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A note on independent domination in graphs of girth 5

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Abstract

Let G be a simple graph of order n, maximum degree Δ and minimum degree $\delta \geq 2$. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of vertices of G. The *girth* $g(G)$ is the minimum length of a cycle in G. We establish sharp upper and lower bounds, as functions of n, Δ and δ, for the independent domination number of graphs G with $g(G) = 5$.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$, maximum degree Δ and minimum degree $\delta \geq 2$. An *independent set* is a set of pairwise non-adjacent vertices of G. A subset I of V is a *dominating set* if every vertex of $V - I$ has at least one neighbour in I. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of G. An independent set is maximal if and only if it is dominating, so $i(G)$ is also the minimum cardinality of an independent dominating set in G.

A number of previous papers on the parameter $i(G)$ have been focussed upon finding upper bounds, as functions of n and/or δ, for general and regular graphs [2–4, 6, 7, 9, 10]. In [5], the present author proved analogous results for triangle-free graphs. If the *girth* $g(G)$ is the minimum length of a cycle in G, clearly triangle-free graphs satisfy $g(G) \geq 4$ and indeed, all graphs from [5] which are either extremal or conjectured to be extremal have girth exactly 4. We explore a natural extension of this work here, in providing best possible upper and lower bounds, as functions of n, Δ and δ, for the independent domination number of graphs with girth 5. Moreover,

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as we shall demonstrate, it is surprisingly straightforward to find minimum maximal independent sets in the corresponding extremal graphs.

In what follows, we abbreviate $i(G)$ to i and $g(G)$ to g where it is unambiguous. The *open neighbourhood* in G of a vertex $v \in V$ will be denoted by $\Gamma(v) = \{u \in V : uv \in E\}$, and that of a set of vertices $X \subset V$ by $\Gamma(X) = \bigcup_{x \in X} \Gamma(x) \cap (V - X)$. For disjoint vertex sets $X, Y \subset V$, write $e(X, Y)$ for the number of edges with one endvertex in each of X and Y.

2 Results

As shown in [5], it is simple to prove that $\delta \leq n/2$ for triangle-free graphs, for otherwise $|E| > n^2/4 = \left|e\left(K_{n/2,n/2}\right)\right|$ and so $G \supset K_3$ by Turán's Theorem. In order to determine the corresponding range of δ for graphs with girth 5, we require the following well-known result of Tutte [11], which also yields the upper bound above for the case $g = 4$.

Proposition 2.1 *(*Tutte *[11]) Any graph of order n, minimum degree $\delta \geq 3$ and girth $g \geq 3$ satisfies*

$$n \geq \begin{cases} 1 + \frac{\delta}{\delta-2}\left[(\delta-1)^{(g-1)/2} - 1\right] & \text{if } g \text{ is odd;} \\ \frac{2}{\delta-2}\left[(\delta-1)^{g/2} - 1\right] & \text{if } g \text{ is even.} \end{cases}$$

Corollary 2.2 *Any graph of order n, minimum degree $\delta \geq 2$ and girth $g = 5$ satisfies*

$$\delta \leq \begin{cases} (n-1)/2 & \text{if } \delta = 2; \\ (n-1)^{1/2} & \text{if } \delta \geq 3. \end{cases}$$

Proof. If $\delta = 2$ then the proof of Proposition 2.1 (see for example [1], pp.68–9) is easily adapted for this case, as follows. Suppose $g = 5$. Choosing a vertex x, there is no vertex x' for which G contains two distinct x-x' paths of length at most 2, since otherwise G has a cycle of length at most 4. Hence there are at least $\delta = 2$ vertices at distance j from x for $j = 1, 2$. Therefore $n \geq 1 + 2\delta$, as claimed.

Otherwise $\delta \geq 3$, so taking $g = 5$ in the odd case of Proposition 2.1, we obtain $n \geq 1 + \delta^2$, which yields the required upper bound for δ. □

Note that if $\delta = 2$ and $n = g = 5$ then $\delta = (n-1)/2 = (n-1)^{1/2}$. The proofs of Proposition 2.1 and Corollary 2.2 imply that any extremal graph is regular of degree δ; such graphs are called Moore graphs of degree $\delta = (n-1)^{1/2}$ and girth 5. When $\delta = 2$ and $g = 5$, the only Moore graph is the pentagon C_5. In addition, Hoffman and Singleton [8] proved that if there exists a Moore graph of degree $\delta \geq 3$ and girth 5 then $\delta = 3, 7$ or 57. For $\delta = 3$ and 7, the unique graphs of this type are the Petersen graph and the Hoffman-Singleton graph respectively, with the latter constructed as follows. For $j, k, \ell = 0, \ldots, 4$, take five pentagons P_k and five pentagrams Q_ℓ so that vertex j of P_k is adjacent to vertices $j-1, j+1$ of P_k, and vertex j of Q_ℓ is adjacent to vertices $j-2, j+2$ of Q_ℓ; finally, join vertex j of P_k to vertex $k\ell + j$ of Q_ℓ (all indices modulo 5). It is not known whether a Moore graph of degree 57 and girth 5 exists.

We now give sharp lower and upper bounds for the independent domination number of graphs of girth 5. Observe that all such graphs have the following two properties:

(†) G is triangle-free, so no pair of adjacent vertices have a common neighbour.

(‡) G is C_4-free, so every pair of non-adjacent vertices have at most one common neighbour.

We exploit these two neighbourhood conditions in our proofs.

Theorem 2.3 *Any graph of order n, minimum degree $\delta \geq 2$ and girth 5 satisfies $i \geq \delta$.*

Proof. Let I be a minimum maximal independent set. Suppose $i < \delta$. Choose any $x \in V - I$ and let $W = \Gamma(x) \cap I$ and $Z = \Gamma(x) \cap (V - I)$. If $W = I$ then by (†), x must satisfy $Z = \emptyset$ and hence $|\Gamma(x)| = |W| = i < \delta$, a contradiction. We conclude that $I - W \neq \emptyset$ and therefore $Z \neq \emptyset$.

By (†), no vertex of Z can have a common neighbour with x, so $\Gamma(Z) \cap I \subseteq I - W$. In addition, by the maximality of I, all vertices of Z have at least one neighbour in $I - W$. Clearly Z is an independent set, for any pair of adjacent vertices of Z would have x as a common neighbour and thus contradict (†). Consequently by (‡) no pair of vertices of Z can have a common neighbour $x' \neq x$, so we deduce that $|I - W| \geq |\Gamma(Z) \cap I| = e(Z, I - W) \geq |Z|$. Hence

$$i = |W| + |I - W| \geq |W| + |Z| \geq \delta;$$

this contradiction completes the proof. □

Theorem 2.4 *Any graph of order n, maximum degree Δ, minimum degree $\delta \geq 2$ and girth 5 satisfies*

$$i \leq n - \Delta\delta + \Delta - 1,$$

where $\Delta \leq (n-1)/\delta$.

Proof. Choose any vertex x of degree Δ. As G is triangle-free, then $V^* = \Gamma(x)$ is an independent set (though not necessarily maximal), for otherwise any pair of adjacent vertices therein would have x as a common neighbour and thus contradict (†). Now all vertices of V^* are adjacent to x and hence by (‡) have disjoint neighbourhoods in $V - V^* - \{x\}$. Therefore

$$\begin{aligned} i &\leq |V| - |\Gamma(V^*)| \\ &= n - |\{x\}| - |\Gamma(V^*) \cap (V - V^* - \{x\})| \\ &\leq n - 1 - \Delta(\delta - 1), \end{aligned}$$

as claimed. To verify the upper bound for Δ, it is easily seen that

$$\Delta = |V^*| \leq |V| - |\Gamma(V^*)| \leq n - \Delta\delta + \Delta - 1,$$

which rearranges to give the required result. □

From Theorems 2.3 and 2.4, we have $\delta \le i \le n - \Delta\delta + \Delta - 1$. When $\delta = \Delta = (n-1)^{1/2}$, this chain of inequalities becomes an equality, so Theorems 2.3 and 2.4 are best possible, with their respective bounds attained by the Moore graphs of girth 5. Furthermore, our final result reveals these extremal graphs to have an interesting property.

Theorem 2.5 *The Moore graphs of girth* 5 *have independent domination number* δ *and the open neighbourhood of every vertex is a minimum maximal independent set.*

Proof. Let G be a Moore graph of girth 5. Arguing similarly to the proof of Theorem 2.4, the open neighbourhood of every vertex is an independent set of order $\delta = (n-1)^{1/2}$ with $1 + \delta(\delta - 1) = n - \delta$ neighbours, and hence is maximal. By Theorem 2.3, it must be a minimum maximal independent set. □

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