AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **40** (2008), Pages 293–300

A matrix approach to construct magic rectangles of even order

J. P. De Los Reyes

The University of Akron
Department of Statistics
Akron, OH 44325
U.S.A.

Ashish Das

Indian Institute of Technology Bombay
Department of Mathematics
Mumbai 400076
India

Chand K. Midha

The University of Akron
Department of Statistics
Akron, OH 44325
U.S.A.

Abstract

Magic rectangles are well-known for their very interesting and entertaining combinatorics. In a magic rectangle, the numbers 1 to mn are arranged in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N. In the present paper we provide a new systematic method for constructing any even by even magic rectangle. The method proposed is extremely simple as it allows one to arrive at the magic rectangles by simply carrying out some matrix operations. It is also seen that the magic rectangles of lower orders are embedded in a magic rectangle of higher order.

1 Introduction

Magic rectangles are well-known for their very interesting and entertaining combinatorics. A magic rectangle is an arrangement of the numbers 1 to mn in an array of

m rows and n columns so that each row adds to the same total M and each column to the same total N. The totals M and N are termed the magic constants. Since the average value of the numbers is $A = (mn + 1)/2$, we must have $M = nA$ and $N = mA$. The total of all the numbers in the array is $mnA = mM = nN$. If mn is even $mn+1$ is odd and so for $M = n(mn+1)/2$ and $N = m(mn+1)/2$ to be whole numbers n and m must both be even. On the other hand if mn is odd then m and n must both be odd, by simple arithmetic. Therefore, an odd by even magic rectangle is impossible. Also, it is easy to see that a 2×2 magic rectangle is impossible. [2] re-established that, for $m > 1$, $n > 1$, an m by n magic rectangle exists only if one of the following conditions is hold: (a) Both m and n are even and at least one of them is greater than 2; (b) Both m and n are odd.

For an update on available literature on magic rectangles we refer to [1] and [3]. Such magic rectangles have been used in designing experiments. For example, [4], [5] and [6] illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. As highly balanced structures, magic rectangles can be potential tools for use in situations yet unexplored.

In the present paper we provide a method for constructing any even by even magic rectangle. The construction involves some simple matrix operations. The method has been shaped in form of an algorithm that is very convenient for writing a computer program for constructing such rectangles. Furthermore, it is also presented in a ready-to-write form since the magic rectangles of lower orders are embedded in a magic rectangle of higher order. Algorithm to construct odd by odd magic rectangle is more involved and would be discussed in a separate communication.

In Section 2 we construct magic rectangle of sides m and n with $2p = m \leq n = 2q$. The proofs related to the construction are given in the appendix. In Section 3 we illustrate our construction method through some examples of magic rectangles.

2 The construction

For any given positive integers p and q with $p \leq q$, we first define the following matrices:

$$Q_e = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}, \; Q_a = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}, \; Q_b = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{pmatrix}.$$

We also define A to be a $p \times q$ matrix with elements being the sequence of numbers from 0 to $pq - 1$ as given by

$$A = \begin{pmatrix} 0 & 1 & 2 & \cdots & q-1 \\ q & q+1 & q+2 & \cdots & 2q-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (p-1)q & (p-1)q+1 & (p-1)q+2 & \cdots & pq-1 \end{pmatrix}.$$

The matrix A can also be expressed as $A = 1_p \otimes s_q' + 1_q' \otimes q s_p$ where $\otimes$ denotes the Kronecker product symbol, 1_t represents a $t \times 1$ column vector of all ones and

$s'_t = (0\ 1\ 2\ \cdots\ t-1)$ is a row vector of order t with elements being the sequence of numbers from 0 to $t-1$.

Let I_t be an identity matrix of order t and K_t a square matrix of order t given by, $K_t = ((k_{ij}))$ with $k_{ij} = 1$ if $i + j = t + 1$, otherwise $k_{ij} = 0$. Then, let $X = A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (AK_q) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We define next, two matrices B and C such that their sum is the required magic rectangle.

Let B be a $2p \times 2q$ matrix given by

$$B = \begin{cases} 1_p \otimes \left(\begin{array}{c|c|c|c} Q_e & Q_a & 1'_{\frac{q-3}{2}} \otimes (Q_b \mid Q_a) & Q_e \end{array} \right) & \text{for } q \text{ odd,} \\ (1_p 1'_{\frac{q}{2}}) \otimes Q & \text{for } q \text{ even.} \end{cases}$$

Note that, when $q = 3$, B reduces to $1_p \otimes \left(\begin{array}{c|c|c} Q_e & Q_a & Q_e \end{array} \right)$. It is easy to verify that $B1_{2q} = 3q1_{2p}$ and $1'_{2p}B = 3p1'_{2q}$, i.e., the row sums of B are $3q$ and the column sums are equal to $3p$.

Finally, let C be a $2p \times 2q$ matrix given by

$$C = \left(\begin{array}{c|c} Y_1 & (K_p \otimes I_2)Y_2 \end{array} \right),$$

where $Y = 4(X \otimes 1'_2) + 1_{2p}1'_{2q} = \left(\begin{array}{c|c} Y_1 & Y_2 \end{array} \right)$ and for $j = 1, 2$, the matrix Y_j is of order $2p \times q$.

Again, it is easy to verify that the row sums of C are $2q(2pq - 1)$ and column sums are equal to $2p(2pq - 1)$.

To conclude, since $R = B + C$ consists of the $4pq$ distinct numbers from 1 through $4pq$, it is a $2p \times 2q$ magic rectangle with magic constants $M = q(4pq + 1)$ and $N = p(4pq + 1)$.

The proofs related to the construction is given in the appendix.

3 Remarks and some illustrative examples

In this section we first indicate some of our observations. This is followed by examples of magic rectangles of orders 6×8, 8×8, 6×10, 8×10 and 10×10. We observe the following points.

(i) Any magic rectangle of order $2p^* \times 2q$ is embedded in a magic rectangle of order $2p \times 2q$ where $p^* < p$. Let $R = (R_1 \mid R_2)$ be a magic rectangle of order $2p \times 2q$ where for $j = 1, 2$, the matrix R_j is of order $2p \times q$. Then in order to get a magic rectangle R^* of order $2p^* \times 2q$, one simply needs to take the first $2p^*$ rows of R_1 (call the resultant matrix of the rows, R_1^*) and take the last $2p^*$ rows of R_2 (call the resultant matrix of the rows, R_2^*). Then $R^* = (R_1^* \mid R_2^*)$. In other words, $R^* = \{(I_{2p^*} \mid 0_{2p^*,2(p-p^*)})R_1 \mid (0_{2p^*,2(p-p^*)} \mid I_{2p^*})R_2\}$ where $0_{a,b}$ denotes a $a \times b$ matrix of all zeros.

(ii) The method proposed here allows us to arrive at the magic rectangles by simply carrying out some matrix operations. This makes it very convenient to write a program for generating magic rectangles using standard packages like SAS, MATLAB or MAPLE.

(iii) Though we have, without any loss of generality, taken $1 \le p \le q$, the construction method proposed here also holds for $p > q > 1$.

We now provide some examples.

Magic rectangle of order 6×8. Here $p = 3$, $q = 4$. Therefore,

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 8 & 9 & 10 & 11 \\ 11 & 10 & 9 & 8 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 5 & 5 & 9 & 9 & 13 & 13 \\ 13 & 13 & 9 & 9 & 5 & 5 & 1 & 1 \\ 17 & 17 & 21 & 21 & 25 & 25 & 29 & 29 \\ 29 & 29 & 25 & 25 & 21 & 21 & 17 & 17 \\ 33 & 33 & 37 & 37 & 41 & 41 & 45 & 45 \\ 45 & 45 & 41 & 41 & 37 & 37 & 33 & 33 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 & 5 & 5 & 41 & 41 & 45 & 45 \\ 13 & 13 & 9 & 9 & 37 & 37 & 33 & 33 \\ 17 & 17 & 21 & 21 & 25 & 25 & 29 & 29 \\ 29 & 29 & 25 & 25 & 21 & 21 & 17 & 17 \\ 33 & 33 & 37 & 37 & 9 & 9 & 13 & 13 \\ 45 & 45 & 41 & 41 & 5 & 5 & 1 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 0 & 2 & 1 & 3 & 0 & 2 & 1 \end{pmatrix}, \ R = \begin{pmatrix} 1 & 4 & 6 & 7 & 41 & 44 & 46 & 47 \\ 16 & 13 & 11 & 10 & 40 & 37 & 35 & 34 \\ 17 & 20 & 22 & 23 & 25 & 28 & 30 & 31 \\ 32 & 29 & 27 & 26 & 24 & 21 & 19 & 18 \\ 33 & 36 & 38 & 39 & 9 & 12 & 14 & 15 \\ 48 & 45 & 43 & 42 & 8 & 5 & 3 & 2 \end{pmatrix}.$$

Magic rectangle of order 8×8. Here $p = 4$, $q = 4$. Therefore,

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 8 & 9 & 10 & 11 \\ 11 & 10 & 9 & 8 \\ 12 & 13 & 14 & 15 \\ 15 & 14 & 13 & 12 \end{pmatrix},$$

$$Y=\begin{pmatrix} 1&1&5&5&9&9&13&13\\ 13&13&9&9&5&5&1&1\\ 17&17&21&21&25&25&29&29\\ 29&29&25&25&21&21&17&17\\ 33&33&37&37&41&41&45&45\\ 45&45&41&41&37&37&33&33\\ 49&49&53&53&57&57&61&61\\ 61&61&57&57&53&53&49&49 \end{pmatrix}, C=\begin{pmatrix} 1&1&5&5&57&57&61&61\\ 13&13&9&9&53&53&49&49\\ 17&17&21&21&41&41&45&45\\ 29&29&25&25&37&37&33&33\\ 33&33&37&37&25&25&29&29\\ 45&45&41&41&21&21&17&17\\ 49&49&53&53&9&9&13&13\\ 61&61&57&57&5&5&1&1 \end{pmatrix},$$

$$B=\begin{pmatrix} 0&3&1&2&0&3&1&2\\ 3&0&2&1&3&0&2&1\\ 0&3&1&2&0&3&1&2\\ 3&0&2&1&3&0&2&1\\ 0&3&1&2&0&3&1&2\\ 3&0&2&1&3&0&2&1\\ 0&3&1&2&0&3&1&2\\ 3&0&2&1&3&0&2&1 \end{pmatrix}, R=\begin{pmatrix} 1&4&6&7&57&60&62&63\\ 16&13&11&10&56&53&51&50\\ 17&20&22&23&41&44&46&47\\ 32&29&27&26&40&37&35&34\\ 33&36&38&39&25&28&30&31\\ 48&45&43&42&24&21&19&18\\ 49&52&54&55&9&12&14&15\\ 64&61&59&58&8&5&3&2 \end{pmatrix}.$$

Magic rectangle of order 10×10. Here $p = 5$, $q = 5$. Therefore,

$$A=\begin{pmatrix} 0&1&2&3&4\\ 5&6&7&8&9\\ 10&11&12&13&14\\ 15&16&17&18&19\\ 20&21&22&23&24 \end{pmatrix}, X=\begin{pmatrix} 0&1&2&3&4\\ 4&3&2&1&0\\ 5&6&7&8&9\\ 9&8&7&6&5\\ 10&11&12&13&14\\ 14&13&12&11&10\\ 15&16&17&18&19\\ 19&18&17&16&15\\ 20&21&22&23&24\\ 24&23&22&21&20 \end{pmatrix},$$

$$Y=\begin{pmatrix} 1&1&5&5&9&9&13&13&17&17\\ 17&17&13&13&9&9&5&5&1&1\\ 21&21&25&25&29&29&33&33&37&37\\ 37&37&33&33&29&29&25&25&21&21\\ 41&41&45&45&49&49&53&53&57&57\\ 57&57&53&53&49&49&45&45&41&41\\ 61&61&65&65&69&69&73&73&77&77\\ 77&77&73&73&69&69&65&65&61&61\\ 81&81&85&85&89&89&93&93&97&97\\ 97&97&93&93&89&89&85&85&81&81 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 5 & 5 & 9 & 89 & 93 & 93 & 97 & 97 \\ 17 & 17 & 13 & 13 & 9 & 89 & 85 & 85 & 81 & 81 \\ 21 & 21 & 25 & 25 & 29 & 69 & 73 & 73 & 77 & 77 \\ 37 & 37 & 33 & 33 & 29 & 69 & 65 & 65 & 61 & 61 \\ 41 & 41 & 45 & 45 & 49 & 49 & 53 & 53 & 57 & 57 \\ 57 & 57 & 53 & 53 & 49 & 49 & 45 & 45 & 41 & 41 \\ 61 & 61 & 65 & 65 & 69 & 29 & 33 & 33 & 37 & 37 \\ 77 & 77 & 73 & 73 & 69 & 29 & 25 & 25 & 21 & 21 \\ 81 & 81 & 85 & 85 & 89 & 9 & 13 & 13 & 17 & 17 \\ 97 & 97 & 93 & 93 & 89 & 9 & 5 & 5 & 1 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 \\ 3 & 1 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 3 & 7 & 8 & 9 & 90 & 95 & 96 & 97 & 99 \\ 20 & 18 & 14 & 13 & 12 & 91 & 86 & 85 & 84 & 82 \\ 21 & 23 & 27 & 28 & 29 & 70 & 75 & 76 & 77 & 79 \\ 40 & 38 & 34 & 33 & 32 & 71 & 66 & 65 & 64 & 62 \\ 41 & 43 & 47 & 48 & 49 & 50 & 55 & 56 & 57 & 59 \\ 60 & 58 & 54 & 53 & 52 & 51 & 46 & 45 & 44 & 42 \\ 61 & 63 & 67 & 68 & 69 & 30 & 35 & 36 & 37 & 39 \\ 80 & 78 & 74 & 73 & 72 & 31 & 26 & 25 & 24 & 22 \\ 81 & 83 & 87 & 88 & 89 & 10 & 15 & 16 & 17 & 19 \\ 100 & 98 & 94 & 93 & 92 & 11 & 6 & 5 & 4 & 2 \end{pmatrix}.$$

Magic rectangle of order 8×10. Here $p = 4$, $q = 5$. To illustrate the embedding property of magic rectangles, we derive the 8×10 magic rectangle from the 10×10 magic rectangle constructed above. From point (i) of the remarks, we have

$$R = \begin{pmatrix} 1 & 3 & 7 & 8 & 9 & 70 & 75 & 76 & 77 & 79 \\ 20 & 18 & 14 & 13 & 12 & 71 & 66 & 65 & 64 & 62 \\ 21 & 23 & 27 & 28 & 29 & 50 & 55 & 56 & 57 & 59 \\ 40 & 38 & 34 & 33 & 32 & 51 & 46 & 45 & 44 & 42 \\ 41 & 43 & 47 & 48 & 49 & 30 & 35 & 36 & 37 & 39 \\ 60 & 58 & 54 & 53 & 52 & 31 & 26 & 25 & 24 & 22 \\ 61 & 63 & 67 & 68 & 69 & 10 & 15 & 16 & 17 & 19 \\ 80 & 78 & 74 & 73 & 72 & 11 & 6 & 5 & 4 & 2 \end{pmatrix}.$$

Finally, magic rectangle of order 6×10 ($p = 3$, $q = 5$) follows from the embedding property of magic rectangles. To get the 6×10 magic rectangle we may use either of the above constructed magic rectangles of orders 10×10 or 8×10.

$$R = \begin{pmatrix} 1 & 3 & 7 & 8 & 9 & 50 & 55 & 56 & 57 & 59 \\ 20 & 18 & 14 & 13 & 12 & 51 & 46 & 45 & 44 & 42 \\ 21 & 23 & 27 & 28 & 29 & 30 & 35 & 36 & 37 & 39 \\ 40 & 38 & 34 & 33 & 32 & 31 & 26 & 25 & 24 & 22 \\ 41 & 43 & 47 & 48 & 49 & 10 & 15 & 16 & 17 & 19 \\ 60 & 58 & 54 & 53 & 52 & 11 & 6 & 5 & 4 & 2 \end{pmatrix}.$$

The 6×10 and 8×10 magic rectangles could also be constructed directly (without using embedding property of a higher order magic rectangle) following our general method of construction.

Appendix

Proof for row sums and column sums of B.

i) The row sums, $B1_{2q} = 3q1_{2p}$ since for q odd, $(Q_e \mid Q_a \mid Q_e)1_6 = 91_2$, $\{1'_{\frac{q-3}{2}} \otimes (Q_b \mid Q_a)\}1_{2(q-3)} = 3(q-3)1_2$ and for q even, $Q1_4 = 61_2$.

ii) The column sums, $1'_{2p}B = 3p1'_{2q}$ since $1'_2 Q_e = 1'_2 Q_a = 1'_2 Q_b = 31'_2$ and $1'_2 Q = 31'_4$.

Proof for row sums and column sums of C.

i) The row sums, $C1_{2q} = 2q(2pq-1)1_{2p}$ since for q odd,
$Y_1 1_q = 4q^2(s_p \otimes 1_2) + 1_p \otimes \begin{pmatrix} (q-1)^2 + q \\ q(3q-1) - 1 \end{pmatrix}$,
$Y_2 1_q = 4q^2(s_p \otimes 1_2) + 1_p \otimes \begin{pmatrix} q(3q-1) - 1 \\ (q-1)^2 + q \end{pmatrix}$ and
$\begin{pmatrix} Y_1 \mid (K_p \otimes I_2)Y_2 \end{pmatrix} 1_{2q} = 4q^2\{(s_p + K_p s_p) \otimes 1_2\} + 2q(2q-1)1_{2p} = \{4q^2(p-1) + 2q(2q-1)\}1_{2p}$.

Similarly, for q even, the row sums hold since
$Y_1 1_q = 4q^2(s_p \otimes 1_2) + 1_p \otimes \begin{pmatrix} q(q-1) \\ q(3q-1) \end{pmatrix}$,
$Y_2 1_q = 4q^2(s_p \otimes 1_2) + 1_p \otimes \begin{pmatrix} q(3q-1) \\ q(q-1) \end{pmatrix}$ and
$\begin{pmatrix} Y_1 \mid (K_p \otimes I_2)Y_2 \end{pmatrix} 1_{2q} = 4q^2\{(s_p + K_p s_p) \otimes 1_2\} + 2q(2q-1)1_{2p} = \{4q^2(p-1) + 2q(2q-1)\}1_{2p}$.

ii) The column sums, $1'_{2p}C = 2p(2pq-1)1'_{2q}$ since $1'_pA = \frac{qp(p-1)}{2}1'_q + ps'_q$, $1'_{2p}X = p(pq-1)1'_q$ and $1'_{2p}Y = 2p(2pq-1)1'_{2q}$.

Proof for distinct numbers in R from 1 through $4pq$.

Note that every consecutive pair of rows in Y_1 and Y_2 lead to sets of four identical numbers. From one set to another the numbers increase by 4. The full set of numbers look like $1, 5, 9, \ldots, 4pq-3$. It is now sufficient to observe that every consecutive pair of rows in B has sub-matrices involving the numbers $0, 1, 2, 3$ and each of these numbers gets added to each of the sets comprising the four identical numbers.

Acknowledgements

Thanks are due to a referee for some constructive suggestions. Part of this work was carried out while the second author was visiting The University of Akron.

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(Received 7 May 2007; revised 4 Sep 2007)