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A single axiom for semi-Boolean SQS-skeins

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Abstract

In this note, we show that the variety of semi-Boolean SQS-skeins can be defined by a single axiom and exhibit a single axiom for said variety that was found with the aid of the automated theorem-prover Prover9.

1 Introduction

A *Steiner triple* (*quadruple*) *system*, or STS (SQS), is a pair $(X, \mathcal{B})$ where X is a set, the elements of which are called *points*, and $\mathcal{B}$ is a set of 3-subsets (4-subsets) of X, the elements of which are called *blocks*, such that every 2-subset (3-subset) of points is contained in exactly one block.

Given an SQS $(X, \mathcal{B})$ and a point $x \in X$, the *derived* STS *with respect to* x is $(X \setminus \{x\}, \{B \setminus \{x\} : x \in B \in \mathcal{B}\})$. An SQS is *semi-Boolean* if all of its derived STS's are isomorphic to projective spaces over $GF(2)$.

Given an SQS $(X, \mathcal{B})$, we can construct an algebra $(X; q)$ of type (3) as follows: define $q(y, x, x) = q(x, y, x) = q(x, x, y) = y$ for all $x, y \in X$ and also define $q(x, y, z)$ to be the fourth point in the unique block containing $\{x, y, z\}$ for all distinct $x, y, z \in X$. Clearly, this algebra satisfies the four identities below.

$$q(x, x, y) = y \tag{1}$$

$$q(x, y, z) = q(x, z, y) \tag{2}$$

$$q(x, y, z) = q(y, z, x) \tag{3}$$

$$q(x, y, q(x, y, z)) = z \tag{4}$$

Conversely, given an algebra $(X; q)$ of type (3) that satisfies (1), (2), (3), and (4), we can construct an SQS whose points are the elements of X and whose blocks are the 4-subsets of X of the form $\{x, y, z, q(x, y, z)\}$, x, y, and z distinct. Therefore, there is a one-to-one correspondence between SQS's and type (3) algebras satisfying (1), (2), (3), and (4) [7], just as there is a one-to-one correspondence between STS's and idempotent totally symmetric quasigroups (squags). For this reason, type (3) algebras $(X; q)$ satisfying (1), (2), (3), and (4) are known as SQS-*skeins*.

The SQS-skeins corresponding to the semi-Boolean SQS's are known as *semi-Boolean* SQS-*skeins*, or SBSQS-*skeins*. It is known that the SBSQS-skeins are precisely the SQS-skeins that satisfy the additional identity below [4].

$$q(x, u, q(y, u, z)) = q(q(x, u, y), u, z) \tag{5}$$

We will call an identity in a single ternary operation q a *single axiom for* SBSQS-*skeins* if and only if the identity is valid in all SBSQS-skeins and all models of the identity are SBSQS-skeins. Single axioms for several varieties of algebras arising from combinatorial designs are known. For instance, in [2], a single axiom for squags was found and, in [1], single axioms were found for the two subvarieties of squags corresponding to the Hall triple systems (distributive squags) and the affine spaces over $GF(3)$ (medial squags). In [5], single axioms were found for some varieties of groupoids associated with strongly 2-perfect m-cycle systems. Also, in [3], the identity

$$q(x, x, q(y, q(q(q(y, z, u), z, u), v, w), v)) = w \tag{6}$$

was shown to be a single axiom for SQS-skeins. Therefore, it is natural to ask if there exists a single axiom for SBSQS-skeins and, if so, what is the length of a shortest such identity (in terms of the number of variable occurrences) and what is the smallest number of distinct variables among such identities.

In this note, we show that the variety of SBSQS-skeins admits a single axiom (or is *one-based*) by exhibiting a single axiom for SBSQS-skeins. Our investigations were aided by the automated theorem-prover Prover9 [6], a resolution theorem-prover for first-order logic with equality. We also used the scripting language Perl to further automate our search.

2 A Single Axiom for SBSQS-Skeins

In this section, we describe our search for a single axiom for SBSQS-skeins.

We began by generating a large number of identities that are valid in the variety of SBSQS-skeins by running Prover9 on the input

```
set(paramodulation).  % turn on paramodulation inference rule
set(print_kept).  % print derived and retained clauses
assign(max_given, 100).  % terminate after 100 iterations of main loop
clauses(usable).  % usable clauses
q(x,x,q(y,q(q(q(y,z,u),z,u),v,w),v)) = w.  % single axiom for SQS-skeins
end_of_list.
clauses(sos).  % set of support clauses
```

```
q(x,u,q(y,u,z)) = q(q(x,u,y),u,z).  % SBSQS-skein
end_of_list.
```

and simply letting it derive consequences of (5) and (6) for 100 iterations of its main loop. We then extracted from the output (using Perl) some of the derived identities with the following properties: one side consists of a single variable that is not the left-most or right-most variable on the other side, the number of distinct variables is at most six, and the number of variable occurrences is at most 16. This resulted in 1308 identities that are valid in the variety of SBSQS-skeins.

Remark 2.1. The Prover9 version used by the author throughout this paper was Prover9 Sep-2005A for Windows. The reader should note that in versions of Prover9 dated July 2006 or later, the word "formulas" should be substituted for the word "clauses" in all Prover9 input.

Next, we sent each of these identities to Prover9 to search for a proof that it implies (5) and (6), and is therefore a single axiom for SBSQS-skeins. For example, running Prover9 on the input

```
set(auto).  % autonomous mode
assign(max_seconds, 60).  % one minute time limit per identity
clauses(sos).  % set of support
q(x,x,q(y,q(y,z,q(u,z,q(v,z,q(u,z,q(v,z,w))))),z)) = w.  % candidate
                                                             identity
q(a,a,q(b,q(q(q(b,c,d),c,d),e,f),e)) != f |
 q(a,d,q(b,d,c)) != q(q(a,d,b),d,c).  % not SBSQS-skein
end_of_list.
```

produces a proof that the identity above is a single axiom for SBSQS-skeins. Proofs were found for only two identities (both with six distinct variables and 16 variable occurrences).

Theorem 2.2. *The variety of* SBSQS-*skeins is one-based. The identity*

$$q(x, x, q(y, q(y, z, q(u, z, q(v, z, q(u, z, q(v, z, w))))), z)) = w$$

is a single axiom for SBSQS-*skeins.*

Problem 2.3. Find a (all) shortest single axiom(s) for SBSQS-skeins or show that ours is as short as possible (16 variable occurrences).

Problem 2.4. Find a single axiom for SBSQS-skeins with the fewest number of distinct variables or show that ours has as few as possible (six distinct variables).

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