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# Restricted arc-connectivity of generalized tournaments

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**Abstract**

If $D$ is a strongly connected digraph, then an arc set $S$ of $D$ is called a restricted arc-cut of $D$ if $D - S$ has a non-trivial strong component $D_1$ such that $D - V(D_1)$ contains an arc. Recently, Volkmann [12] defined the restricted arc-connectivity $\lambda'(D)$ as the minimum cardinality over all restricted arc-cuts $S$. A strongly connected digraph $D$ is called $\lambda'$-*connected* when $\lambda'(D)$ exists. Let $k \geq 2$ be an integer. An arc set $S$ of $D$ is a $k$-restricted arc-cut of $D$ if $D - S$ contains at least $k$ non-trivial strong components. Volkmann [*Inform. Process. Lett.* 103 (2007), 234–239] also defined the $k$-restricted arc-connectivity $\lambda'_k(D)$ as the minimum cardinality over all $k$-restricted arc-cuts $S$. A strongly connected digraph $D$ is called $\lambda'_k$-connected when $\lambda'_k(D)$ exists.

In this paper we characterize all $\lambda'$-connected tournaments, multipartite tournaments, local tournaments and in-tournaments. In addition, we determine the $\lambda'_2$-connected tournaments and local tournaments.

## 1 Terminology and preliminary results

We consider finite digraphs without loops, multiple arcs and directed cycles of length two. For any digraph $D$ the vertex set is denoted by $V(D)$ and the arc set by $E(D)$. We define the *order* of $D$ by $n = n(D) = |V(D)|$ and the *size* by $m = m(D) = |E(D)|$.

If $uv$ is an arc of a digraph $D$, then $v$ is a *positive neighbor* of $u$ and $u$ a *negative neighbor* of $v$, and we also say that $u$ *dominates* $v$. If $A$ and $B$ are two disjoint subdigraphs of $D$ such that every vertex of $A$ dominates every vertex of $B$, then we say $A$ *dominates* $B$, denoted by $A \to B$. The *outset* $N^+(u) = N^+_D(u)$ and the *inset* $N^-(u) = N^-_D(u)$ of a vertex $u$ is the set of positive neighbors and negative neighbors

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of $u$, respectively. The numbers $d^+(u) = d^+_D(u) = |N^+(u)|$ and $d^-(u) = d^-_D(u) = |N^-(u)|$ are the *out-degree* and the *in-degree* of the vertex $u$. By a *cycle* of a digraph we mean a directed cycle. A cycle of length $p$ is also called a $p$-cycle. A digraph $D$ is *vertex pancyclic* if every vertex of $D$ is contained in a $p$-cycle for all $p$ between 3 and $|V(D)|$. If $D$ is a digraph and $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by $X$. Two vertices $u$ and $v$ of a digraph are *adjacent* if $u \to v$ or $v \to u$. Two vertex-disjoint subdigraphs $A$ and $B$ of a digraph $D$ are *complementary*, if $V(D) = V(A) \cup V(B)$. A digraph is called *cycle complementary*, if it has two complementary cycles. If $C = x_1x_2 \ldots x_nx_1$ is a cycle, then the *second power* of the cycle $C$ consists of $C$ and the arcs $x_ix_{i+2}$ for $i = 1, 2, \ldots, n$ where $x_{n+j} = x_j$ for $j = 1, 2$. If we replace every arc $uv$ by $vu$ in a digraph $D$, then we call the resulting digraph the *converse* of $D$.

A digraph $D$ is *strongly connected* or simply *strong* if for every pair $u, v$ of vertices there exists a directed path from $u$ to $v$ in $D$. A digraph $D$ with at least $k+1$ vertices is *$k$-connected* if for every set $A$ of at most $k-1$ vertices, the subdigraph $D - A$ is strong. The *connectivity* of a digraph $D$, denoted by $\kappa(D)$, is then defined to be the largest value $k$ such that $D$ is $k$-connected. A digraph $D$ is $k$-arc-connected if for any set $S$ of at most $k-1$ arcs the subdigraph $D-S$ is strong. The *arc-connectivity* $\lambda(D)$ of a digraph $D$ is defined as the largest value of $k$ such that $D$ is $k$-arc-connected.

A *$c$-partite* or *multipartite tournament* is an orientation of a complete $c$-partite graph. A *tournament* is a $c$-partite tournament with exactly $c$ vertices. A digraph $D$ is a *local tournament*, if for every vertex $u$ the out-neighborhood as well as the in-neighborhood of $u$ induce tournaments. A digraph $D$ is an *in-tournament*, if for every vertex $u$ the in-neighborhood of $u$ induces a tournament. For other graph theory terminology we follow Bang-Jensen and Gutin [2].

For strongly connected digraphs $D$, Volkmann [12] defined the following kinds of restricted arc-connectivity.

An arc set $S$ of $D$ is a *restricted arc-cut* of $D$ if $D - S$ has a non-trivial strong component $D_1$ such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is the minimum cardinality over all restricted arc-cuts $S$. A strongly connected digraph $D$ is called *$\lambda'$-connected*, if $\lambda'(D)$ exists.

Let $k \geq 2$ be an integer. An arc set $S$ of $D$ is a *$k$-restricted arc-cut* of $D$ if $D - S$ contains at least $k$ non-trivial strong components. The $k$-restricted arc-connectivity $\lambda'_k(D)$ is the minimum cardinality over all $k$-restricted arc-cuts $S$. A strongly connected digraph $D$ is called *$\lambda'_k$-connected*, if $\lambda'_k(D)$ exists.

**Proposition 1.1 (Volkmann [12] 2007).** *Let $k \geq 2$ be an integer. A strongly connected digraph $D$ is $\lambda'_k$-connected, if and only if $D$ contains at least $k$ pairwise vertex-disjoint cycles.*

**Observation 1.2.** It is well-known (cf. Bang-Jensen and Gutin [2], p. 554) that the problem of finding at least $k \geq 2$ vertex-disjoint cycles in a digraph is **NP**-complete. Applying Proposition 1.1, we observe that the recognition problem, whether $\lambda'_k(D)$ exists for a strongly connected digraph $D$, is **NP**-complete too.

In this paper we will characterize the $\lambda_2'$-connected local tournaments and tournaments. These characterizations (cf. Theorem 3.1 and Corollary 3.2) show that the recognition problem, whether a strongly connected local tournament or tournament of order $n$ and size $m$ is $\lambda_2'$-connected, is solvable in time $O(n(n+m))$ (cf. Remark 4.1).

In addition, we characterize all $\lambda'$-connected tournaments, multipartite tournaments, local tournaments and in-tournaments.

The following results play an important role in our investigations.

**Theorem 1.3 (Moon [9] 1966).** *Every strong tournament is vertex pancyclic.*

**Theorem 1.4 (Bondy [4] 1976).** *Each strong c-partite tournament contains an m-cycle for each* $m \in \{3, 4, \ldots, c\}$.

Let $T_R$ be the 3-regular tournament of order seven consisting of the cycle $x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_1$ such that

$$x_1 \to x_3 \to x_5 \to x_1 \to x_6 \to x_2 \to x_7 \to x_3 \to x_6 \to x_4 \to x_2 \to x_5 \to x_7 \to x_4 \to x_1.$$

Notice that $T_R$ is the unique Hadamard tournament of order 7 which contains no transitive subtournament of order 4.

**Theorem 1.5 (Reid [10] 1985).** *Let $T$ be a 2-connected tournament of order $n \geq 6$. If $T \neq T_R$, then $T$ contains two vertex-disjoint cycles of lengths 3 and $n-3$.*

**Theorem 1.6 (Bang-Jensen [1] 1990).** *Let $D$ be a strongly connected local tournament, and let $S$ be a minimal separating set of $D$. The strong components of $D-S$ are tournaments and they can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i \to D_{i+1}$ for $i = 1, 2, \ldots, p-1$.*

**Theorem 1.7 (Bang-Jensen, Huang, Prisner [3] 1993).** *An in-tournament is Hamiltonian if and only if it is strong.*

**Theorem 1.8 (Bang-Jensen, Huang, Prisner [3] 1993).** *Let $D$ be a strong in-tournament, and let $S$ be a minimal separating set of $D$. The strong components of $D-S$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and there exists a vertex $x_i \in V(D_i)$ such that $x_i \to D_{i+1}$ for $i = 1, 2, \ldots, p-1$.*

**Theorem 1.9 (Guo, Volkmann [5] 1994).** *Every partite set of a strongly connected c-partite tournament $D$ contains at least one vertex that lies on cycles of each length $m$ for $m \in \{3, 4, \ldots, c\}$.*

Let $D^1_{GV}$ be the local tournament of order 6 consisting of the cycle $u_1u_2u_3u_4u_5u_6u_1$ such that $u_1 \to u_3 \to u_6 \to u_2 \to u_4 \to u_6$ and $u_2 \to u_5 \to u_3$.

Let $D^2_{GV}$ be the local tournament of order 7 consisting of the cycle $v_1v_2v_3v_4v_5v_6v_7v_1$ such that $v_3 \rightarrow v_5 \rightarrow v_7 \rightarrow v_2 \rightarrow v_5$, $v_6 \rightarrow v_1 \rightarrow v_3 \rightarrow v_6 \rightarrow v_4 \rightarrow v_2$ and $v_1 \rightarrow v_4 \rightarrow v_7$.

**Theorem 1.10 (Guo, Volkmann [6], [7] 1994, 1996).** *Let $D$ be a 2-connected local tournament of order $n \geq 6$. Then $D$ is cycle complementary, if and only if $D \neq T_R, D^1_{GV}, D^2_{GV}$ and $D$ is not the second power of an odd cycle.*

## 2 $\lambda'$-connectedness

In view of Theorem 1.3, every strongly connected tournament $T_n$ of order $n \geq 5$ is $\lambda'$-connected. In our first result we will characterize all $\lambda'$-connected multipartite tournaments.

**Theorem 2.1.** *Let $V_1, V_2, \ldots V_c$ be the partite sets of a strongly connected $c$-partite tournament $D$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c \geq 2$ and $n(D) \geq 5$, then $D$ is $\lambda'$-connected, if and only if $c \geq 4$ or $c = 3$ and $|V_2| \geq 2$ or $c = 2$ and $|V_1| \geq 3$.*

**Proof.** If $c \geq 5$, then, by Theorem 1.4, there exists a 3-cycle $C$ through exactly 3 partite sets. Hence $D - V(C)$ is at least 2-partite and contains thus an arc. Since $n(D) \geq 5$, we deduce in the case $c = 4$ that $|V_4| \geq 2$. Applying Theorem 1.9, we observe that there is a 3-cycle through a vertex of $V_4$. Therefore $D - V(C)$ contains an arc.

According to Theorem 1.4, there exists a 3-cycle $C$ through all partite sets when $c = 3$. The hypothesis $2 \leq |V_2| \leq |V_3|$ shows that there exists an arc in $D - V(C)$. Obviously, $D$ is not $\lambda'$-connected when $|V_1| = |V_2| = 1$.

In the case $c = 2$ it is well-known and easy to see that $|V_1| \geq 2$, and that $D$ contains a 4-cycle $C'$ such that $|V(C') \cap V_i| = 2$ for $i = 1, 2$. If $3 \leq |V_1| \leq |V_2|$, then $D - V(C')$ contains at least two adjacent vertices and so $D$ is $\lambda'$-connected. However, if $|V_1| = 2$, then $D - V(C)$ is the empty graph for each cycle $C$ in $D$. □

It is easy to see that the following family $H_1$ of in-tournaments is not $\lambda'$-connected.

Let $C = x_1x_2 \ldots x_nx_1$ be a cycle with $n \geq 5$. The family $H_1$ consists of the cycle $C$ and the cycle $C$ together with any of the arcs $x_ix_{i+2}$ such that the following conditions are fulfilled. If $x_i \rightarrow x_{i+2}$ and $x_{i+1} \rightarrow x_{i+3}$, then the arc $x_{i+2}x_{i+4}$ is not admissible, and if $x_i \rightarrow x_{i+2} \rightarrow x_{i+4}$, then the arc $x_{i+1}x_{i+3}$ is not admissible. All subscripts are taken modulo $n$.

**Theorem 2.2.** *Let $D$ be a strongly connected in-tournament of order $n \geq 5$. Then $D$ is $\lambda'$-connected with exception of the case that $D$ is a member of the family $H_1$.*

**Proof.** Assume first that $\delta^+(D) \geq 2$. According to Theorem 1.7, $D$ has a Hamiltonian cycle $C = x_1x_2 \ldots x_nx_1$. If $x_i \rightarrow x_{i+t}$ for any $3 \leq t \leq n-2$, then there exists the cycle $C' = x_ix_{i+t}x_{i+t+1} \ldots x_i$ and $D - V(C')$ contains the arc $x_{i+1}x_{i+2}$, and thus $D$ is $\lambda'$-connected. Otherwise, $\delta^+(D) \geq 2$ implies that $x_1 \to x_3$, $x_2 \to x_4$ and $x_3 \to x_5$ and therefore $D$ has the cycle $C'' = x_1x_3x_5x_6 \ldots x_1$ such that $x_2x_4$ is an arc of $D - V(C'')$.

Assume second that $\delta^+(D) = 1$. Then $D$ has a cut-vertex $x_1$. In view of Theorem 1.8, the strong components of $D - x_1$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and there exits an arc with tail in $D_i$ and head in $D_{i+1}$ for $i = 1, 2, \ldots, p-1$. Since there is at least one arc from $D_p$ to $x_1$ and one arc from $x_1$ to $D_1$, it is easy to see that $D$ is $\lambda'$-connected when one of the strong components $D_1, D_2, \ldots, D_p$ is non-trivial. Hence it remains the case that $p = n-1$. Now we define $x_{i+1} = D_i$ for $1 \leq i \leq n-1$. If $x_i \to x_{i+t}$ for any $3 \leq t \leq n-2$, then we observe as above that $D$ is $\lambda'$-connected. Finally, it is straightforward to verify that $D$ is $\lambda'$-connected or $D$ is a member of the family $H_1$.
□

Since local tournaments are also in-tournaments and all members of the family $H_1$ are even local tournaments, Theorem 2.2 immediately yields the next result.

**Corollary 2.3.** *Let $D$ be a strongly connected local tournament of order $n \geq 5$. Then $D$ is $\lambda'$-connected with exception of the case that $D$ is a member of the family $H_1$.*

## 3 All strong local tournaments that are $\lambda_2'$-connected

Firstly we will characterize all strongly connected local tournaments of order $n \geq 6$ which are $\lambda_2'$-connected. It is a simple matter to verify that the following members of the family $F^*$ of strongly connected local tournaments are not $\lambda_2'$-connected.

**The family $F^*$ of local tournaments.** Let $D'$ be a strong local tournament with a cut-vertex $x$. Then Theorem 1.6 implies that the strong components of $D' - x$ are tournaments and they can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i \to D_{i+1}$ for $1 \leq i \leq p-1$.

(i) If $D_1, D_2, \ldots, D_p$ are all trivial such that $x \to D_1$, $D_p \to x$ and arbitrary arcs between $x$ and $\{D_2, D_3, \ldots, D_{p-1}\}$ as well as arbitrary arcs from $D_i$ to $D_j$ for $1 \leq i < j \leq p$ such that the resulting digraph is a local tournament, then we arrive at the first family $F_1$.

Next assume that all strong components of $D' - x$ are trivial with exception of $D_t$.

(ii) In the case that $2 \leq t \leq p-1$, let $D_t$ be a 3-cycle, $x \to D_1$ and $D_p \to x$. If we assume that there are no arcs from $D_i$ to $D_j$ for $1 \leq i \leq t-1$ and $t+1 \leq j \leq p$, no arcs from $D_i$ to $x$ for $2 \leq i \leq t-1$ and no arcs from $x$ to $D_j$ for $t+1 \leq j \leq p$ and arbitrary further arcs such that the resulting digraph is a local tournament, then we arrive at the second family $F_2$.

If $t = 1$, then assume that $D_1$ has a cut-vertex $u$ such that $x \to u$, $D_p \to x$ and that there is no arc from $x$ to $\{D_2, D_3, \ldots, D_{p-1}\}$.

(iii) If $D_1$ is a 3-cycle $uu_1u_2u$, then we arrive at the third family $F_3$, where $x$ and $u_2$ are adjacent and the other arcs are arbitrary such that the resulting digraph is a local tournament.

If $D_1$ has at least four vertices, then assume that the strong components of $D_1 - u$ consist of single vertices $u_1, u_2, \ldots, u_s$ such that $u_i \rightarrow u_j$ for $1 \leq i < j \leq s$, $u \rightarrow u_1$, $u_s \rightarrow u$, $D_i \rightarrow D_j$ for $1 \leq i < j \leq p$ and $D_i \rightarrow x$ for $2 \leq i \leq p$.

(iv) If $u_s \rightarrow x$, $x \rightarrow u_1$, $\{u_2, u_3, \ldots, u_{s-1}\} \rightarrow u$ and there are no arcs from the cut-vertex $x$ to $\{u_2, u_3, \ldots, u_{s-1}\}$ such that the resulting digraph is a local tournament, then we conclude that $\{u_2, u_3, \ldots, u_{s-1}\} \rightarrow x$, and we arrive at the fourth family $F_4$.

(v) Next assume that $u_s \rightarrow x$ and $u_1 \rightarrow x$ or $u_1$ and $x$ are not adjacent. If there are no arcs from $x$ to $\{u_2, u_3, \ldots, u_{s-1}\}$ and arbitrary arcs between $u$ and $\{u_2, u_3, \ldots, u_{s-1}\}$ such that the resulting digraph is a local tournament, then we arrive at the fifth family $F_5$.

(vi) If $x \rightarrow u_s$, then the sixth family $F_6$ consists of tournaments with the propositions $\{u_1, u_2, \ldots, u_{s-1}\} \rightarrow x$ and $u \rightarrow \{u_2, u_3, \ldots, u_{s-1}\}$.

Finally, we define $F^*$ as the union of $F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6$ together with the converse of these local tournaments.

**Theorem 3.1.** *A strongly connected local tournament $D$ of order $n \geq 6$ is $\lambda_2'$-connected if and only if $D$ is not the second power of an odd cycle, $D \neq D_{GV}^1$ and $D$ is not a member of the family $F^*$.*

**Proof.** If $D$ is 2-connected, then the desired result follows from Theorem 1.10, since $T_R$ has the two vertex-disjoint 3-cycles $x_2x_3x_6x_2$ and $x_4x_5x_7x_4$, $D_{GV}^2$ has the two vertex-disjoint 3-cycles $v_1v_3v_6v_1$ and $v_2v_5v_7v_2$ and since the shortest cycle of the second power of an odd cycle of length $2m+1$ has length $m+1$ for $m \geq 2$.

In the case that $D$ is not 2-connected, assume that $x$ is a cut-vertex of $D$. According to Theorem 1.6, the strong components of $D - x$ are tournaments and they can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and $D_i \rightarrow D_{i+1}$ for $i = 1, 2 \ldots, p-1$. In addition, there is at least one arc from $D_p$ to $x$ and one arc, say $xu$, from $x$ to $D_1$. If all components of $D - x$ are trivial, then we arrive at the family $F_1$. If $D - x$ has at least two non-trivial strong component, then $D$ is $\lambda_2'$-connected. Thus assume in the following that there is exactly one non-trivial strong component $D_t$.

Assume that $2 \leq t \leq p-1$. If $|V(D_t)| \geq 4$, then, in view of Theorem 1.3, the tournament $D_t$ contains a 3-cycle $C_3$. If $x_t \in V(D_t) - V(C_3)$, then $C_3$ is vertex-disjoint to the cycle $xD_1D_2 \ldots D_{t-1}x_tD_{t+1} \ldots D_px$ and thus $D$ is $\lambda_2'$-connected. Hence assume now that $D_t$ is a 3-cycle. If there is an arc from $D_i$ to $D_j$ for $1 \leq i \leq t-1$ and $t+1 \leq j \leq p$ or an arc from $D_i$ to $x$ for $2 \leq i \leq t-1$ or an arc from $x$ to $D_j$ for $t+1 \leq j \leq p-1$, then it is easy to see that $D$ is $\lambda_2'$-connected. Otherwise, we obtain a member of the family $F_2$ or its converse.

Next we assume, without loss of generality, that $D_1$ is a non-trivial strong component. If there is an arc $xD_i$ for $2 \leq i \leq p$, then there are two vertex-disjoint cycles in $D$, one in $D_1$ and the other one is $xD_iD_{i+1}\dots D_{p-1}D_px$, and consequently $D$ is $\lambda_2'$-connected. Hence we assume in the following that there is no arc from $x$ to $\{D_2, D_3, \dots, D_{p-1}\}$. If $u$ is not a cut-vertex of $D_1$, then $D$ contains the cycle $xuD_2D_3\dots D_{p-1}D_px$ and each cycle in the strong tournament $D_1 - u$ is vertex-disjoint to this cycle, and so $D$ is $\lambda_2'$-connected. If $D_1 - u$ contains a non-trivial strong component $H$, then $H$ and the cycle $xuD_2D_3\dots D_{p-1}D_px$ are vertex-disjoint, and thus $D$ is $\lambda_2'$-connected. Hence we now investigate the case that $u$ is a cut-vertex of $D_1$ such that the strong components of $D_1 - u$ consist of single vertices $u_1, u_2, \dots, u_s$ such that $u_i \to u_{i+1}$ for $i = 1, 2\dots, s-1$ and that there is no arc from $u_j$ to $u_i$ for $1 \le i < j \le s$. Since $D_1$ is a tournament, it follows that $u_i \to u_j$ for $1 \le i < j \le s$.

If $D_1$ is a 3-cycle, then it is a simple matter to verify that $D$ belongs to $F_3$ or its converse, and thus $D$ is not $\lambda_2'$-connected.

Assume now that $|V(D_1)| \ge 4$. Since $D$ is a local tournament, we observe that $x$ and $u_s$ are adjacent.

First assume that $u_s \to x$. Since $D$ is a local tournament, we conclude that $D_i \to D_j$ for $1 \le i < j \le p$ and $D_i \to x$ for $2 \le i \le p$. If there exists a vertex $u_r$ with $x \to u_r$ for $2 \le r \le s-1$, then there are the vertex-disjoint 3-cycles $xu_rD_px$ and $uu_1u_su$, and $D$ is $\lambda_2'$-connected. Hence we assume next that there is no arc from $x$ to $\{u_2, u_3, \dots, u_{s-1}\}$.

If $x \to u_1$ and there exists a vertex $u_r$ such that $u \to u_r$ for $2 \le r \le s-1$, then there are the vertex-disjoint 3-cycles $xu_1D_px$ and $uu_ru_su$, and $D$ is $\lambda_2'$-connected. Using the fact that $D_1$ is a tournament, we arrive at the family $F_4$ or its converse in the remaining cases.

If $u_1 \to x$ or $u_1$ and $x$ are not adjacent, then we arrive at the family $F_5$ or its converse, and $D$ is not $\lambda_2'$-connected.

Finally, assume that $x \to u_s$. Since $D$ is a local tournament, we deduce that $u_i$ is adjacent to $x$ for $1 \le i \le s-1$. If there exists a vertex $u_r$ such that $x \to u_r$ for any $2 \le r \le s-1$, then there exist the vertex-disjoint 3-cycles $uu_1u_su$ and $xu_rD_px$, and $D$ is $\lambda_2'$-connected. Hence we assume now that $\{u_2, u_3, \dots u_{s-1}\} \to x$. If there exists a vertex $u_r$ such that $u_r \to u$ for any $2 \le r \le s-1$, then there exist the vertex-disjoint 3-cycles $uu_1u_ru$ and $xu_sD_px$, and $D$ is $\lambda_2'$-connected. Hence we assume next that $u \to \{u_2, u_3, \dots u_{s-1}\}$. If $x \to u_1$, then there are the two vertex-disjoint 3-cycles $uu_2u_su$ and $xu_1D_px$, and $D$ is $\lambda_2'$-connected. Consequently there remains the case that $D$ belongs to the family $F_6$ of tournaments or its converse, and then $D$ is not $\lambda_2'$-connected. □

If we reduce the exceptional digraphs in Theorem 3.1 to tournaments, then we obtain immediately the following result.

**Corollary 3.2.** *A strongly connected tournament $T$ of order $n \ge 6$ is $\lambda_2'$-connected if and only if $T$ is not a member of the family $T^*$, described below.*

**The family $T^*$ of tournaments.** Let $T'$ be a strong tournament with a cut-vertex $x$. Then it is well-known that the strong components of $T' - x$ can be ordered in a unique way $D_1, D_2, \ldots, D_p$ such that $D_i \to D_j$ for $1 \le i < j \le p$.

(a) If $D_1, D_2, \ldots, D_p$ are all trivial such that $x \to D_1$, $D_p \to x$ and arbitrary arcs between $x$ and $\{D_2, D_3, \ldots, D_{p-1}\}$, then we arrive at the first family $T_1$ corresponding to $F_1$.

Next assume that $D_1$ is a non-trivial strong component with a cut-vertex $u$ such that $x \to u$ and $D_2, D_3, \ldots, D_p$ are trivial strong components such that $D_i \to x$ for $2 \le i \le p$.

(b) If $D_1$ is a 3-cycle $uu_1u_2u$, then we arrive at the second family $T_3$ corresponding to $F_3$, where the arcs between $u_1$ and $x$ as well as between $u_2$ and $x$ are arbitrary.

If $D_1$ has at least four vertices, then assume that the strong components of $D_1 - u$ consists of single vertices $u_1, u_2, \ldots, u_s$ such that $u_i \to u_j$ for $1 \le i < j \le s$, $u_i \to x$ for $2 \le i \le s-1$, $u \to u_1$ and $u_s \to u$.

(c) If $u_s \to x \to u_1$ and $\{u_2, u_3, \ldots, u_{s-1}\} \to u$, then we obtain the third family $T_4$ corresponding to $F_4$.

(d) If $\{u_1, u_s\} \to x$ and there are arbitrary arcs between $u$ and $\{u_2, u_3, \ldots, u_{s-1}\}$, then we arrive at the fourth family $T_5$ corresponding to $F_5$.

(e) If $u_1 \to x \to u_s$ and $u \to \{u_2, u_3, \ldots, u_{s-1}\}$, then we obtain the fifth family $T_6$ corresponding to $F_6$.

As above, we define $T^*$ as the union $T_1 \cup T_3 \cup T_4 \cup T_5 \cup T_6$ together with the converse of these tournaments.

## 4 Concluding remarks

In the following remark we determine the complexity of the recognition problem, whether a strongly connected local tournament is $\lambda_2'$-connected.

**Remark 4.1.** To decide whether a strongly connected local tournament $D$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a member of $F^*$ we can perform the following steps.

1. For $j = 1, 2, \ldots, n$ determine the strong components of $D - v_j$;

   a) if there is an index $j$ such that $D - v_j$ is not strong, let $i$ be the minimal such index;

   b) otherwise $D - v_j$ is strong for each $j$ and thus $D$ is 2-connected and therefore not a member of $F^*$.

2. Determine the strong decomposition $D_1, D_2, \ldots, D_p$ of $D - v_i$, where $p \ge 2$;

   a) if $|V(D_j)| \ge 3$ for at least two indices $j$, then $D$ is $\lambda_2'$-connected;

   b) if $|V(D_j)| = 1$ for each index $j$, then $D$ is a member of $F_1$;

   c) otherwise $|V(D_t)| \ge 3$ for a single index $t$.

3. Determine the single index $t$ with $|V(D_t)| \geq 3$;

   a) if $2 \leq t \leq p-1$ check all arcs between $v_i$ and $V(D) - V(D_t)$ to determine whether $D$ is a member of $F_2$;

   b) otherwise assume, without loss of generality, that $t = 1$.

4. Determine $|V(D_1)|$ and check the arcs between $v_i$ and $D_p$;

   a) if $D_p \not\rightarrow v_i$, then $D$ is $\lambda_2'$-connected;

   b) if $D_p \rightarrow v_i$ and $|V(D_1)| = 3$, then $D$ is a member of $F_3$;

   c) otherwise $D_p \rightarrow v_i$ and $|V(D_1)| \geq 4$.

5. Determine the strong components of $D_1 - u$, where $u$ is an out-neighbor of $v_i$ in $D_1$;

   a) if $D_1 - u$ is strong, then $D$ is $\lambda_2'$-connected;

   b) otherwise $D_1 - u$ is not strong.

6. Determine the strong decomposition $A_1, A_2, \ldots, A_q$ of $D_1 - u$, where $q \geq 2$;

   a) if $|V(A_j)| \geq 3$ for an index $j$, then $D$ is $\lambda_2'$-connected;

   b) if $|V(A_j)| = 1$ for each index $j$, check all arcs of $D[\{v_i\} \cup V(D_1)]$ to determine whether $D$ is a member of $F_4 \cup F_5 \cup F_6$.

Let $m$ be the size of $D$. It is well-known that there exist algorithms to determine the strong components of a digraph in time $O(n+m)$ (see Tarjan [11]) and the acyclic ordering of an acyclic connected digraph in time $O(n+m)$ (see Bang-Jensen and Gutin [2]). Therefore we can check whether a local tournament $D$ is a member of $F^*$ in time $O(n(n+m))$.

The next result is a generalization of Theorem 1.10.

**Theorem 4.2 (Meierling, Volkmann [8]).** *Let $D$ be a 2-connected in-tournament of order $n \geq 6$. Then $D$ is cycle complementary if and only if $D \neq T_R, D_{GV}^1, D_{GV}^2$ or $D$ is not the second power of an odd cycle.*

Theorem 4.2 shows that all 2-connected in-tournaments $D$ of order $n \geq 6$ are $\lambda_2'$-connected with exception of the case that $D = D_{GV}^1$ or $D$ is the second power of an odd cycle.

**Remark 4.3.** The same method used in the proof of Theorem 3.1 also leads to a similar result for strongly connected in-tournaments. But the proof is a very clumsy and boring case analysis and thus the result would not be very attractive and is therefore not mentioned here in detail. Similar observations as in Remark 4.1 lead to the conclusion that the recognition problem, whether a strongly connected in-tournament is $\lambda_2'$-connected, is also solvable in polynomial time.

Two vertex disjoint cycles $C$ and $C'$ of a multipartite tournament are called *weakly complementary*, if they contain vertices from all partite sets. The main theorem in [13] says

**Theorem 4.4 (Volkmann, Winzen [13]).** *Let $D$ be a $c$-partite tournament with $c \geq 3$, $n(D) \geq 6$ and $\kappa(D) \geq 3$. Then $D$ is weakly cycle complementary unless $D$ is isomorphic to $T_R$.*

This theorem implies that all $c$-partite tournaments with $c \geq 3$, $n(D) \geq 6$ and $\kappa(D) \geq 3$ are $\lambda'_2$-connected.

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