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On removable series classes in connected matroids

Y. M. Borse*

Department of Mathematics
S. P. College, Pune
India
ymborse@math.unipune.ernet.in

B. N. Waphare†

Department of Mathematics
University of Pune, Pune
India
bnwaph@math.unipune.ernet.in

Abstract

A series class P of a connected matroid M is *removable* if $M \setminus P$ is connected. In this paper, we prove that a connected matroid M with $r^*(M) \geq 2$ has at least $r^*(M) + 1$ removable series classes. Further, we obtain certain results from which the following result of Oxley and its graph theoretic version follow: If C is a circuit of a connected matroid M with $C \neq M$ such that $M \setminus x$ is not connected for all $x \in C$, then C contains at least two nontrivial series classes of M.

1 Introduction

All graphs considered here are loopless. Given a connected graph G with n vertices, let $r^*(G) = |E(G)| - n + 1$, where $E(G)$ is the set of edges of G. An *ear* in a graph G is a maximal path whose all internal vertices have degree two in G. Given a matroid M, let $E(M)$, $r(M)$ and $r^*(M)$ respectively denote the ground set, the rank and the corank of M. Let M be a matroid and let $x, y \in E(M)$. We say that x and y are in *series* if $\{x, y\}$ is a 2-cocircuit. A *series class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct elements of A, then a and b are in series. A series class P of a connected matroid M is *removable* if $M \setminus P$ is connected. An ear P

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of a 2-connected graph G is *removable* if $G - P$ is 2-connected, where $G - P$ is the graph with edge set $E(G) - E(P)$ and vertex set $V(G) - \{\text{internal vertices of } P\}$. A connected matroid M is *minimally connected* if $M \setminus x$ is not connected for all $x \in E(M)$. We follow the terminologies of Oxley [4] and West [8].

Zhang and Guo [10] proved that a 2-connected graph G with $r^*(G) \geq 2$ has at least $r^*(G) + 1$ removable ears. We extend this result to matroids as follows.

Theorem 1.1. *Let M be a connected matroid with $r^*(M) \geq 2$. Then M has at least $r^*(M) + 1$ removable series classes.*

Since every removable series class of a minimally connected matroid is nontrivial, it follows from Theorem 1.1 that a minimally connected matroid M has at least $r^*(M) + 1$ nontrivial removable series classes. This strengthens the result of Oxley [3] which states that a minimally connected matroid M with $r^*(M) \geq 2$ has at least $r^*(M) + 1$ nontrivial series classes. This result of Oxley is an improvement on the results of Seymour [6], [7], Murty [2] and White [9] regarding existence of a 2-cocircuit in a minimally 2-connected matroid.

Further, we obtain the following two results.

Theorem 1.2. *Let C be a circuit of a connected matroid M with $|E(M)| \geq 2$ and let $e \in C$. Let B be a component of $M \setminus e$ with $|B| \geq 2$ such that no two elements of $B \cap C$ are in series in M. Then there exists an element f in $B \cap C$ such that $M \setminus f$ is connected.*

Theorem 1.3. *Let C be a cycle of a 2-connected graph G and let e be an edge of C. Let B be a block of $G - e$ with $|E(B)| \geq 2$ such that B contains no vertex of C whose degree is two in G. Then there exists an edge f in $E(B) \cap E(C)$ such that $G - f$ is 2-connected.*

Oxley [3] proved the following result, and its graph theoretic version, which extends a result of Dirac [1] and Plummer [5] for minimally 2-connected graphs.

Theorem 1.4 . *Let C be a circuit of a connected matroid M with $C \neq M$ such that $M \setminus x$ is not connected for all $x \in C$. Then C contains at least two distinct nontrivial series classes of M.*

We deduce Theorem 1.4 from Theorem 1.2, and the graph theoretic version of Theorem 1.4 from Theorem 1.3.

2 Proofs

An element e of a connected matroid M is *removable* if $M \setminus e$ is connected.

Proof of Theorem 1.1. We prove the result by induction on $r^*(M)$. First note that the result can be easily checked if $r^*(M) = 2$. Thus we may assume that $r^*(M) > 2$. Contract all but one element from each nontrivial series class of M to get a connected matroid N with $r^*(N) = r^*(M)$ and having no nontrivial series class. If e is an element of N such that $N \setminus e$ is connected, then the series class of M containing e is removable. Thus the number of removable series classes of M is at least the number of removable elements of N. If N is minimally connected, then it has a nontrivial series class, which is a contradiction. Hence N has an element x such that $N \setminus x$ is connected. Obviously, $r^*(N \setminus x) = r^*(N) - 1 \geq 2$. Apply the induction assumption to $N \setminus x$. It has at least $r^*(N \setminus x) + 1$ removable series classes. Let P be a removable series class of $N \setminus x$.

Claim. There exists an element of P which is removable in N.

Let $y \in P$. Suppose that $P = \{y\}$. Since N does not have a nontrivial series class, there is a circuit C of N containing x but not containing y. Therefore C intersects $N \setminus x \setminus y$. This implies that $N \setminus y$ is connected. Suppose that $|P| \geq 2$. Let $y_1 \in P - y$. As y and y_1 do not belong to the same series class of N, there exists a circuit C_1 of N containing y_1 such that $y \notin C_1$. Hence $x \in C_1$. This implies that x and $P - y$ belong to the same component of $N \setminus y$. Therefore, if C_1 contains an element of $N \setminus x \setminus P$, then $N \setminus y$ is connected.

Suppose that C_1 does not contain an element of $N \setminus x \setminus P$. We prove that $N \setminus y_1$ is connected. By the above arguments, $P - y_1$ and x belong to the same component of $N \setminus y_1$. Let y_2 be an element of $N \setminus x \setminus P$. Let C_2 be a circuit of $N \setminus x$ containing y_1 and y_2. There exists a circuit C_3 such that $y_2 \in C_3 \subseteq (C_1 \cup C_2) - y_1$. As C_3 intersects $C_1 - y_1$, and P is a series class of $N \setminus x$, it follows that $x \in C_3$. Therefore C_3 and $P - y_1$ belong to the same component of $N \setminus y_1$. This implies that that $N \setminus y_1$ is connected.

From the above claim it follows that there are at least $r^*(N \setminus x) + 1$ elements of $N \setminus x$ which are removable in N. Thus N has at least $r^*(N) + 1$ removable elements. Hence M has at least $r^*(M) + 1$ removable series classes. $\square$

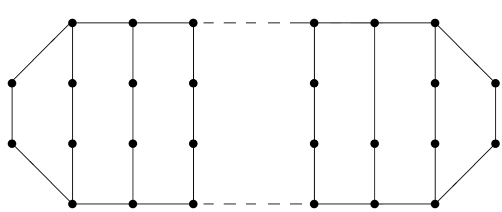

Fig. 1

The graph G of Figure 1, which is constructed by Zhang and Guo [10], has precisely $r^*(G) + 1$ removable ears. The cycle matroid M of this graph has precisely $r^*(M) + 1$ removable series classes. Hence the lower bound for the number of removable series classes given in Theorem 1.1 is sharp. As far as an upper bound is concerned, note that *every* element of the uniform matroid $U_{m,n}$ with $m \geq 1$, $n \geq m + 2$ is a removable series class.

We now prove Theorem 1.2 and derive Theorem 1.4 from it.

The proof of the following lemma is trivial.

Lemma 2.1. *Let C be a circuit of a connected matroid M with $M \neq C$. Let P be a series class of M such that $P \cap C \neq \emptyset$. Then every component of $M \setminus P$ intersects with C. In particular, if $e \in C$, then every component of $M \setminus e$ intersects with C.* □

Lemma 2.2. *Let C be a circuit of a connected matroid M with $|E(M)| \geq 2$ and let $e \in C$. Let B be a component of $M \setminus e$ with $|B| \geq 2$ such that no two elements of $B \cap C$ are in series in M. For $e_1 \in B \cap C$, if $M \setminus e_1$ is not connected, then $M \setminus e_1$ has a component B_1 such that $B_1 \subseteq B - e_1$, $|B_1| \geq 2$, $C \cap B_1 \neq \emptyset$.*

Proof. By Lemma 2.1, $B \cap C \neq \emptyset$. Let $e_1 \in B \cap C$. Since B is connected, it has a circuit C_1 containing e_1. There exists a circuit C_2 of M such that $e \in C_2 \subseteq (C_1 \cup C) - e_1$. By Lemma 2.1, C_2 intersects all components of $M \setminus e$. This implies that C_2 and $M \setminus B$ belong to the same component of $M \setminus e_1$. Suppose that $M \setminus e_1$ is not connected. Let B_1 be a component of $M \setminus e_1$ with $B_1 \cap C_2 = \emptyset$. Then $B_1 \subseteq B - e_1$. By Lemma 2.1, $B_1 \cap C \neq \emptyset$. Since no two elements of $C \cap B$ are in series, $|B_1| \geq 2$. □

Proof of Theorem 1.2. By Lemma 2.1, $C \cap B \neq \emptyset$. Let e_1 be any element of $C \cap B$. If $M \setminus e_1$ is connected, then the result follows. Otherwise, by Lemma 2.2, there exists a component B_1 of $M \setminus e_1$ such that $B_1 \subseteq B - e_1$, $|B_1| \geq 2, C \cap B_1 \neq \emptyset$. Further, no two elements of $B_1 \cap C$ are in series in M.

Let $e_2 \in B_1 \cap C$. If $M \setminus e_2$ is connected, then the result holds. Otherwise, by Lemma 2.2, there exists a component B_2 of $M \setminus e_2$ such that $B_2 \subseteq B_1 - e_2 \subseteq B - \{e_1, e_2\}$, $|B_2| \geq 2$, $C \cap B_2 \neq \emptyset$. Obviously, no two elements of $B_2 \cap C$ are in series in M.

Since B is finite, continuing the above process we get an element, say e_i in $C \cap B$ such that $M \setminus e_i$ has a component B_i with the property that $B_i \subset B$ and $C \cap B_i \neq \emptyset$ and $M \setminus f$ is connected for all $f \in C \cap B_i$. □

Corollary 2.3. *Let C be a circuit of a connected matroid M with $|E(M)| \geq 2$. Suppose that no two elements of C are in series in M. Then there exist distinct elements f_1 and f_2 of C such that both $M \setminus f_1$ and $M \setminus f_2$ are connected.*

Proof. Since M is connected, $|C| \geq 2$. Let $e \in C$. Let D be any component of $M \setminus e$. It suffices to prove that D contains an element f of C such that $M \setminus f$ is connected. By Lemma 2.1, $D \cap C \neq \emptyset$. Since no two elements of C are in series in M, $|D| \geq 2$. By Theorem 1.2, there is $f \in D \cap C$ such that $M \setminus f$ is connected. □

Proof of Theorem 1.4. By Corollary 2.3, at least two elements x and y of C are in series in M. Let P be the series class of M containing x and y. Then $P \subset C$. Since $M \neq C$, $M \setminus x$ has a nontrivial component D. Obviously, $D \cap P = \emptyset$. By Lemma 2.1, $D \cap C \neq \emptyset$. If no two elements of $D \cap C$ are in series in M, then, by Theorem 1.2, there exists $f \in C$ such that $M \setminus f$ is connected, which is a contradiction. Hence at least two elements of $D \cap C$ are in series in M. Since D is connected, the series class P' of M containing this pair of elements is a subset of $D \cap C$. Thus P and P' are distinct nontrivial series classes of M. □

Now, we prove Theorem 1.3.

Lemma 2.4. *Let C be a cycle in a 2-connected graph G and let e be an edge of C. Let B be a block of $G-e$ with $|E(B)| \geq 2$ such that B contains no vertex of C whose degree is two in G. For $e_1 \in E(B) \cap E(C)$, if $G - e_1$ is not 2-connected, then it has a block B_1 such that $B_1 \subseteq B - e_1$, $|E(B_1)| \geq 2$, and $E(B_1) \cap E(C) \neq \emptyset$.*

Proof. Obviously, B contains at least one edge of C. Let $e_1 \in E(B) \cap E(C)$. Suppose that $G - e_1$ is not 2-connected. Then $G - e_1$ is connected and has exactly two end blocks. Each end block of $G - e_1$ contains an end vertex of e_1 as an internal vertex. Since the end vertices of e_1 belong to $V(B) \cap V(C)$, by a hypothesis, both have degree at least 3 in G. This implies that each end block of $G - e_1$ contains at least two edges. Let C_1 be a cycle in B containing e_1. Then there exists a cycle C_2 in G containing e and avoiding e_1. As $e \in E(C_2)$, C_2 contains at least one edge of every block of $G-e$. This implies that the union of C_2 and all blocks of $G-e$ that are different from B is a 2-connected graph and hence it is contained in a block B' of $G - e_1$. Let B_1 be an end block of $G - e_1$ such that $B_1 \neq B'$. Obviously, $|E(B_1)| \geq 2$ and B_1 is contained in $B - e_1$. Since $e_1 \in E(C)$ and B_1 is a block of $G - e_1$, $E(B_1) \cap E(C) \neq \emptyset$. □

Proof of Theorem 1.3. The proof follows from Lemma 2.4 in the same way just as the proof of Theorem 1.2 follows from Lemma 2.2. □

Corollary 2.5. *Let C be a cycle of a 2-connected graph G such that $d_G(v) \geq 3$ for all $v \in V(C)$. Then there are two edges e_1 and e_2 of C such that both $G - e_1$ and $G - e_2$ are 2-connected.*

Proof. If $G-f$ is 2-connected for every edge f of C, then the result follows. Suppose that $G-e$ is not 2-connected for some edge $e \in E(C)$. Then $G-e$ is connected and has exactly two end blocks B_1, B_2. Each of B_1 and B_2 contains an end vertex of e as an internal vertex. Since the end vertices of e have degree at least two in $G-e$, $|E(B_i)| \geq 2$ for $i = 1, 2$. By Theorem 1.3, B_i contains an edge e_i of C such that $G - e_i$ is 2-connected for $i = 1, 2$. □

The following result of Oxley [3], which corresponds to Theorem 1.4 for 2-connected graphs, follows from Theorem 1.3 and Corollary 2.5.

Theorem 2.6. *Let G be a 2-connected graph and let C be a cycle in G with $G \neq C$ such that for every edge e of C, the graph $G-e$ is not 2-connected. Then C contains at least two nontrivial ears of G.*

Proof. By Corollary 2.5, at least one vertex v_1 of C has degree two in G. Let P_1 be the ear of G containing v_1. Since both the edges that are incident with v_1 belong to $E(C)$, $E(P_1) \subset E(C)$. Let $e \in E(P_1)$. Then $G-e$ is connected and has exactly two end blocks. Since $G \neq C$, there exists a block B of $G-e$ such that $|E(B)| \geq 2$ and $E(B) \cap E(P_1) = \emptyset$. Further, B contains at least one edge of C. By Theorem 1.3, B contains a vertex v_2 of C such that $d_G(v_2) = 2$. Let P_2 be the ear of G containing v_2. Then $E(P_2) \subset E(C) \cap E(B)$. Thus C contains distinct nontrivial ears P_1 and P_2 of G. □

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