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Diameter of paired domination edge-critical graphs

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Abstract

A paired dominating set of a graph G without isolated vertices is a dominating set of G whose induced subgraph has a perfect matching. The paired domination number $\gamma_{\rm pr}(G)$ of G is the minimum cardinality amongst all paired dominating sets of G. The graph G is paired domination edge-critical ($\gamma_{\rm pr}$EC) if for every $e \in E(\overline{G})$, $\gamma_{\rm pr}(G+e) < \gamma_{\rm pr}(G)$.

We investigate the diameter of $\gamma_{\rm pr}$EC graphs. To this effect we characterize $\gamma_{\rm pr}$EC trees. We show that for arbitrary even $k \geq 4$ there exists a $k_{\rm pr}$EC graph with diameter two. We provide an example which shows that the maximum diameter of a $k_{\rm pr}$EC graph is at least $k-2$ and prove that it is at most $\min\{2k-6, 3k/2+3\}$.

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1 Introduction

Criticality is a fundamental concept for many graph parameters. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. Sumner and Blitch [28] began the study of those graphs, called domination edge-critical graphs, where the (ordinary) domination number decreases on the addition of any edge. This concept was further investigated in [4, 8, 9, 11, 27, 29, 30, 31] and elsewhere. The study of total domination edge-critical graphs, defined analogously, was initiated by Van der Merwe [32] and continued in [12, 15, 16, 17, 18, 19, 32] and elsewhere.

We investigate paired domination edge-critical graphs, first studied by Edwards [6]; in particular, we obtain results on the diameter of these graphs.

2 Definitions

For notation and graph theory terminology we generally follow [13]. Specifically, for a graph $G = (V, E)$ and $v \in V$, the *open* and *closed neighbourhoods* of v are, respectively, $N(v) = \{u \in V : uv \in E\}$ and $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

For sets $S, R \subseteq V$, we say that S *dominates* R (abbreviated $S \succ R$) if $R \subseteq N[S]$, S *totally dominates* R if $R \subseteq N(S)$, and S *pairwise dominates* R (abbreviated $S \succ_{\text{pr}} R$) if S dominates R and $G[S]$ contains a perfect matching. If $R = V$ in the above sentence, then S is, respectively, a *dominating set*, a *total dominating set* (TDS), and a *paired dominating set* (PDS) of G. We then write $S \succ G$ and $S \succ_{\text{pr}} G$, respectively. If $S = \{u\}$ and $R = \{v\}$ we also write $u \succ v$, $S \succ v$, $u \succ R$, etc.

Two vertices of a PDS S with perfect matching M are said to be *paired* (by M), or *partners* in S, if they are joined by an edge of M. Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The *total domination number* $\gamma_t(G)$ (*paired domination number* $\gamma_{\text{pr}}(G)$, respectively) of G is the minimum cardinality of a TDS (a PDS, respectively). Note that these parameters are defined if and only if G has no isolated vertices. A PDS of cardinality $\gamma_{\text{pr}}(G)$ is called a γ_{pr}-set of G; a γ_t-set is defined similarly. Since every PDS of G is a TDS, $2 \leq \gamma_t(G) \leq \gamma_{\text{pr}}(G)$ for all graphs G.

Paired domination was introduced by Haynes and Slater [20] as a model for assigning backups to guards for security purposes, and is studied, for example, in [1, 2, 3, 6, 7, 10, 14, 21, 22, 23, 25, 26].

The graph G is *paired domination edge-critical*, or γ_{pr}*EC*, if for every edge $e \in E(\overline{G})$, $\gamma_{\text{pr}}(G+e) < \gamma_{\text{pr}}(G)$. If G is γ_{pr}EC and $\gamma_{\text{pr}}(G) = k$, we say that G is k_{pr}*EC*. A *total domination edge-critical* (γ_t*EC*) graph and a k_t*EC* graph are defined similarly. For example, the 5-cycle is 3_tEC and 4_{pr}EC. Note that since $\gamma_t(G), \gamma_{\text{pr}}(G) \geq 2$, the complete graphs are the only 2_tEC graphs and also the only 2_{pr}EC graphs.

If $\operatorname{diam} G = k$ and the distance $d(u, v) = k$, then we say that u and v are *peripheral* vertices, and a shortest $u-v$ path is called a *diametrical path* of G. A vertex adjacent to an end-vertex is called a *support* vertex.

It is intuitively clear that graphs with fixed paired domination number cannot have arbitrary diameter. This idea also suggests that $k_{\rm pr}$EC graphs have smaller diameter than the maximum diameter realized by graphs with $\gamma_{\rm pr} = k$. We shall show that these notions are indeed correct.

After giving some preliminary results in Section 3, we characterize $\gamma_{\rm pr}$EC trees in Section 4. It will follow that $\gamma_{\rm pr}$EC trees have diameter four. In Section 5 we provide an example which shows that the maximum diameter of a $k_{\rm pr}$EC graph is at least $k_{\rm pr} - 2$, and in Section 6 we obtain upper bounds for the diameter of $\gamma_{\rm pr}$EC graphs. Some open problems are given in Section 7.

3 Preliminary results

We first present some preliminary results. A set S is a *minimal PDS* if S is a PDS and no proper subset of S is a PDS.

Observation 1 [3] *A PDS S of a graph G is a minimal PDS if and only if any two vertices $x, y \in S$ satisfy one of the following conditions:*

(i) *$G[S - \{x, y\}]$ does not contain a perfect matching,*

(ii) *without loss of generality, x is an end-vertex in $G[S]$ adjacent to y,*

(iii) *there exists a vertex $u \in V - S$ such that $N(u) \cap S \subseteq \{x, y\}$.*

Observation 2 (i) *Each support vertex in a graph G is contained in every PDS of G.*

(ii) *Every vertex in a $\gamma_{\rm pr}$EC graph is adjacent to at most one end-vertex.*

Observation 3 *If G is $\gamma_{\rm pr}$EC and $uv \in E(\overline{G})$, then every $\gamma_{\rm pr}$-set S of $G + uv$ contains at least one of u and v, and if $\{u, v\} \subseteq S$, then u and v are paired in S.*

Observation 4 (i) *If G is a $\gamma_{\rm pr}$EC graph, then $\gamma_{\rm pr}(G+e) = \gamma_{\rm pr}(G) - 2$ for every $e \in E(\overline{G})$.*

(ii) [16] *If G is a γ_tEC graph, then $\gamma_t(G) - 2 \leq \gamma_t(G + e) \leq \gamma_t(G) - 1$ for every $e \in E(\overline{G})$. Moreover, the lower bound holds if and only if G is the disjoint union of two or more complete graphs.*

If the lower bound holds in Observation 4(ii), then G is called *γ_t-super-edge-critical* or γ_tSEC. As mentioned in Section 2, C_5 is 3_tEC and $4_{\rm pr}$EC. That this is no coincidence was shown in [6].

Theorem 1 [6, Theorem 3.7] *If $2k < \gamma_t(G) \leq 2(k + 1) = \gamma_{\rm pr}(G)$ and G is $\gamma_{\rm pr}$EC, then G is γ_tEC or γ_tSEC.*

Corollary 2 [6, Corollary 3.9] *The class of* $4_{\rm pr}$*EC graphs is the union of the classes of* 3_t*EC graphs and* 4_t*SEC graphs.*

We show in the next section that Corollary 2 does not extend to $k_{\rm pr}$EC graphs where $k \geq 6$ (see the remark following the proof of Theorem 5).

In preparation for results on the diameter of $\gamma_{\rm pr}$EC graphs we now bound the diameter of graphs with $\gamma_{\rm pr} = k$. Let x be a peripheral vertex of a graph G with $\operatorname{diam} G = m$. Define the *levels* $V_0, V_1, \ldots, V_m$ of G *with respect to* x by $V_i = \{v \in G : d(x, v) = i\}$. Notice that $V_0 = \{x\}$ and $V_m \neq \phi$.

Lemma 3 *If S is a PDS of a graph G and a and b are paired in S, then $\{a, b\}$ dominates at most four levels of G.*

Proof. Suppose $\{a, b\}$ dominates at least five levels $V_j, V_{j+1}, \cdots, V_{j+l}$ $(l \geq 4)$ of G. Then every vertex in $W = V_j \cup V_{j+1} \cup \cdots \cup V_{j+l}$ is adjacent to at least one of a or b. For $u \in V_j$ and $v \in V_{j+l}$, $d(u, v) \geq l \geq 4$. But each of u and v is adjacent to a or b, hence $d(u, v) \leq 3$, a contradiction. ■

Proposition 4 *If G is connected and $\gamma_{\rm pr}(G) = k$, then* $\operatorname{diam} G \leq 2k - 1$.

Proof. Let $m = \operatorname{diam} G$ and $V_0, V_1, \ldots, V_m$ be the levels of G with respect to a peripheral vertex x. Let S be a $\gamma_{\rm pr}$-set of G.

By Lemma 3, every set of partners $\{a, b\} \subseteq S$ dominates at most four levels of G. Since there are $k/2$ such pairs in S, at most $2k$ levels of G are dominated. Hence if $\operatorname{diam} G \geq 2k$, at least one level of G is undominated. Therefore $\operatorname{diam} G \leq 2k - 1$. ■

4 Trees

We begin the study of the diameter of $\gamma_{\rm pr}$EC graphs by characterizing $\gamma_{\rm pr}$EC trees. The *subdivided star* S_{2n+1} is obtained from $K_{1,n}$ by subdividing each edge exactly once.

Theorem 5 *A tree $T \neq K_2$ is $\gamma_{\rm pr}$EC if and only if $T = S_{2n+1}$, $n \geq 3$.*

Proof. The sufficiency is straightforward to verify. To prove the necessity, let T be a $\gamma_{\rm pr}$EC tree. If $\operatorname{diam} T \leq 3$, then T is a star or a double star. But then $\gamma_{\rm pr}(T) = 2$. Since the complete graphs are the only $2_{\rm pr}$EC graphs, $T = K_2$. Thus we may assume that $\operatorname{diam} T \geq 4$.

Let $P : v_0, v_1, v_2, \ldots, v_d$ be a diametrical path in T and suppose firstly that $\operatorname{diam} T = d \geq 5$. By Observation 2$(ii)$, $\deg v_1 = \deg v_{d-1} = 2$. Let $e = v_2 v_{d-1} \in E(\overline{T})$ and consider the tree $T' = T + e$. Since T is $\gamma_{\rm pr}$EC, $\gamma_{\rm pr}(T') = \gamma_{\rm pr}(T) - 2$. Let S' be a $\gamma_{\rm pr}$-set of T'. By Observation 2(i), S' contains the two support vertices v_1 and v_{d-1}.

If $v_2 \notin S'$, let $S = S'$. If $v_2 \in S'$, then (Observation 3) v_2 is paired with $v_{d-1} \in S'$, hence v_1 is paired with v_0 in S', and we let $S = (S' \setminus \{v_0\}) \cup \{v_d\}$. In both cases S is a PDS of T with $|S| = |S'| = \gamma_{\mathrm{pr}}(T')$. (In the latter case observe that v_1 is paired with v_2, v_{d-1} is paired with v_d while all other pairings in S remain unchanged from those in S'.) Thus $\gamma_{\mathrm{pr}}(T) \le \gamma_{\mathrm{pr}}(T')$, a contradiction. Hence $\operatorname{diam} T \le 4$ and so $d = \operatorname{diam} T = 4$.

If $T = P_5$, then $T + v_0v_4 = C_5$. But $\gamma_{\mathrm{pr}}(P_5) = \gamma_{\mathrm{pr}}(C_5) = 4$, contradicting the fact that T is γ_{pr}EC. Hence $\Delta(T) \ge 3$. Thus, by Observation 2(ii), T is obtained from $K_{1,n}$, $n \ge 3$, by subdividing every edge, except for possibly one edge. Let v be the central vertex of $K_{1,n}$ and $N(v) = \{\ell_1, ..., \ell_n\}$. Suppose $v\ell_i$ has been subdivided by the vertex u_i, $i = 1, ..., n-1$, but not $v\ell_n$. Then $\bigcup_{i=2}^{n-1}\{u_i, \ell_i\} \cup \{v, u_1\}$ is a γ_{pr}-set of T and of $T + u_1\ell_n$, a contradiction. Therefore each edge $v\ell_i$ has been subdivided by u_i, so that $T = S_{2n+1}$ and $\bigcup_{i=1}^{n}\{u_i, \ell_i\}$ is a γ_{pr}-set of T. ■

Thus all γ_{pr}EC trees have diameter 4. As shown in the proof of Theorem 5, $\gamma_{\mathrm{pr}}(S_{2n+1}) = 2n$ and it is easy to see that S_{2n+1} is $2n_{\mathrm{pr}}$EC. It is also easy to see that $\gamma_t(S_{2n+1}) = n + 1$ and that S_{2n+1} is not γ_tEC. Thus Corollary 2 does not extend to k_{pr}EC graphs where $k \ge 6$.

5 γ_{pr}-Edge critical graphs with small/large diameter

The only graphs with diameter 1 are the complete graphs, and, except for K_1, they are vacuously 2_{pr}EC. In this section we provide constructions of, firstly, a k_{pr}EC graph with diameter two for each $k \ge 4$, and secondly, γ_{pr}EC graphs with large diameter. The latter result shows that the maximum diameter of a γ_{pr}EC graph is at least $\gamma_{\mathrm{pr}}(G) - 2$.

Proposition 6 *For every even $k \ge 4$ there exists a k_{pr}EC graph of diameter* 2.

Proof. For $k = 2l$, $l \ge 2$, consider the Cartesian product of the graph K_k with itself, i.e. the graph $G_k = K_k \times K_k$. We can think of G_k as having k disjoint copies of K_k in "rows" and k disjoint copies of K_k in "columns". In other words, we consider the vertices of G_k as a matrix, where vertex v_{ij} is in the i^{th} row (copy of K_k) and the j^{th} column (copy of K_k). For ease of discussion we shall use the words row and column to mean a "copy of K_k".

We show first that $\gamma_{\mathrm{pr}}(G_k) = k$. Since $\{v_{11}, v_{21}, \ldots, v_{k1}\}$ is a dominating set with a perfect matching, $\gamma_{\mathrm{pr}}(G_k) \le k$. Suppose $\gamma_{\mathrm{pr}}(G_k) \le k - 2$. Then for any γ_{pr}-set S there exists an i such that S does not have a vertex in row i. Any vertex in S dominates only one vertex of row i, implying that at most $k - 2$ of the k vertices of row i are dominated, a contradiction. Thus $\gamma_{\mathrm{pr}}(G_k) = k$.

We now show that $\operatorname{diam} G = 2$. Clearly, for $k \ge 2$, G_k is not complete and so $\operatorname{diam} G \ge 2$. Consider distinct vertices $x, y \in V(G_k)$. If x and y are in the same row, i.e. $x = v_{ij}$ and $y = v_{ik}$, then $d(x, y) = 1$; this is also true if x and y are in the same column. If x and y are not in the same row or column, i.e. $x = v_{hi}$ and $y = v_{jk}$ where

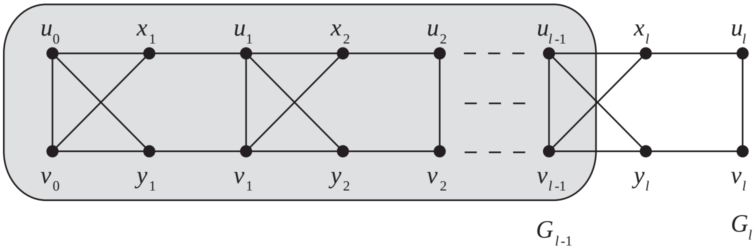

Figure 1: A $2(l+1)_{\rm pr}$-edge critical graph with diameter $2l$

$h \neq j$ and $i \neq k$, let $z = v_{hk}$. Then $d(x,z) = 1$ and $d(z,y) = 1$ and so $d(x,y) = 2$. It follows that $\operatorname{diam} G_k = 2$.

If G_k is $k_{\rm pr}$EC, then we are finished. Otherwise, as can be seen by adding edges to G_k without changing $\gamma_{\rm pr}$, G_k is a spanning subgraph of some $k_{\rm pr}$EC graph G' and, since G' is not complete, $\operatorname{diam} G' = 2$. ■

We next construct a $(2l+2)_{\rm pr}$-EC graph with diameter $2l$, where $l \geq 1$. For each $i = 1, \ldots, l$, let $H_i \cong P_4$ with vertex sequence x_i, u_i, v_i, y_i. Construct G_l recursively as follows. Let $G_0 = K_2$ with $V(G_0) = \{u_0, v_0\}$, and once G_{i-1} has been constructed, let G_i be the graph with $V(G_i) = V(G_{i-1}) \cup V(H_i)$ and $E(G_i) = E(G_{i-1}) \cup E(H_i) \cup \{u_{i-1}x_i, u_{i-1}y_i, v_{i-1}x_i, v_{i-1}y_i\}$. See Figure 1.

Proposition 7 *For any $l \geq 1$, G_l is a $(2l+2)_{\rm pr}$-EC graph with diameter $2l$.*

Proof. It is obvious that $\operatorname{diam} G_l = 2l$. We first prove by induction that $\gamma_{\rm pr}(G_l) = 2(l+1)$. Since $D = \bigcup_{i=0}^{l} \{u_i, v_i\}$ is a PDS of G_l with $|D| = 2(l+1)$, it remains to show that $\gamma_{\rm pr}(G_l) \geq 2(l+1)$. This is easy to verify for G_1. For $l \geq 2$, assume $\gamma_{\rm pr}(G_{l-1}) \geq 2l$ and let S be any minimal PDS of G_l.

Suppose firstly that $S \cap \{x_l, y_l\} = \phi$. Then $S' = S \cap V(G_{l-1})$ is a PDS of G_{l-1}, hence by assumption $|S'| \geq 2l$. Since S' does not dominate u_l or v_l, $\{u_l, v_l\} \subseteq S$ to pairwise dominate $\{u_l, v_l\}$, hence $|S| \geq 2(l+1)$.

Now assume without loss of generality that $x_l \in S$. Then x_l is paired in S with $w \in \{u_{l-1}, v_{l-1}, u_l\}$.

Suppose $w \neq u_l$. By symmetry we may then assume that $w = u_{l-1}$. To dominate v_l, $S \cap \{u_l, v_l, y_l\} \neq \phi$. If $v_{l-1} \notin S$, or if $v_{l-1} \in S$ and v_{l-1} is not partnered by y_l, then $|S \cap \{u_l, v_l, y_l\}| \geq 2$. In the latter case, v_{l-1} is partnered by y_{l-1}; moreover $x_{l-1} \notin S$, otherwise $\{u_{l-1}, x_l\}$ satisfies none of the conditions of Observation 1. Define S^* by

$$S^* = \begin{cases} (S - \{x_l, u_l, v_l, y_l\}) \cup \{x_{l-1}\} & \text{if } v_{l-1} \in S \text{ is partnered by } y_{l-1} \\ (S - \{x_l, u_l, v_l, y_l\}) \cup \{v_{l-1}\} & \text{otherwise.} \end{cases}$$

In the first instance of the definition of S^*, each of the sets $\{u_{l-1}, x_{l-1}\}$ and $\{v_{l-1}, y_{l-1}\}$ is a pair, and in the second instance $\{u_{l-1}, v_{l-1}\}$ is a pair; in either case $|S^*| \leq |S| - 2$.

Also, S^* is a PDS of G_{l-1}, hence by the induction hypothesis $|S^*| \geq 2l$ and so $|S| \geq 2(l+1)$.

Suppose $w = u_l$. To pairwise dominate y_l, $|S \cap \{u_{l-1}, v_{l-1}, y_l, v_l\}| \geq 1$. If $S \cap \{u_{l-1}, v_{l-1}\} \neq \phi$, then $S' = S - \{x_l, u_l\} \succ_{\rm pr} G_{l-1}$, hence $|S'| \geq 2l$ and $|S| \geq 2(l+1)$. If $S \cap \{u_{l-1}, v_{l-1}\} = \phi$, then $\{y_l, v_l\} \subseteq S$. In this case $(S - \{u_l, v_l, x_l, y_l\}) \cup \{u_{l-1}, v_{l-1}\} \succ_{\rm pr} G_{l-1}$ and again $|S| \geq 2(l+1)$. It follows that $\gamma_{\rm pr}(G_l) = 2(l+1)$.

We next prove by induction that G_l is $\gamma_{\rm pr}$EC, the case $l = 1$ being easy to verify. For $l \geq 2$, assume that G_{l-1} is $\gamma_{\rm pr}$EC and let $e = ab \in E(\overline{G}_l)$.

If $e \in E(\overline{G}_{l-1})$, let S' be any $\gamma_{\rm pr}$-set of $G_{l-1} + e$ and $S = S' \cup \{u_l, v_l\}$. Then $S \succ_{\rm pr} G_l + e$ and $|S| = |S'| + 2 = 2l$ by the induction hypothesis.

If $e \in E(\overline{H}_l) = \{x_l y_l, x_l v_l, y_l u_l\}$, let $S = (\bigcup_{i=0}^{l-2}\{u_i, v_i\}) \cup \{a, b\}$. Then $S \succ_{\rm pr} G_l + e$ and $|S| = 2l$.

Hence assume $a \in V(G_{l-1})$ and $b \in \{x_l, u_l, v_l, y_l\}$. We may assume without loss of generality that $a \in \{u_0, u_1, ..., u_{l-1}\} \cup \{x_1, x_2, ..., x_{l-1}\}$. By symmetry we may also assume without loss of generality that $b \in \{y_l, v_l\}$. If $b = v_l$, then regardless of a, let $S = \bigcup_{i=0}^{l-1}\{u_i x_{i+1}\}$, and if $b = y_l$, let $S = (\bigcup_{i=0}^{l-2}\{u_i x_{i+1}\}) \cup \{x_l, u_l\}$. In either case $S \succ_{\rm pr} G_l + e$ and $|S| = 2l$. Therefore G_l is $\gamma_{\rm pr}$EC. ■

6 Bounds on the diameter

In general, the diameter of a $k_{\rm pr}$EC graph is smaller than the general upper bound established in Proposition 4 for graphs with $\gamma_{\rm pr} = k$. In this section we establish upper bounds on the diameter of connected $k_{\rm pr}$EC graphs.

In Section 5 we exhibited a $4_{\rm pr}$EC graph with diameter 2. However, this is not the maximum diameter amongst $4_{\rm pr}$EC graphs, because, as shown in [16], $2 \leq \operatorname{diam} G \leq 3$ whenever G is a connected 3_tEC graph, and the bounds are sharp. The corresponding result for connected $4_{\rm pr}$EC graphs follows from Observation 4(*ii*) and Corollary 2.

We henceforth consider connected $k_{\rm pr}$EC graphs with $k \geq 6$.

Theorem 8 *If G is a connected $k_{\rm pr}$EC graph with $k \geq 6$, then* $\operatorname{diam} G \leq 2k - 6$.

Proof. Suppose to the contrary that G is a $k_{\rm pr}$EC graph, $k \geq 6$, such that $\operatorname{diam} G \geq 2k - 5$. Since G is $k_{\rm pr}$EC, $\gamma_{\rm pr}(G + e) = k - 2$ for any $e \in E(\overline{G})$. Let $k = 2l$ and consider the nonempty levels $V_0, V_1, \ldots, V_{2k-5}$ with respect to a peripheral vertex v. Let $u \in V_4$. Then $uv \in E(\overline{G})$, $\gamma_{\rm pr}(G + uv) = 2l - 2$ and, by Observation 3, any $\gamma_{\rm pr}$-set D of $G + uv$ contains at least one of u and v, where u and v are paired in D if D contains both.

Suppose first that $\{u, v\} \subseteq D$. Then the pair $\{u, v\}$ dominates only vertices in $V_0 \cup V_1 \cup V_3 \cup V_4 \cup V_5$. Let u', v' be paired in D such that $\{u', v'\}$ dominates vertices in V_2. Then by Lemma 3, $\{u', v'\}$ does not dominate any vertices in V_t, $t \geq 6$. Hence the remaining $2(l-3)$ vertices in D dominate all of $\bigcup_{t=6}^{2k-5} V_t$, a total of $4l - 10$ levels of G. (Note that $2k - 10 > 0$ because $k \geq 6$.) Using Lemma 3 and the pigeonhole principle, we see that this is impossible.

If $\{u, v\} \cap D = \{u\}$, then u is paired with a vertex $u' \in V_3 \cup V_4 \cup V_5$. By Lemma 3, $\{u, u'\}$ dominates only (some) vertices in $V_0 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$. At least one more pair of vertices in D is needed to dominate V_1, and by Lemma 3, this pair does not dominate any vertex in V_t, $t \geq 7$. Hence the remaining $2(l - 3)$ vertices in D dominate at least $4l - 11$ levels. By the pigeonhole principle, at least one pair dominates at least five levels, contradicting Lemma 3.

Hence we conclude that $u \notin D$ and so $v \in D$. Necessarily, v is paired with the vertex $v' \in V_1$. Then the pair $\{v, v'\}$ dominates only vertices in $V_0 \cup V_1 \cup V_2 \cup V_4$. At least one more pair is needed to dominate V_3, and by Lemma 3, no such pair dominates any vertices in V_t, $t \geq 7$. This leads to a contradiction as before.

Therefore $\operatorname{diam} G \leq 2k - 6$. ■

It is not known whether the bound in Theorem 8 is the best possible if G is 6_{pr}EC; the graph constructed in Section 5 for $k = 6$ has diameter 4, and we have not found a 6_{pr}EC graph with diameter 5 or 6. The bound can be improved to $2k - 7$ and $2k - 8$ for $k = 8$ and $k = 10$, respectively, which suggests that the coefficient of k in the bound in Theorem 8 is incorrect. We show next that this coefficient can be decreased from 2 to $\frac{3}{2}$; the bound in Theorem 10 is better than the one in Theorem 8 if $k > 18$.

We start with a lemma.

Lemma 9 *Let G be a connected graph with $S_1, S_2 \subseteq V$. If there exist sets $D_1, D_2 \subseteq V$ such that $D_i \succ_{\text{pr}} S_i$, $i = 1, 2$, then there exists a set $D \subseteq V$ such that $|D| \leq |D_1| + |D_2|$ and $D \succ_{\text{pr}} S_1 \cup S_2$.*

Proof. Let M_i be a perfect matching in $\langle D_i \rangle$, $i = 1, 2$. Partition M_2 into three sets P_1, P_2, P_3 as follows. Let

$$\begin{aligned} P_1 &= \{ab \in M_2 : a, b \notin D_1\}, \\ P_2 &= \{ab \in M_2 : \text{ without loss of generality, } a \in D_1 \text{ and } b \notin D_1\} \\ \text{and } P_3 &= \{ab \in M_2 : a, b \in D_1\}. \end{aligned}$$

Say $P_2 = \{a_1 b_1, a_2 b_2, ..., a_t b_t\}$ for some $0 \leq t \leq |D_1 \cap D_2|$. If $t = 0$, let $D = D_1 \cup D_2$ and $M = M_1 \cup P_1$. Then $|D| = |D_1| + |D_2| - |D_1 \cap D_2| \leq |D_1| + |D_2|$ and M is a perfect matching in $\langle D \rangle$; thus $D \succ_{\text{pr}} S_1 \cup S_2$ and we are done. So assume $t \geq 1$. Let $T_0 = D_1 \cup D_2 - \{b_1, ..., b_t\}$. Then $M_1 \cup P_1$ is a perfect matching in $\langle T_0 \rangle$ and so $T_0 \succ_{\text{pr}} S_1 \cup S_2 - \bigcup_{i=1}^{t} N[b_i]$. For $1 \leq i \leq t$, we inductively define T_i as follows.

If $N(b_i) \subseteq T_{i-1}$, then let $T_i = T_{i-1}$. Note that since G is connected, b_i has at least one neighbour and that neighbour is in T_i.

Otherwise, there exists a vertex $c_i \in V - T_{i-1}$ such that $b_i c_i \in E(G)$. In this case, let $T_i = T_{i-1} \cup \{b_i, c_i\}$.

In either case $T_i \succ N[b_i]$. Let $D = T_t$. Then

$$\begin{aligned} |D| &\le |D_1| + |D_2| - |D_1 \cap D_2| - |\{b_1, ..., b_t\}| + 2t \\ &\le |D_1| + |D_2| - t - t + 2t \quad = \quad |D_1| + |D_2|. \end{aligned}$$

Since $T_0 \subseteq D$, we have $D \succ S_1 \cup S_2 - \bigcup_{i=1}^{t} N[b_i]$ and since $T_i \subseteq D$ for all $1 \le i \le t$, we have $D \succ N[b_i]$ for all $1 \le i \le t$; thus $D \succ S_1 \cup S_2$. Let

$$M = M_1 \cup P_1 \cup \{b_i c_i : N(b_i) \nsubseteq T_{i-1},\ 1 \le i \le t\}.$$

Then M is a perfect matching in $\langle D \rangle$ and so $D \succ_{\mathrm{pr}} S_1 \cup S_2$. ■

Theorem 10 *If G is $k_{\mathrm{pr}}EC$, then* $\operatorname{diam} G \le \frac{3k}{2} + 3$.

Proof. Assume $k \ge 8$ (we already have this for small k). For $m = \operatorname{diam} G$, let $V_0, V_1, ..., V_m$ be the levels of G with respect to a peripheral vertex v_0. Suppose $m \ge \frac{3k}{2} + 4$ (≥ 16). Let $v_4 \in V_4$. Then $e = v_0 v_4 \in E(\overline{G})$ and thus by the criticality of G, $\gamma_{\mathrm{pr}}(G + e) = k - 2$. Let A be a γ_{pr}-set of $G + e$. Then (Observation 3) $\{v_0, v_4\} \cap A \neq \phi$.

If $\{v_0, v_4\} \subseteq A$, then v_0, v_4 are paired in A. Note that $\{v_0, v_4\}$ only dominates vertices in $V_0 \cup V_1 \cup V_3 \cup V_4 \cup V_5$ and so there exists another pair of vertices in A that dominates vertices in V_2. This pair does not dominate any vertices in V_i for all $i \ge 7$ (Lemma 3).

If $\{v_0, v_4\} \cap A = \{v_0\}$, then v_0 is necessarily paired with a vertex $v_1 \in V_1$. Note that $\{v_0, v_1\}$ only dominates vertices in $V_0 \cup V_1 \cup V_2 \cup V_4$ and so there exists another pair of vertices in A that dominates vertices in V_3. This pair does not dominate any vertices in V_i for all $i \ge 7$.

If $\{v_0, v_4\} \cap A = \{v_4\}$, then v_4 is paired with a vertex $w \in V_3 \cup V_4 \cup V_5$. Hence $\{v_4, w\}$ does not dominate any vertices in V_1 and so there exists another pair of vertices that dominates vertices in V_1. Note that neither pair dominates any vertices in V_i for all $i \ge 7$.

Thus in any of the three possibilities listed above, there exist two pairs of vertices in A that do not dominate any vertices in V_i for all $i \ge 7$. Thus there exists a set $D_1 \subseteq A$ such that $|D_1| \le k - 6$ and $D_1 \succ_{\mathrm{pr}} \bigcup_{i=7}^{m} V_i$ in $G + e$ and thus in G. We generalize this result as follows.

$$\text{For each } j \in \{1, ..., \lceil \tfrac{k}{6} \rceil\} \text{ there exists a set } D_j \subseteq V(G) \text{ such that} \\ |D_j| \le k - 6j \text{ and } D_j \succ_{\mathrm{pr}} \bigcup_{i=9(j-1)+7}^{m} V_i \text{ in } G. \tag{1}$$

We prove (1) by induction on j. The base case holds for $j = 1$ as shown above. Thus let $j \in \{2, ..., \lceil \frac{k}{6} \rceil\}$ and assume that (1) holds for $j - 1$; i.e. there exists a set

$D_{j-1} \subseteq V(G)$ such that $|D_{j-1}| \leq k - 6(j-1)$ and $D_{j-1} \succ_{\rm pr} \bigcup_{i=9(j-2)+7}^{m} V_i$ in G. Since k is even, $j \leq \lceil \frac{k}{6} \rceil \leq \frac{k}{6} + \frac{2}{3} \leq \frac{m-4}{9} + \frac{2}{3}$. Thus

$$9(j-1) + 7 \leq m. \tag{2}$$

Consider vertices $w_1 \in V_{9(j-1)}$ and $w_2 \in V_{9(j-1)+4}$. Then $e = w_1w_2 \in E(\overline{G})$ and thus by the criticality of G, $\gamma_{\rm pr}(G+e) = k-2$. Let B_j be a $\gamma_{\rm pr}$-set of $G+e$. By Observation 3, $\{w_1, w_2\} \cap B_j \neq \phi$.

- We show, in each of three cases, that there are two pairs of vertices in B_j that do not dominate any vertices of G in levels V_i, for all integers i in the intervals $I_1 = [0{,}9(j-2)+6]$ and $I_2 = [9(j-1)+7, m]$. The endpoints of I_2 have been chosen to match those of the union in (1), while the endpoints of I_1 have been chosen not necessarily to maximise the length of the interval, but to facilitate the proof of (1).

If $\{w_1, w_2\} \subseteq B_j$, then w_1, w_2 are paired in B_j. Note that $\{w_1, w_2\}$ only dominates (some) vertices in

$$\left(\bigcup_{i=-1}^{5} V_{9(j-1)+i} \right) - V_{9(j-1)+2}$$

and so there exists another pair of vertices in B_j that dominates vertices in $V_{9(j-1)+2}$. This pair dominates at most four levels of G and hence the two pairs of vertices do not dominate any vertices in V_i for all $i \in I_1 \cup I_2$.

If $\{w_1, w_2\} \cap B_j = \{w_1\}$, then w_1 is paired with a vertex $w \in V_{9(j-1)-1} \cup V_{9(j-1)} \cup V_{9(j-1)+1}$. Then $\{w_1, w\}$ does not dominate any vertices in $V_{9(j-1)+3}$ and so there exists another pair of vertices in B_j that dominates vertices in $V_{9(j-1)+3}$. These two pairs of vertices do not dominate any vertices in V_i, $i \in I_1 \cup I_2$.

If $\{w_1, w_2\} \cap B_j = \{w_2\}$, then w_2 is paired with $u \in V_{9(j-1)+3} \cup V_{9(j-1)+4} \cup V_{9(j-1)+5}$. In any case, $\{w_2, u\}$ does not dominate any vertices in $V_{9(j-1)+1}$ and so there exists another pair of vertices that dominates vertices in $V_{9(j-1)+1}$. Again these two pairs of vertices do not dominate any vertices in V_i, $i \in I_1 \cup I_2$.

Thus in any of the three possibilities listed above, there exist two pairs of vertices in B_j that do not dominate any vertices in V_i for all $i \in I_1 \cup I_2$. Therefore there exists a set $C_j \subseteq B_j$ such that $|C_j| \leq k-6$ and

$$C_j \succ_{\rm pr} \bigcup_{i \in I_1 \cup I_2} V_i$$

in $G+e$ and thus in G.

Suppose there exists a set $D' \subseteq C_j$ such that $|D'| \leq 6(j-1) - 2$ and $D' \succ_{\rm pr} \bigcup_{i \in I_1} V_i$. Then by Lemma 9 and the induction hypothesis, there exists a set $D'' \subseteq V(G)$ such that $|D''| \leq |D'| + |D_{j-1}| \leq k-2$ and

$$D'' \succ_{\rm pr} \left(\bigcup_{i=0}^{9(j-2)+6} V_i \right) \cup \left(\bigcup_{i=9(j-2)+7}^{m} V_i \right) = V(G);$$

a contradiction since $\gamma_{\rm pr}(G) = k$. Thus at least $6(j-1)$ vertices in C_j are required to pairwise dominate the vertices in $\bigcup_{i \in I_1} V_i$. Since none of these vertices dominates any vertices in $\bigcup_{i \in I_2} V_i$ (Lemma 3), at most $k - 6 - 6(j-1) = k - 6j$ vertices in C_j remain to dominate the vertices in $\bigcup_{i \in I_2} V_i$. It follows that there exists a set $D_j \subseteq C_j$ such that $|D_j| \le k - 6j$ and $D_j \succ_{\rm pr} \bigcup_{i \in I_2} V_i$, and thus (1) holds.

Now for $j = \lceil \frac{k}{6} \rceil$, $9(j-1) + 7 \le m$ by (2) and thus $\bigcup_{i \in I_2} V_i \ne \phi$. However, by (1), $|D_j| \le k - 6j = k - 6\lceil \frac{k}{6} \rceil \le 0$ and $D_j \succ_{\rm pr} \bigcup_{i \in I_2} V_i$, which is absurd. Thus $\operatorname{diam} G \le \frac{3k}{2} + 3$. ■

If $k \equiv 0 \pmod 6$, then the same proof shows that if G is $k_{\rm pr}$EC, then $\operatorname{diam} G \le \frac{3k}{2} - 3$, which generalizes the bound in Theorem 8 for the case $k = 6$.

7 Open Problems

We conclude with a few open problems.

1. As remarked above it is not known whether the bound in Theorem 8 is the best possible if G is $6_{\rm pr}$EC, and the graph constructed in Section 5 for $k = 6$ has diameter 4. Find a $6_{\rm pr}$EC graph with diameter 5 or 6, or improve this bound.

2. In general, let d_k be the maximum value of the diameter for a $k_{\rm pr}$EC graph. Find a sharp upper bound for d_k, or at least improve the bound in Theorem 10.

3. We showed in Section 5 that the minimum value for the diameter of a noncomplete $\gamma_{\rm pr}$EC graph is 2, and that $k_{\rm pr}$EC graphs satisfying this diameter exist for all even $k \ge 4$. What is the spectrum of diameters for $k_{\rm pr}$EC graphs? In particular, is it true that there exists a $k_{\rm pr}$EC graph of diameter l for every $2 \le l \le d_k$?

4. In Section 4 we characterized $\gamma_{\rm pr}$EC trees. It is evident that they have diameter 4, regardless of the value of $\gamma_{\rm pr}$. Characterize bipartite $\gamma_{\rm pr}$EC graphs and determine or bound their diameter.

5. All the above questions may be also be asked (with obvious modifications) for *paired domination vertex-critical* graphs, i.e. graphs G for which $\gamma_{\rm pr}(G - v) < \gamma_{\rm pr}(G)$ for all $v \in V$. See [6, 24] for results on these graphs.

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