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# On a problem of Fronček and Kubesa

Ngo Dac Tan

*Institute of Mathematics*
*18 Hoang Quoc Viet Road*
*10307 Hanoi*
*Vietnam*
ndtan@math.ac.vn

**Abstract**

Let $n$ be a positive integer and $T$ be a tree of order $2n$. We say that the complete graph $K_{2n}$ of order $2n$ has a $T$-factorization if there are spanning trees $T_1, \ldots, T_n$ of $K_{2n}$, all isomorphic to $T$, such that each edge of $K_{2n}$ belongs to exactly one of $T_1, \ldots, T_n$. Fronček and Kubesa have raised the following question. Suppose that $K_{2n}$ has a $T$-factorization. Is it true that $T$ possesses a set $X$ of $n$ vertices such that $\sum_{x \in X} \deg_T(x) = 2n-1$? In this paper, we show that the above question has a positive answer if one of the following conditions holds: (i) The degree set $D$ of $T$ has the cardinality at most 3; (ii) The maximum degree $\Delta$ of $T$ is at most 4 or it is at least $n-3$.

## 1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ for short) will denote its vertex set and its edge set, respectively. For a vertex $v \in V(G)$, the degree of $v$, denoted by $\deg_G(v)$, is the number of neighbours of $v$. The maximum degree of $G$, denoted by $\Delta(G)$ (or $\Delta$ for short if $G$ is clear from the context), is the number $\max\{\deg_G(v) \mid v \in V(G)\}$. The degree set of $G$, denoted by $D(G)$ or $D$ for short, is the set $\{\deg_G(v) \mid v \in V(G)\}$. The complete graph of order $n$ is denoted by $K_n$. If graphs $G_1$ and $G_2$ are isomorphic, then we write $G_1 \cong G_2$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

Let $n$ be a positive integer and $T$ be a tree of order $2n$. We say that the complete graph $K_{2n}$ of order $2n$ has a $T$-factorization if there are spanning trees $T_1, T_2, \ldots, T_n$ of $K_{2n}$, all isomorphic to $T$, such that each edge of $K_{2n}$ belongs to exactly one of $T_1, T_2, \ldots, T_n$. The study of $T$-factorizations of $K_{2n}$ was begun not long ago by several authors (see, for example, [2]–[5]). First attempts show that even for very simple classes of trees like caterpillars and lobsters the task is very complex.

At the workshop in Krynica in 2004, Fronček and Kubesa raised the following question, which also appeared recently in [6]. Suppose that $T$ is a tree of order

$2n$ and $K_{2n}$ has a $T$-factorization. Is it then true that the vertex set of $T$ can be decomposed into two subsets $X$ and $Y$ such that $|X| = |Y| = n$ and $\sum_{x \in X} \deg_T(x) = \sum_{y \in Y} \deg_T(y)$? It is clear that this question is equivalent to the following one. Suppose that $T$ is a tree of order $2n$ and $K_{2n}$ has a $T$-factorization. Is it then true that $T$ possesses a set $X$ of $n$ vertices such that $\sum_{x \in X} \deg_T(x) = 2n - 1$? We shall adopt the latter formulation of the question for further consideration.

In this paper, we shall prove that the question of Fronček and Kubesa has a positive answer if one of the following conditions holds: (i) The degree set $D$ of $T$ has cardinality at most 3; (ii) The maximum degree $\Delta$ of $T$ is at most 4 or is at least $n - 3$.

## 2 Results

First of all, we prove the following Lemma 1 for a tree $Q$ with the degree set $D(Q) = \{a, b, 1\}$, where $a$ and $b$ are integers with $a > 1$, $b > 1$ and $a \neq b$. This result is needed later for the proof of Theorem 2, one of our main results in this paper. We note that in this lemma we do not require the tree $Q$ to factorize a complete graph.

**Lemma 1.** *Let $Q$ be a tree with the degree set $D(Q) = \{a, b, 1\}$, where $a$ and $b$ are integers with $a > 1$, $b > 1$ and $a \neq b$. Further, let $t_a, t_b$ and $t_1$ be the numbers of vertices of degrees $a$, $b$, and 1 in $Q$, respectively. Then*

$$t_1 = (a - 2)t_a + (b - 2)t_b + 2. \tag{2.1}$$

*Proof.* We prove this lemma by induction on $t_a + t_b$.

It is clear that the smallest value for $t_a + t_b$ is 2. Furthermore, in the case $t_a + t_b = 2$, we must have $t_a = t_b = 1$ and the vertex of degree $a$ is adjacent to the vertex of degree $b$. So, $T$ has $(a-1)+(b-1)$ vertices of degree 1 and Formula (2.1) is true in this case.

Suppose that Formula (2.1) has been proved to be true for any tree $Q'$ with the degree set $D(Q') = \{a, b, 1\}$ and the sum of the numbers of vertices of degrees $a$ and $b$ in $Q'$ that is less than or equal to an integer $k \geq 2$. We show that Formula (2.1) is also true for any tree $Q$ with the degree set $D(Q) = \{a, b, 1\}$ and $t_a + t_b = k + 1$. Let $\overline{Q}$ be the graph obtained from $Q$ by deleting all vertices of degree 1. Then $\overline{Q}$ is a tree of order $k+1 \geq 3$ and therefore it has at least two vertices of degree 1. Further, since $k + 1 \geq 3$, at least one of $t_a$ and $t_b$ is greater than or equal to 2. From the above remarks for $\overline{Q}$, $t_a$ and $t_b$, it is not difficult to see that we can find a vertex $u$ in $Q$ with the following properties:

(i) $\deg(u)$ is greater than 1;

(ii) among the neighbours of $u$, there is exactly one neighbour with the degree greater than 1;

(iii) there exists in $Q$ another vertex with the degree equal to $\deg(u)$.

For definiteness, without loss of generality we may assume that $\deg(u) = a$. Let $S$ be the subgraph of $Q$ induced by $u$ and all neighbours of degree 1 of $u$. Further, let $Q^*$ be the graph obtained from $Q$ by replacing $S$ by a vertex $u^* \notin V(Q)$. Then

$Q^*$ is a tree and by the properties (i)–(iii) of the chosen vertex $u$, we can see that $D(Q^*) = \{a, b, 1\}$. Denote by $t_a^*$, $t_b^*$ and $t_1^*$ the numbers of vertices of degrees $a$, $b$ and 1 in $Q^*$, respectively. Then by the construction of $Q^*$ we have $t_a^* = t_a - 1$, $t_b^* = t_b$ and $t_1^* = t_1 - (a-1) + 1 = t_1 - (a-2)$. So we have $t_a^* + t_b^* = k$ and therefore by the induction hypothesis, $t_1^* = (a-2)t_a^* + (b-2)t_b^* + 2$. It follows that $t_1 - (a-2) = (a-2)(t_a - 1) + (b-2)t_b + 2$ if and only if $t_1 = (a-2)t_a + (b-2)t_b + 2$ and Formula (2.1) is true for $Q$.

The proof of Lemma 1 is complete. □

Now we formulate and prove our first main result.

**Theorem 2.** *Let $n$ be a positive integer and $T$ be a tree of order $2n$ such that the cardinality of the degree set $D$ of $T$ is at most 3. Further, let $K_{2n}$ have a $T$-factorization. Then $T$ possesses a set $X$ of $n$ vertices such that $\sum_{x \in X} \deg_T(x) = 2n - 1$.*

*Proof.* Since $K_{2n}$ has a $T$-factorization, there exist in $K_{2n}$ spanning trees $T_1, T_2, \ldots, T_n$, all isomorphic to $T$, such that each edge of $K_{2n}$ belongs to exactly one of $T_1, T_2, \ldots, T_n$. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of $K_{2n}$. Consider the following matrix $M$ with $2n$ rows and $n$ columns. The $i$-th row of $M$ is labelled by $v_i$ and the $j$-th column of $M$ is labelled by $T_j$. The $(i, j)$-entry of $M$ is $\deg_{T_j}(v_i)$. Since each edge of $K_{2n}$ belongs to exactly one of the spanning trees $T_1, T_2, \ldots, T_n$ of $K_{2n}$, for each $i$-th row of $M$, where $i \in \{1, 2, \ldots, 2n\}$, we have

$$\sum_{j=1}^{n} \deg_{T_j}(v_i) = \deg_{K_{2n}}(v_i) = 2n - 1, \tag{2.2}$$

where all summands $\deg_{T_j}(v_i)$ are positive.

Let $D$ be the degree set of $T$. Since $T$ is a tree of order at least 2, the number 1 must be in $D$. We consider the following cases.

*Case 1.* $|D| = 1$.

In this case, $D = \{1\}$. Since the only tree $T$ with $D = \{1\}$ is $K_2$, the theorem is trivially true in this case.

*Case 2.* $|D| = 2$.

In this case, $D = \{a, 1\}$ with an integer $a > 1$. Then entries of $M$ are only $a$ or 1. Let $t_a$ and $t_1$ be the number of vertices of degrees $a$ and 1 in $T$, respectively, and let $x_i$ be the number of entries $a$ in the $i$-th row of $M$. For any $i \in \{1, 2, \ldots, 2n\}$, Equality (2.2) becomes $x_i a + (n - x_i) = 2n - 1$, that is, $x_i = \frac{n-1}{a-1}$. It is clear that $x_1 = x_2 = \cdots = x_{2n}$ and therefore the total number of entries $a$ in $M$, if we count them by rows, is $x_1 + x_2 + \cdots + x_{2n} = 2nx_1$. On the other hand, since $T_1 \cong T_2 \cong \cdots \cong T_n \cong T$, it is clear that each column of $M$ has exactly $t_a$ entries $a$ and $t_1$ entries 1. So the total number of entries $a$ in $M$, if we count them by columns, is $nt_a$. So $2nx_1 = nt_a$, that is, $2x_1 = t_a$. Hence, $t_1 = 2n - t_a = 2n - 2x_1 = 2(n - x_1)$. In particular, we get $x_1 < t_a$ and $n - x_1 < t_1$. Therefore, we can choose $x_1$ different vertices of degree $a$, say $u_1, u_2, \ldots, u_{x_1}$, and $n - x_1$ different vertices of degree 1, say $u_{x_1+1}, \ldots, u_n$, in $T$. For these $n$ chosen vertices $u_1, \ldots, u_{x_1}, u_{x_1+1}, \ldots, u_n$, we have

$$\sum_{i=1}^{n} \deg_T(u_i) = ax_1 + (n - x_1) = 2n - 1.$$

The last equality holds because of (2.2). Thus, the theorem is true in this case.

*Case 3.* $|D| = 3$.

In this case, $D = \{a, b, 1\}$, where $a$ and $b$ are integers, $a > 1, b > 1$ and $a \neq b$. Then entries of $M$ are $a$, $b$ or 1. Let $t_a, t_b$ and $t_1$ be the numbers of vertices of degrees $a$, $b$ and 1 in $T$, respectively. Further, let $x_i$ and $y_i$ be the numbers of entries $a$ and $b$ in the $i$-th row of $M$, respectively. Then $x_i$ and $y_i$ are nonnegative integers. It is also clear that the number of entries 1 in the $i$-th row of $M$ is $n - x_i - y_i$. For each $i \in \{1, 2, \ldots, 2n\}$, by (2.2) we have

$$ax_i + by_i + (n - x_i - y_i) = 2n - 1. \tag{2.3}$$

This implies that

$$n = (a - 1)x_i + (b - 1)y_i + 1. \tag{2.4}$$

Now we count the number of entries $a$ in $M$ in two ways: by columns and by rows. Since $T_1 \cong T_2 \cong \cdots \cong T_n \cong T$, it is clear that each column of $M$ has exactly $t_a$ entries $a$. So the total number of entries $a$ in $M$, if we count them by columns, are $nt_a$. On the other hand, if we count them by rows, then the total number of entries $a$ in $M$ is $x_1 + x_2 + \cdots + x_{2n}$. Thus,

$$x_1 + x_2 + \cdots + x_{2n} = nt_a. \tag{2.5}$$

From (2.5) it is not difficult to see that among $x_1, x_2, \ldots, x_{2n}$ there are at least $n+1$ numbers that are less than or equal to $t_a$. By similar arguments, we can show that among $y_1, y_2, \ldots, y_{2n}$ there are at least $n + 1$ numbers that are less than or equal to $t_b$. Therefore, there is at least one $i \in \{1, \ldots, 2n\}$ such that both

$$x_i \leq t_a \quad \text{and} \quad y_i \leq t_b. \tag{2.6}$$

hold.

Now we consider the number $n - x_i - y_i$. Since $T$ is a tree with the degree set $D = \{a, b, 1\}$, by Lemma 1, $t_1$ can be calculated by Formula (2.1). Therefore, by using first (2.4), then (2.6) and finally (2.1), we get

$$\begin{aligned} n - x_i - y_i &= [(a-1)x_i + (b-1)y_i + 1] - x_i - y_i \\ &= (a-2)x_i + (b-2)y_i + 1 \\ &\leq (a-2)t_a + (b-2)t_b + 2 = t_1. \end{aligned}$$

Thus, we also have

$$n - x_i - y_i \leq t_1. \tag{2.7}$$

By (2.6) and (2.7), we can choose in $T$ $x_i$ different vertices of degree $a$, say $u_1, u_2, \ldots, u_{x_i}$, $y_i$ different vertices of degree $b$, say $u_{x_i+1}, u_{x_i+2}, \ldots, u_{x_i+y_i}$, and $n - x_i - y_i$ different vertices of degree 1, say $u_{x_i+y_i+1}, \ldots, u_n$. For these $n$ chosen vertices, we have

$$\sum_{i=1}^{n} \deg_T(u_i) = ax_i + by_i + (n - x_i - y_i) = 2n - 1.$$

The last equality holds because of (2.3). Thus, the theorem is also true in Case 3.

The proof of Theorem 2 is complete. □

Now we prove the second result of the paper.

**Theorem 3.** *Let $n$ be a positive integer and $T$ be a tree of order $2n$ such that either $\Delta \leq 4$ or $\Delta \geq n-3$, where $\Delta$ is the maximum degree of $T$. Further, let $K_{2n}$ have a $T$-factorization. Then $T$ possesses a set $X$ of $n$ vertices such that $\sum_{x\in X} \deg_T(x) = 2n-1$.*

*Proof.* We divide the proof of this theorem into two cases.

*Case 1.* $\Delta \leq 4$.

If $\Delta \leq 3$, then the degree set $D$ of $T$ has the cardinality at most 3. Therefore, by Theorem 2, if $\Delta \leq 3$ or $\Delta = 4$ and $|D| \leq 3$, then Theorem 3 is true. So we may assume further that $\Delta = |D| = 4$. It follows that $D = \{1, 2, 3, 4\}$. Let $t_1, t_2, t_3$ and $t_4$ be the numbers of vertices in $T$ of degree 1, 2, 3 and 4, respectively. Then

$$t_1 > 1, t_2 \geq 1, t_3 \geq 1, t_4 \geq 1 \text{ and}$$
$$t_1 + t_2 + t_3 + t_4 = 2n.$$

Further, since $\sum_{v\in V(T)} \deg_T(v) = 2|E(T)|$, it is clear that

$$t_1 + 2t_2 + 3t_3 + 4t_4 = 2(2n-1).$$

Let $u_1^i, u_2^i, \ldots, u_{t_i}^i$ be the vertices of degree $i$ in $T$, $i \in \{1, 2, 3, 4\}$. Since the number of vertices of odd degrees in a graph must be even, $t_1 + t_3$ is an even number. Therefore, $t_2 + t_4$ is also even because $|V(T)| = 2n$. We consider separately the following subcases.

*Subcase 1.1. $t_4$ is even.*

In this subcase, $t_2$ is even because $t_2 + t_4$ is even. If $t_3$ is even, then $t_1$ is also even. For this situation, let

$$X = \{u_1^1, \ldots, u_{t_1/2}^1, u_1^2, \ldots, u_{t_2/2}^2, u_1^3, \ldots, u_{t_3/2}^3, u_1^4, \ldots, u_{t_4/2}^4\}.$$

Then

$$|X| = \frac{t_1}{2} + \frac{t_2}{2} + \frac{t_3}{2} + \frac{t_4}{2} = \frac{t_1 + t_2 + t_3 + t_4}{2} = \frac{2n}{2} = n, \text{ and}$$

$$\begin{aligned}\sum_{u_j^i\in X}\deg_T(u_j^i) &= \frac{t_1}{2}+2\frac{t_2}{2}+3\frac{t_3}{2}+4\frac{t_4}{2}\\ &= \frac{t_1+2t_2+3t_3+4t_4}{2}\\ &= \frac{2(2n-1)}{2}=2n-1.\end{aligned}$$

So Theorem 3 is true in this situation. If $t_3$ is odd, then $t_1$ is also odd. Since $t_2 \geq 1$ is even, it is at least 2. Let

$$X=\{u_1^1,\dots,u_{(t_1+1)/2}^1,u_1^2,\dots,u_{(t_2-2)/2}^2,u_1^3,\dots,u_{(t_3+1)/2}^3,u_1^4,\dots,u_{t_4/2}^4\}.$$

Then

$$|X|=\frac{t_1+1}{2}+\frac{t_2-2}{2}+\frac{t_3+1}{2}+\frac{t_4}{2}=\frac{t_1+t_2+t_3+t_4}{2}=\frac{2n}{2}=n \text{ and}$$

$$\begin{aligned}\sum_{u_j^i\in X}\deg_T(u_j^i) &= \frac{t_1+1}{2}+2\frac{t_2-2}{2}+3\frac{t_3+1}{2}+4\frac{t_4}{2}\\ &= \frac{t_1+2t_2+3t_3+4t_4}{2}\\ &= \frac{2(2n-1)}{2}=2n-1.\end{aligned}$$

So Theorem 3 is again true.

*Subcase 1.2. $t_4$ is odd.*

In this subcase, $t_2$ is odd because $t_2+t_4$ is even. If $t_3$ is even, then $t_1$ is also even. For this situation, let

$$X=\{u_1^1,\dots,u_{t_1/2}^1,u_1^2,\dots,u_{(t_2-1)/2}^2,u_1^3,\dots,u_{(t_3+2)/2}^3,u_1^4,\dots,u_{(t_4-1)/2}^4\}.$$

Then

$$|X|=\frac{t_1}{2}+\frac{t_2-1}{2}+\frac{t_3+2}{2}+\frac{t_4-1}{2}=\frac{t_1+t_2+t_3+t_4}{2}=\frac{2n}{2}=n \text{ and}$$

$$\begin{aligned}\sum_{u_j^i\in X}\deg_T(u_j^i) &= \frac{t_1}{2}+2\frac{t_2-1}{2}+3\frac{t_3+2}{2}+4\frac{t_4-1}{2}\\ &= \frac{t_1+2t_2+3t_3+4t_4}{2}\\ &= \frac{2(2n-1)}{2}=2n-1.\end{aligned}$$

So Theorem 3 is true in this situation. If $t_3$ is odd, then $t_1$ is also odd. For this situation, let

$$X=\{u_1^1,\dots,u_{(t_1+1)/2}^1,u_1^2,\dots,u_{(t_2-1)/2}^2,u_1^3,\dots,u_{(t_3-1)/2}^3,u_1^4,\dots,u_{(t_4+1)/2}^4\}.$$

| No | k | $\deg_{T_1}(v_i)$ | $\deg_{T_2}(v_i)$ | $\deg_{T_3}(v_i)$ | $\deg_{T_4}(v_i)$ | $\deg_{T_5}(v_i)$ | $\deg_{T_6}(v_i)$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | $n$ | 1 | 1 | 1 | 1 | 1 | $\cdots$ |
| 2 | 1 | $n-1$ | 2 | 1 | 1 | 1 | 1 | $\cdots$ |
| 3 | 2 | $n-2$ | 3 | 1 | 1 | 1 | 1 | $\cdots$ |
| 4 | 2 | $n-2$ | 2 | 2 | 1 | 1 | 1 | $\cdots$ |
| 5 | 3 | $n-3$ | 4 | 1 | 1 | 1 | 1 | $\cdots$ |
| 6 | 3 | $n-3$ | 3 | 2 | 1 | 1 | 1 | $\cdots$ |
| 7 | 3 | $n-3$ | 2 | 2 | 2 | 1 | 1 | $\cdots$ |

Table 1: Possibilities for the $i$-th row

Then

$$|X| = \frac{t_1+1}{2} + \frac{t_2-1}{2} + \frac{t_3-1}{2} + \frac{t_4+1}{2} = \frac{t_1+t_2+t_3+t_4}{2} = \frac{2n}{2} = n \text{ and}$$

$$\begin{aligned}\sum_{u_j^i \in X} \deg_T(u_j^i) &= \frac{t_1+1}{2} + 2\frac{t_2-1}{2} + 3\frac{t_3-1}{2} + 4\frac{t_4+1}{2} \\ &= \frac{t_1+2t_2+3t_3+4t_4}{2} \\ &= \frac{2(2n-1)}{2} = 2n-1.\end{aligned}$$

So Theorem 3 is again true. Case 1 is completely considered.

*Case 2.* $\Delta \geq n-3$.

Let $\{T_1, T_2, \ldots, T_n\}$ be a $T$-factorization and $v_1, v_2, \ldots, v_{2n}$ be the vertices of $K_{2n}$. We form the matrix $M$ as in the proof of Theorem 2. Then Equality (2.2) holds for each $i \in \{1, 2, \ldots, 2n\}$. There exists a row of $M$ with an entry $\Delta$. For this row, say the $i$-th row, for definiteness let $\deg_{T_1}(v_i) = \Delta$. Then Equality (2.2) becomes

$$\Delta + \deg_{T_2}(v_i) + \cdots + \deg_{T_n}(v_i) = 2n-1. \tag{2.8}$$

Since $T$ factorizes $K_{2n}$, it is necessary that $\Delta \leq n$. So for this case $\Delta = n-k$ with $k \in \{0, 1, 2, 3\}$.

By Case 1, we may assume further that $\Delta = n-k \geq 5$. Also, without loss of generality, we may assume that in the $i$-th row of $M$

$$\Delta = \deg_{T_1}(v_i) \geq \deg_{T_2}(v_i) \geq \cdots \geq \deg_{T_n}(v_i).$$

For each $k \in \{0, 1, 2, 3\}$, we list all possibilities for the $i$-th row of $M$ in Table 1. We need further the following claim 2.1 which is a well known fact. Therefore, we omit its proof here.

**Claim 2.1.** *If a tree $S$ possesses a vertex of degree $k$, then $S$ has at least $k$ vertices of degree* 1.

Now we consider the possibilities for the $i$-th row of $M$, that are listed in Table 1, in turn.

For the possibility 1, by Claim 2.1 the number of vertices of degree 1 in $T$ is at least $n$. Therefore, we can choose in $T$ a vertex $u_1$ of degree $n$ and $n-1$ different vertices of degree 1, say $u_2, u_3, \ldots, u_n$.

For the possibilities 2 (respectively, 3), since $T_1 \cong T_2 \cong T$, there exist in $T$ a vertex $u_1$ of degree $n-1$ (respectively, $n-2$) and a vertex $u_2$ of degree 2 (respectively, 3). Further, by Claim 2.1, we can choose $n-2$ different vertices $u_3, u_4, \ldots, u_n$ of degree 1 in $T$.

For the possibility 6, since $T_1 \cong T_2 \cong T_3 \cong T$, there exist in $T$ a vertex $u_1$ of degree $n-3$, a vertex $u_2$ of degree 3 and a vertex $u_3$ of degree 2. By Claim 2.1, we can choose $n-3$ different vertices of degree 1 in $T$, say $u_4, u_5, \ldots, u_n$.

For each of the possibilities 1, 2, 3 and 6, by (2.8), the sum of the degrees of all $n$ chosen vertices is $2n-1$. So Theorem 3 is true in these situations.

Now we consider the possibility 4. Since Theorem 3 is true if the possibility 3 happens, we may assume further that in every row of $M$, that contains an entry $\Delta = n-2$, there are exactly one entry $\Delta$, two entries 2 and $n-3$ entries 1. But the number of entries $\Delta$ in $M$ is at least $n$ because each column contains at least one entry $\Delta$. So there are at least $n$ rows of $M$ with an entry $\Delta$. It follows that the number of entries 2 in $M$, if we count them by rows, is at least $2n$. Hence, since all columns have the same number of entries 2, each column of $M$ has at least two entries 2. This means that $T$ has at least two vertices of degree 2. By Claim 2.1 the number of vertices of degree 1 in $T$ is at least $n-2$. So we can choose in $T$ a vertex $u_1$ of degree $n-2$, two vertices of degree 2, say $u_2$ and $u_3$, and $n-3$ vertices of degree 1, say $u_4, u_5, \ldots, u_n$. For these $n$ chosen vertices, by (2.8)

$$\sum_{i=1}^{n} \deg_T(u_i) = (n-2) + 2 + 2 + \underbrace{1 + \cdots + 1}_{n-3 \text{ times}} = 2n-1$$

and the theorem is true in this situation.

Next, we consider the possibility 5. Since $T_1 \cong T_2 \cong T$, we can choose in $T$ a vertex $u_1$ of degree $n-3$ and a vertex $u_2$ of degree 4. Let $w_1, w_2, \ldots, w_{n-3}$ be the neighbours of $u_1$. Denote by $\overline{T}$ the graph obtained from $T$ by deleting all edges incident with $u_1$. Then $\overline{T}$ has the connected components $\overline{T}_0, \overline{T}_1, \ldots, \overline{T}_{n-3}$, where $V(\overline{T}_0) = \{u_1\}$ and $V(\overline{T}_i)$ contains $w_i, i = 1, \ldots, n-3$. Without loss of generality we may assume that the vertex $u_2$ of degree 4 chosen above is in $\overline{T}_1$. Then the degree of $u_2$ in $\overline{T}_1$ is at least 3 ($\deg_{\overline{T}_1}(u_2) = 3$ iff $u_2 = w_1$). Since $\overline{T}_1$ is a tree, by Claim 2.1, $\overline{T}_1$ has at least 3 vertices of degree 1 and therefore at least two of them are different from $w_1$. It follows that $\overline{T}_1$ contains at least two vertices of degree 1 in $T$. For the remaining components $\overline{T}_2, \ldots, \overline{T}_{n-3}$, it is not difficult to see that each of these components contains at least one vertex of degree 1 of $T$. Therefore, in total $T$ has at least $n-2$ vertices of degree 1. So we can choose $n-2$ different vertices of degree 1 in $T$, say $u_3, \ldots, u_n$. For these chosen vertices $u_1, u_2, \ldots, u_n$, by (2.8)

$$\sum_{i=1}^{n} \deg_T(u_i) = (n-3) + 4 + \underbrace{1 + \cdots + 1}_{n-2 \text{ times}} = 2n - 1.$$

and the theorem is again true.

Finally, we consider the possibility 7. Since Theorem 3 has been proved above to be true if the possibility 5 or the possibility 6 happens, we may assume further that in every row of $M$, that contains an entry $\Delta = n-3$, there are exactly one entry $\Delta$, three entries 2 and $n-4$ entries 1. Further we can use arguments similar to those for the possibility 4 to see that Theorem 3 is also true for the possibility 7.

The proof of Theorem 3 is complete. □

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