq n + |D|.$$

Proof. Suppose D satisfies the conditions of the lemma. In $G\Box K_3$, let $S = X_1 \cup Y_2 \cup D_3$ and consider $H := G\Box K_3 - S$. Certainly H_i is acyclic for $i = 1, 2, 3$. Further, there is no edge between H_1 and H_2 as $X \cap Y = \emptyset$. Let $\{u_i\}$ be a component of H_i of order 1. Then u_i is an isolated vertex of H when $u \in D$; otherwise $\deg_H(u_i) = 1$, for $i = 1, 2$, so that u_i is not a vertex of any cycle in H.

Suppose now H contains a cycle C and let u_jv_j be an edge of C in H_j, where $j \in \{1, 2\}$. Then by condition (i) at least one of u or v is in D, say $u \in D$. But then u_j is the only copy of u in H and hence only one copy of the edge uv is in H. It follows that identifying the three copies of v for each vertex $v \in V(C)$ gives a cycle C_0 in G. Then by condition (ii), there is a component K of $\langle X\rangle$ or $\langle Y\rangle$, say $V(K) \subseteq X$, such that a maximal path in $\langle V(C) \cap V(K)\rangle$ has an endvertex $w \in D$. But then w has a neighbour $y \in Y$ on C_0. In H, C contains a copy w_2 of w and y_1 of y. But w_2 is the only copy of w in H and since $y_2 \notin H$, there is no edge in H between w_2 and any copy of y, a contradiction. Thus H is acyclic and S is a decycling set for $G\Box K_3$, giving $\phi(G\Box K_3) \leq |S| = n + |D|$. ■

Several well-known graph families satisfy $\phi(G\Box K_3) = n + \phi(G)$.

Theorem 2.3 *Let G be a bipartite graph of order $n \geq 2$. Then*

$$\phi(G\Box K_3) = n + \phi(G).$$

Proof. Take (X, Y) as the bipartition of $V(G)$ and D as any ϕ-set in G in Lemma 2.2. Then every component of $\langle X\rangle$ and $\langle Y\rangle$ is trivial, and hence the conditions of Lemma 2.2 are satisfied with $|D| = \phi(G)$ giving $\phi(G\Box K_3) \leq n + \phi(G)$. The result then follows from Lemma 2.1. ■

Corollary 2.4 *Let F be a forest of order n. Then $\phi(F\Box K_3) = n$.* ■

The problem of determining $\phi(G)$ when G is bipartite is known to be NP-hard, see [6]. Theorem 2.3 implies the following sharp upper bound on $\phi(G\Box K_3)$ for a connected bipartite graph G, achieved by all complete bipartite graphs.

Corollary 2.5 *Let G be a connected bipartite graph with partite sets of order n_1, n_2, where $n_1 \geq n_2 \geq 1$. Then*

$$\phi(G\Box K_3) \leq n_1 + 2n_2 - 1. \quad ■$$

Lemma 2.6 *Let G be a graph of order $n \geq 2$ with $\Delta(G) = 3$ and D be a ϕ-set for G. Then no component of $\langle D\rangle$ has order greater than 2.*

Proof. Suppose otherwise. Then $\langle D\rangle$ contains a path P of order 3. Let $P := uvw$. Then v has at most one neighbour in $G-u-w$ and hence any cycle through v in G also contains either u or w. Thus $D\setminus\{v\}$ is a decycling set for G of cardinality $\phi(G)-1$, a contradiction. ■

Theorem 2.7 *Let G be a graph of order $n\geq 2$ with $\Delta(G)=3$. Then*

$$\phi(G\Box K_3)=n+\phi(G).$$

Proof. We may assume that G contains at least one odd cycle, since otherwise the result is true by Theorem 2.3. Let D be a ϕ-set for G and D^* be a minimal subset of D such that $H := \langle G-D^*\rangle$ is bipartite. Let I denote the set of isolates in H. Give $V(H\setminus I)$ a proper 2-colouring with colours c_1, c_2. Let $v\in D^*$ be an uncoloured vertex. By Lemma 2.6, v has at most one neighbour in D^* and hence at least two of the neighbours of v have already been coloured. If v has at least two neighbours in the same colour, give v the other colour; otherwise, colour v arbitrarily with c_1 or c_2. When every vertex of D^* has been coloured, colour the vertices of I by the same rule.

Let X, Y be the sets of vertices coloured c_1 and c_2 respectively. Let K be a non-trivial component of $\langle X\rangle$ or $\langle Y\rangle$. Clearly D contains a vertex cover of K. Further, it is easily seen that if K has order 3 or more, then K contains no path of the form xuy or $xuvy$ where $u, v\in D$ and $x, y\in V(G)\setminus D$. In particular, K is acyclic and condition (ii) of Lemma 2.2 is satisfied. Thus (X, Y) and D satisfy all the conditions of that lemma, giving $\phi(G\Box K_3)\leq n+\phi(G)$. The result follows from Lemma 2.1. ■

Since the problem of determining $\phi(G)$ when G is cubic is polynomial, see [9] and [16], the exact value of $\phi(G\Box K_3)$ can be determined from Theorem 2.7 in polynomial time. In the more general case, Alon et al. [1] prove that if G is a connected graph with $\Delta(G)=3$, where $G\neq K_4$, then

$$\phi(G)\leq\lfloor(|E(G)|+1)/4\rfloor.$$

This gives the following corollary to Theorem 2.7.

Corollary 2.8 *Let G be a connected graph of order $n\geq 2$ with $\Delta(G)=3$. If $G\neq K_4$, then*

$$\phi(G\Box K_3)\leq n+\lfloor(|E(G)|+1)/4\rfloor. \ ■$$

Punnim [15] (Lemma 3.4) has shown that if G is a connected triangle-free graph of order n with $\Delta(G)=3$ and such that G is not cubic, then $\phi(G)\leq n/3$. Thus we have the following additional corollary to Theorem 2.7.

Corollary 2.9 *Let G be a connected triangle-free graph of order $n\geq 2$ with $\Delta(G)=3$. If G is not cubic, then*

$$\phi(G\Box K_3)\leq 4n/3.$$

A *cactus* is a simple connected graph with the property that no two cycles have an edge in common.

Theorem 2.10 *Let G be a cactus of order $n \geq 2$. Then*

$$\phi(G \Box K_3) = n + \phi(G).$$

Proof. We may assume that G contains at least one odd cycle, since otherwise the result is true by Theorem 2.3. Let D be a ϕ-set for G and let D^* be a minimum subset of D containing a vertex of each odd cycle. Let $C_1, \ldots, C_k$, $k \geq 1$, be the odd cycles in G. For $i = 1, \ldots, k$, let u_i be a vertex of D^* incident with C_i (note that the vertices $u_1, \ldots, u_k$ are not necessarily distinct). For each cycle C_i make an arbitrary choice of one of the two edges of C_i incident with u_i and label it $e_i \in E(C_i)$. Then $H := G - \{e_1, \ldots, e_k\}$ is bipartite with $c(H) = c(G)$. Let (X, Y) be a bipartition of $V(H)$. Let $xy \in E(G)$. Then x, y are in the same set in this bipartition only if one of them is in D^*. Further, if C is a cycle in G, then $V(C) \cap X$ or $V(C) \cap Y$ contains an isolated vertex of D when C is even; and when C is odd, C intersects some non-trivial component of X or Y in a path of order 2 containing a vertex of D^*. Thus all the conditions of Lemma 2.2 are satisfied and hence $\phi(G \Box K_3) \leq n + \phi(G)$. The result then follows from Lemma 2.1. ■

The maximum number of vertex disjoint cycles in a cactus G of order n is $\lfloor n/3 \rfloor$ and it follows that $\phi(G) \leq \lfloor n/3 \rfloor$. This upper bound is attained by the family of graphs constructed from a cubic tree (that is, a tree in which every internal vertex has degree 3) by replacing each vertex of T with a copy of K_3.

Corollary 2.11 *Let G be a cactus of order $n \geq 2$. Then*

$$\phi(G \Box K_3) \leq \lfloor 4n/3 \rfloor. \quad ■$$

We show in Proposition 3.6 that $\phi(K_n \Box K_3) = 3(n-2)$, for all $n \geq 4$, and hence the lower bound $3\phi(G)$ of Lemma 2.1 is also sharp.

It is easily seen that K_4 has a vertex partition and ϕ-set D satisfying the conditions of Lemma 2.2. However, in any partition (X, Y) of the vertex set of K_5, one of $\langle X \rangle$ or $\langle Y \rangle$ contains a 3-cycle and hence this lemma cannot be applied to any graph with $\omega(G) \geq 5$. We next establish a sharp upper bound on the value of $\phi(G \Box K_3)$ applicable to any graph.

Lemma 2.12 *Let G be a connected graph of order $n \geq 2$. Then*

$$\phi(G \Box K_3) \leq n + 2\phi(G).$$

Proof. Let D be a ϕ-set in G and (X, Y) be the bipartition of $V(G_0)$, where $G_0 := G - D$. Then $X \cup D$, $Y \cup D$ are both vertex covers of G. Hence $S := X_1 \cup D_1 \cup Y_2 \cup D_2 \cup D_3$ is a decycling set for G of cardinality $|V(G)| + 2|D|$ and the result follows. ■

The upper bound on $\phi(G \Box K_3)$ of Lemma 2.12 is sharp: it is attained, for example, by the family of graphs formed by the join of a path of order $r \geq 4$ and an isolated vertex.

3 Decycling $G\Box K_r$, for $r \geq 4$

Lemma 3.1 *Let G be a graph of order $n \geq 2$. Then for $r \geq 4$,*

$$\phi(G\Box K_r) \geq \max\{n(r-2), r\phi(G)\}.$$

Proof. To decycle the copy of K_r at each vertex of G requires at least $n(r-2)$ vertices. But to decycle the copy of G at each vertex of K_r requires at least $r\phi(G)$ vertices, and the result follows. ■

We use the following notation. For any $r \geq 3$, we call a ϕ-set in $(G\Box K_r)$ a *ϕ_r-set.*

Lemma 3.2 *Let r, s be integers with $r > s \geq 3$ and G be a graph of order n. Then*

$$\phi(G\Box K_r) \leq \phi(G\Box K_s) + n(r-s).$$

Proof. Let D be a ϕ_s-set for $G\Box K_s$ and let $D_i := G_i \cap D$, for $1 \leq i \leq s$. Now construct a decycling set Q in $G\Box K_r$ by setting $Q_i := D_i$ when $1 \leq i \leq s$, and $Q_i := V(G)$ when $s+1 \leq i \leq r$. ■

Lemma 3.3 *Let G be a graph of order $n \geq 2$ that admits a partition (X, Y) of $V(G)$ such that $\langle X\rangle$ and $\langle Y\rangle$ are acyclic. Let D be an independent decycling set for G. If*

(i) *$D \cap K$ contains a vertex cover of K for each non-trivial component K of $\langle X\rangle$ or $\langle Y\rangle$; and*

(ii) *for each cycle C in G, there is a component K of $\langle X\rangle$ or $\langle Y\rangle$ such that a maximal path in $\langle V(C) \cap V(K)\rangle$ has an endvertex in D;*

then for all $r \geq 4$,

$$\phi(G\Box K_r) = n(r-2).$$

Proof. Suppose D satisfies the conditions of the lemma. Let $W := V(G)\setminus D$ and note that since D is independent, W is also a decycling set for G. Define a set Q in $G\Box K_4$ by $Q_1 := X_1$, $Q_2 = Y_2$, $Q_3 = D_3$, $Q_4 = W_4$. For $1 \leq i \leq 4$, let $S_i = V(G_i)\setminus Q_i$. Then since $D^* := Q_1 \cup Q_2 \cup Q_3$ and (X, Y) satisfy the conditions of Lemma 2.2 in $G\Box K_3$, $\langle S_1 \cup S_2 \cup S_3\rangle$ is a forest F. However, $S_4 = D_4$ is a set of independent vertices, each of which has degree 1 in $G\Box K_4 \setminus Q$ and hence $\langle S_1 \cup S_2 \cup S_3 \cup S_4\rangle$ is a forest in $G\Box K_4$. Thus Q is a decycling set in $G\Box K_4$ of cardinality $2n$, giving $\phi(G\Box K_4) = 2n$, by Lemma 3.1. The result then follows from Lemma 3.2 with $s = 4$ and Lemma 3.1. ■

Theorem 3.4 *Let G be a bipartite graph of order $n \geq 2$. Then for all $r \geq 4$,*

$$\phi(G\Box K_r) = n(r-2).$$

Proof. This result follows from Lemma 3.3 by taking (X,Y) as a bipartition of $V(G)$ and $D := Y$. ■

Theorem 3.5 *Let G be a cactus of order $n \geq 2$. Then for all $r \geq 4$,*

$$\phi(G\Box K_r) = n(r-2).$$

Proof. We may assume that G contains at least one odd cycle, since otherwise the result is true by Theorem 3.4. Choose a decycling set D for G with the property that each odd cycle contains just one vertex of D and no two vertices of D are adjacent (the set D will not necessarily be a ϕ-set for G). Let $C_1, \ldots, C_k$, $k \geq 1$, be the odd cycles in G. For $i = 1, \ldots, k$, let u_i be the unique vertex of D incident with C_i. Make an arbitrary choice of one of the two edges of C_i incident with u_i and label it e_i. Then $H := G - \{e_1, \ldots, e_k\}$ is bipartite with $c(H) = c(G)$. Let (X,Y) be a bipartition of $V(H)$ and let K be a non-trivial component of $\langle X\rangle$ or $\langle Y\rangle$. Then D contains a vertex cover of K. Further, for each cycle C in G, there is a component K of $\langle V(C)\cap X\rangle$ or $\langle V(C)\cap Y\rangle$ which has order 2 and one of its endvertices in D when C is odd; and is a single vertex of D when C is even. Let $V(G)\setminus D := W$. Then W is also a decycling set for G and hence (X,Y) and D satisfy the conditions of Lemma 3.3. The result follows. ■

Proposition 3.6 *Let n, r be integers with $n \geq r \geq 3$. Then*

$$\phi(K_r\Box K_n) = \begin{cases} r(n-2) & \text{when } n > r \\ (r-1)^2 & \text{when } n = r \end{cases}.$$

Proof. By Lemma 3.1, $\phi(K_r\Box K_n) \geq r(n-2)$. When $n > r$, let $u^1, u^2, \ldots, u^{r+1}$ be distinct vertices of $G \cong K_n$ and set $S := \{u_1^1, u_1^2, u_2^2, u_2^3, u_3^3, u_3^4, \ldots, u_r^{r+1}\}$. Clearly $\langle S\rangle$ is a path. Thus $D := V(K_n\Box K_r)\setminus S$ is a decycling set for $K_n\Box K_r$ and $\phi(K_r\Box K_n) \leq |D| = nr - 2r$ and the result follows.

Now suppose $n = r$ and let $S \subseteq V(K_r\Box K_r)$ be a maximum set such that $\langle S\rangle$ is acyclic. Since S contains at most two copies of each vertex of K_r, $|S| \leq 2r$. However, S_i contains copies of no more than two distinct vertices for $1 \leq i \leq r$, so that if $|S| = 2r$, then $\langle S\rangle$ is 2-regular, a contradiction. Hence $|S| \leq 2r - 1$. Let $V(K_r) := \{u^1, u^2, \ldots, u^r\}$. Setting $S := \{u_1^1, u_1^2, u_2^2, u_2^3, u_3^3, u_3^4, \ldots, u_r^r\}$, $\langle S\rangle$ is a path. Hence $|S| = 2r - 1$. Then $D := V(K_r\Box K_r)\setminus S$ is a ϕ-set for $K_r\Box K_r$ of cardinality $r^2 - 2r + 1$. ■

Lemma 3.7 *Let $G \neq K_4$ be a connected cubic graph. Then G contains an independent decycling set.*

Proof. Partition $V(G)$ into two sets (D_0, X_0) so that D_0 is an independent set and among all such choices of D_0, $\langle X_0\rangle$ contains as few cycles as possible. Without loss of generality, we may assume that D_0 is maximal independent. If $\langle X_0\rangle$ is acyclic, then put $D := D_0$ and D is an independent decycling set for G.

Assume therefore that $\langle X_0 \rangle$ contains an r-cycle C_0. Each vertex of C_0 has two neighbours in X_0 and hence its neighbour in $V(G) \setminus V(C_0)$ is in D_0. Let $y \in V(C_0)$ and u be the neighbour of y in D_0 and $N_G(u) := \{y, v_1, v_2\}$. If G contains no v_1v_2-path P_1 such that $V(P_1) \subseteq X_0$, put $D := (D_0 \cup \{y\}) \setminus \{u\}$ and $X := V(G) \setminus D$. Then D is independent and $\langle X \rangle$ contains one less cycle than $\langle X_0 \rangle$, contradicting the choice of the partition (D_0, X_0). Suppose then that there is a v_1v_2-path P_1 such that $V(P_1) \subseteq X_0$. Then either $\{v_1, v_2\} \subset V(C_0) \setminus \{y\}$, or $\{v_1, v_2\} \cap V(C_0) = \emptyset$.

Suppose $\{v_1, v_2\} \subset V(C_0) \setminus \{y\}$. Note that $r > 3$, since otherwise $G \cong K_4$, contrary to hypothesis. Thus $\{y, v_1, v_2\}$ contains at least one pair of non-adjacent vertices. Denote such a pair by $\{x', x''\}$ and put $D := (D_0 \cup \{x', x''\}) \setminus \{u\}$, $X := V(G) \setminus D$. Then D is an independent set and $\langle X \rangle$ contains one less cycle than $\langle X_0 \rangle$, again contradicting the choice of (D_0, X_0). We may therefore assume that $\{v_1, v_2\} \cap V(C_0) = \emptyset$ and hence $V(P_1) \cap V(C_0) = \emptyset$. Thus each vertex of C_0 has a distinct neighbour in D_0. Let C_1 denote the cycle uP_1. Put $D_1 := (D_0 \cup \{y\}) \setminus \{u\}$ and $X_1 := V(G) \setminus D_1$. Now C_1 is a cycle in $\langle X_1 \rangle$ and repeating the arguments above with C_1, D_1 in place of C_0, D_0, we may conclude that each vertex of C_1 has a distinct neighbour in D_1 and hence each vertex of $V(C_1) \setminus \{u\}$ has a distinct neighbour in $D_0 \setminus \{u\}$. Now suppose there is a vertex $w \in D_0$ such that w has neighbours $z \in V(C_1)$ and $z' \in V(C_0)$. But then putting $D := (D_0 \cup \{z, z'\}) \setminus \{u\}$ and $X := V(G) \setminus D$, D is independent and $\langle X \rangle$ contains one less cycle than $\langle X_0 \rangle$, contradicting the choice of the partition (D_0, X_0). Hence each vertex of $V(C_0) \cup V(C_1) \setminus \{u\}$ has a distinct neighbour in $D_0 \setminus \{u\}$.

Put $y := y_0$, $u := u_1$. Suppose we have extended the sequence C_0, C_1 to a sequence $C_0, C_1, \ldots, C_k$ of distinct cycles, where $k \geq 1$, with the following properties:

1. for $1 \leq i \leq k$, C_i contains a unique vertex $u_i \in D_0$;
2. for $0 \leq i \leq k-1$, C_i contains a vertex y_i adjacent to u_{i+1};
3. each vertex of $V(C_0) \cup V(C_1) \cdots \cup V(C_k) \setminus \{u_1, \ldots, u_k\}$ has a distinct neighbour in $D_0 \setminus \{u_1, \ldots, u_k\}$.

Now let $y_k \in V(C_k) \setminus \{u_k\}$ and u_{k+1} be the vertex of D_0 adjacent to y_k. Let $N_G(u_{k+1}) = \{y_k, w_1, w_2\}$. Assume there is a w_1w_2-path P_{k+1} with $V(P_{k+1}) \subseteq X_0$. Since $u_{k+1} \in D_0$, either $\{w_1, w_2\} \subset V(C_i) \setminus \{u_i\}$ for some $i \in \{0, \ldots, k\}$, or $\{w_1, w_2\} \cap V(C_i) = \emptyset$ for $0 \leq i \leq k$. However, the first alternative implies that two vertices of $V(C_i) \setminus \{u_i\}$ have the same neighbour in $D_0 \setminus \{u_i\}$, contrary to property 3. Hence $\{w_1, w_2\} \cap V(C_0) = \emptyset$ and G contains a cycle $C_{k+1} := u_{k+1}P_{k+1}$ distinct from C_i, for $i = 0, 1, \ldots, k$. Now let $D_1 := (D_0 \cup \{y_0, \ldots, y_{k-1}\}) \setminus \{u_1, \ldots u_k\}$ and $X_1 := V(G) \setminus D_1$. Then C_k is a cycle in $\langle X_1 \rangle$. Repeating the arguments above for C_k,C_{k+1},D_1 in place of C_0,C_1,D_0, we conclude that each vertex of C_{k+1} has a distinct neighbour in D_1 and hence each vertex of $V(C_{k+1}) \setminus \{u_{k+1}\}$ has a distinct neighbour in $D_0 \setminus \{u_1, \ldots, u_k\}$.

Now suppose that there is a vertex $s \in D_0 \setminus \{u_1, \ldots, u_k\}$ having neighbours t, t', where $t \in V(C_{k+1})$ and $t' \in V(C_i)$ for some $i \in \{0, \ldots, k\}$. But putting $D := (D_0 \cup \{t, t'\}) \setminus \{u_i, u_{k+1}\}$ when $i \geq 1$ and $D := (D_0 \cup \{t, t'\}) \setminus \{u_{k+1}\}$ when $i = 0$, and putting $X := V(G) \setminus D$, we obtain an independent set D such that $\langle X \rangle$ contains one less cycle than $\langle X_0 \rangle$, contradicting the choice of the partition (D_0, X_0), so that this does not occur. Thus the cycle C_{k+1} has properties 1 to 3 and the sequence of distinct cycles $C_0, \ldots, C_k$ can be extended to $C_0, \ldots, C_k, C_{k+1}$. Since the sequence exists for $k = 1$, it can be extended indefinitely, by induction. But this is impossible, since G is finite. Hence G contains an independent decycling set. ■

Lemma 3.8 *Let $G \neq K_4$ be a connected graph with $\Delta(G) = 3$. Then G contains an independent decycling set.*

Proof. By Lemma 3.7, the result is true when G is cubic, so we may also assume that G contains at least one vertex of degree less than 3. Let H be the graph constructed from K_4 by subdividing one of its edges. If G has a vertex x of degree 2, construct a new graph from G by joining x to the vertex of degree 2 in a copy of H. Repeat this procedure, joining each vertex of degree 2 in G to the vertex of degree 2 in a distinct copy of H. If G contains a leaf y, join y to the vertex of degree 2 in each of two distinct copies of H. Repeat this procedure for each leaf of G. The resulting graph is cubic and hence contains an independent decycling set S by Lemma 3.7. But then $D := S \cap V(G)$ is an independent decycling set for G, proving the result. ■

Theorem 3.9 *Let $G \neq K_4$ be a connected graph with $\Delta(G) = 3$. Then for all $r \geq 4$,*

$$\phi(G \square K_r) = n(r-2).$$

Proof. By Lemma 3.8, G admits an independent decycling set D. Let D^* be a minimal subset of D such that $H := \langle G - D \rangle^*$ is bipartite. Obtain a partition (X, Y) of $V(G)$ as in Theorem 2.7. Then the conditions of Lemma 3.3 are satisfied and the result follows. ■

References

[1] N. Alon, D. Mubayi and R. Thomas, Large induced forests in sparse graphs, *J. Graph Theory* **38** (2001), 113–123.

[2] Sheng Bau and Lowell W. Beineke, The decycling number of graphs, *Australas. J. Combin.* **25** (2002), 285–298.

[3] Sheng Bau, Lowell W. Beineke and Robert C. Vandell, Decycling snakes, *Congressus Numer.* **134** (1998), 79–87.

[4] Sheng Bau, Nicholas C. Wormald and Sanming Zhou, Decycling numbers of random regular graphs, *Random Structures and Algorithms* **21** (2002), 397–413.

[5] Lowell W. Beineke and Robert C. Vandell, Decycling graphs, *J. Graph Theory* **25** (1997), 59–77.

[6] R.M. Karp, *Reducibility among combinatorial problems, Complexity of Computer Computations*, (eds. R.E. Miller and J.W. Thatcher), Plenum Press, New York, London (1972), 85–103.

[7] Y.D. Liang, On the feedback vertex set problem in permutation graphs, *Inf. Process Lett.* **52** (1994), 123–129.

[8] Y.D. Liang and M.-S. Chang, Minimum feedback vertex sets in cocomparability graphs and convex bipartite graphs, *Acta Inf.* **34** (1997), 337–346.

[9] D.-M. Li and Y.-P. Liu, A polynomial algorithm for finding the mimimum feedback vertex set of a 3-regular simple graph, *Acta Math. Sci.* **19** no. 4 (1999), 375–381.

[10] J.-P. Liu and C. Zhao, A new bound on the feedback vertex sets in cubic graphs, *Discrete Math.* **148** (1996), 119–131.

[11] David A. Pike, Decycling hypercubes, *Graphs Combin.* **19** no. 4 (2003), 547–550.

[12] David A. Pike and Yubo Zou, Decycling cartesian product of two cycles, *SIAM J. Discrete Math.* **19** (2005) no. 3, 551–663.

[13] Narong Punnim, The decycling number of cubic graphs, *Combinatorial Geometry and Graph Theory*, pp. 141–145, Lec. Notes Comp. Sci. **3330**, Springer, Berlin, 2005.

[14] Narong Punnim, Decycling regular graphs, *Australas. J. Combin.* **32** (2005), 147–162.

[15] Narong Punnim, Decycling connected regular graphs, *Australas. J. Combin.* **35** (2006), 155–169.

[16] S. Ueno, Y. Kajitani and S. Gotoh, On the non-separating independent set problem and feedback set problem for graphs with no vertex degree exceeding three, *Discrete Math.* **72** (1988), 355–360.

[17] R.J. Wilson, *Introduction to graph theory* (4th ed.), Addison-Wesley/Longman, Reading, MA (1996).

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