MINIMUM SETS OF PARTIAL POLYOMINOES

Heiko Harborth and Hartmut Weiss

Technische Universität Braunschweig, Germany

Abstract. The smallest size $f(n)$ of a set $C(n)$ of n-ominoes such that every sufficiently large polyomino contains at least one n-ominoe of $C(n)$ is calculated for $n \leq 8$, and asymptotically lies between $(2.205\ldots)^n$ and $(2.241\ldots)^n$. For $n \leq 6$ the minimum sets $C(n)$ are proved to be unique.

Unit squares having their vertices at integer points in the Cartesian plane are called cells. A point set equal to a union of n distinct cells which is simply connected and not separable by the removal of a finite set of points is called an n-omino or polyomino [1]. Two polyominoes are considered to be isomorphic if they are congruent under translation, rotation, and reflection. For example, there are 12 different 5-ominoes, and 4460 different 10-ominoes without holes [2,4]. However, no general formula is known for the number of nonisomorphic n-ominoes.

Many problems about polyominoes have been discussed. We introduce a new one: Determine the smallest size $f(n)$ of a cut-set $C(n)$ of n-ominoes, such that every sufficiently large polyomino contains at least one n-omino of $C(n)$ as a partial polyomino. This question may be of interest in biology or pharmacy, for example, if every infinite cell growing of polyominoes can be avoided by the control of all n-ominoes for fixed n then it would suffice to control only those of $C(n)$. A minimum cut-set $C(n)$ will be denoted by $T(n)$.

A first observation is that only snakes have to be considered. A snake is a polyomino where two squares, the endsquares, have a side in common with one, and all other squares have two sides in common with two of the other squares of the polyomino. Any growing polyomino contains also growing snakes. So we have to look for a minimum cut-set $T(n)$, so that every infinite snake contains at least one n-snake of $T(n)$ as a partial polyomino.

The first numbers $s(n)$ of different snakes with n squares are

$$s(n) = 1, 1, 2, 3, 7, 13, 30, 64 \quad for \quad n = 1, 2, 3, 4, 5, 6, 7, 8.$$

This also may be deduced from the enumeration of filaments in [5].
Here we will determine bounds for $f(n)$ and exact values for small n up to $n = 8$.

Theorem 1. *The smallest numbers $f(n)$ of n-ominoes such that any sufficiently large polyomino has one of them as a partial polyomino are* $f(1) = f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 4$, $f(6) = 6$, $f(7) = 10$, $f(8) = 17$.

Proof. $f(1) = f(2) = 1$ is trivial. Both snakes of order 3 have to belong to $T(3)$ since each of them determines one of the infinite periodic snakes S_1 and S_2 in Figure 2. This implies $f(3) = 2$. Each of the 3 snakes with 4 squares determines

Figure 2. This implies $f(3) = 2$. Each of the 3 snakes with 4 squares determines S_1, S_2 or S_3 of Figure 2, so that $f(4) = 3$. The unique minimum cut-sets $T(n)$ for $n \leq 4$ are shown in Figure 1.

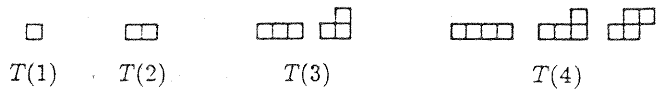

Figure 1. Minimum cut-sets $T(n)$ for $n \leq 4$.

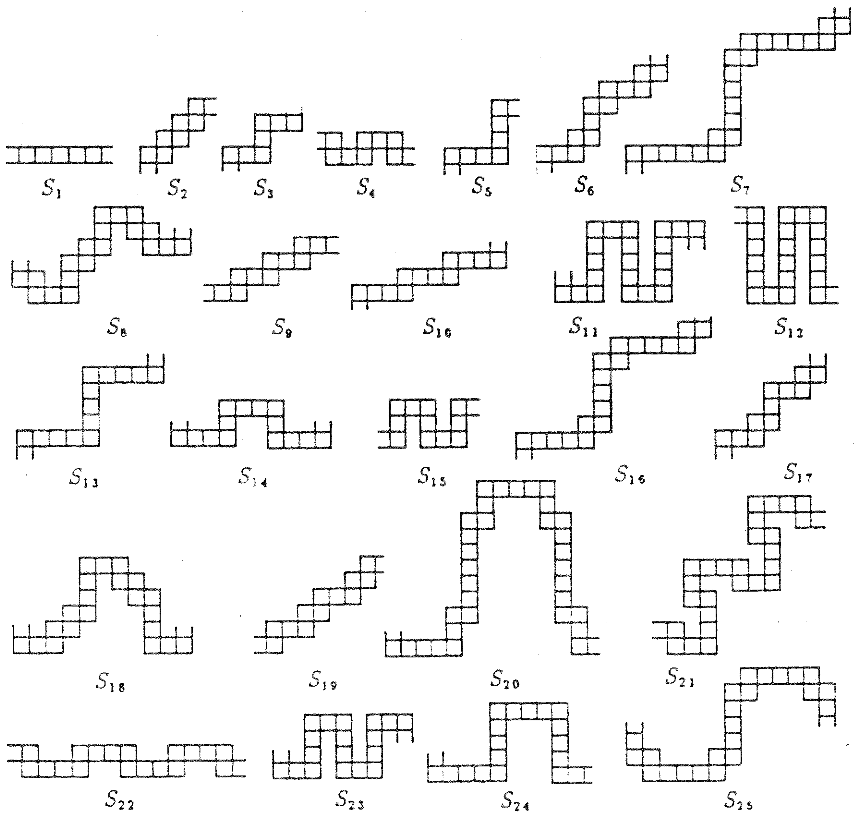

Figure 2. Infinite snakes for lower bounds of $f(n)$.

In general a minimum cut-set $T(n)$ is constructed by the following procedure:
(1) From $T(n-1)$ a cut-set $C_1(n)$ is obtained if one square is added in all possible ways to one end of every snake of $T(n-1)$, and thus $f(n) \leq |C_1(n)|$ is known.

(2) One looks for a largest possible subset $C_2(n)$ of $C_1(n)$ such that each snake of $C_2(n)$ is part of an infinite periodic snake, and no pair of these infinite snakes has a partial snake with n squares in common. Then $f(n) \geq |C_2(n)|$ holds.

(3) To every snake of $C_1(n)$ which is not an element of $C_2(n)$, squares are added in all possible ways to one endsquare provided a snake from $C_2(n)$ is not formed. If no further square can be added, then $C_2(n) = T(n)$ is a minimum cut-set. and $f(n) = |C_2(n)|$.

$f(5) = 4$: The set $C_1(5)$ is given in Figure 3.

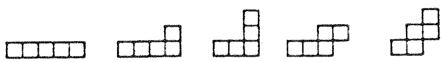

Figure 3. $C_1(5) = \{s_1(5), \ldots, s_5(5)\}$.

The infinite snakes S_1, S_3, S_4, S_2 of Figure 2 determine $C_2(5) = \{s_1(5), s_3(5), s_4(5), s_5(5)\}$ which equals $T(5)$ since additional squares to the right end of $s_2(5)$ lead to $s_3(5)$ or $s_4(5)$.

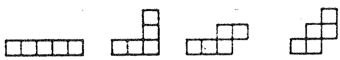

Figure 4. $T(5)$.

$f(6) = 6$: The first step of the procedure leads to $C_1(6)$ in Figure 5.

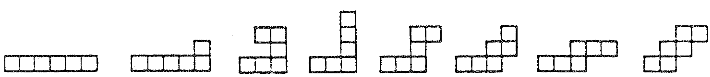

Figure 5. $C_1(6) = \{s_1(6), \ldots, s_8(6)\}$.

The infinite snakes S_1, S_5, S_3, S_6, S_4, S_2 of Figure 2 determine $C_2(6) = \{s_1(6), s_4(6), s_5(6), s_6(6), s_7(6), s_8(6)\}$ which gives $T(6)$ in Figure 6 since for $s_2(6)$ one additional square to the right end leads to snake $s_4(6)$, and two additional squares lead to $s_6(6)$ or $s_7(6)$, and for $s_3(6)$, one additional square to the left end leads to $s_4(6)$ or $s_5(6)$.

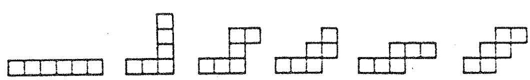

Figure 6. $T(6)$.

$\underline{f(7) = 10}$: From $T(6)$ we obtain $C_1(7)$ in Figure 7.

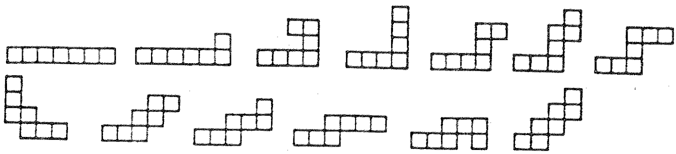

Figure 7. $C_1(7) = \{s_1(7), \ldots, s_{13}(7)\}$.

The infinite snakes S_1, S_{11}, S_5, S_3, S_7, S_8, S_9, S_{10}, S_4, S_2, determine $T(7)$ in Figure 8, since $s_1(7)$ leads to $s_3(7)$, $s_4(7)$, or $s_5(7)$, snake $s_5(7)$ to $s_6(7)$ or $s_7(7)$, and snake $s_6(7)$ to $s_{10}(7)$ or a snake which leads to $s_8(7)$ or $s_9(7)$.

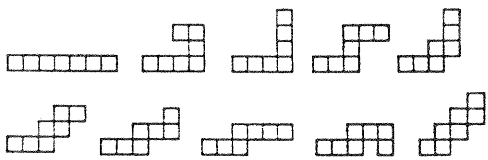

Figure 8. $T(7)$.

$\underline{f(8) = 17}$: We leave out the somewhat tedious step (3) of the proof and give only $T(8)$ in Figure 9, so as the corresponding infinite periodic snakes of step (2) which are, in this sequence, S_1, S_{12}, S_{13}, S_5, S_3, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{19}, S_{20}, S_9, S_4,S_{10}, S_{21}, S_2 of Figure 2.

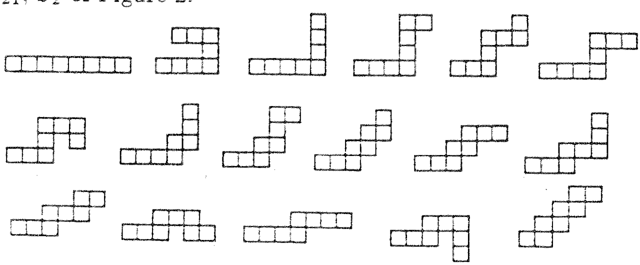

Figure 9. $T(8)$.

Theorem 2. *The minimum cut sets $T(n)$ are unique for $n \leq 6$.*

Proof. For $n \leq 4$ this is trivial. - The set $S(5)$ of all snakes for $n = 5$ is given in Figure 10.

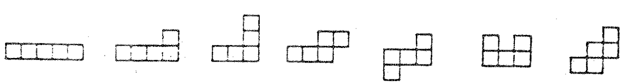

Figure 10. $S(5) = \{a_1(5), \ldots, a_7(5)\}$.

Then S_1 and S_2 of Figure 2 force $a_1(5)$ and $a_7(5)$ to belong to any cut-set $C(5)$. Assume $a_2(5)$ is contained in $C(5)$. Then $|C(5)| \geq 5 > f(5)$ follows since S_3 with snakes $a_3(5)$ and $a_5(5)$, S_4 with snakes $a_4(5)$ and $a_6(5)$, and S_9 with snakes $a_4(5)$ and $a_5(5)$ force at least two further snakes to belong to $C(5)$. Thus $a_2(5)$ is not an element of $T(5)$. Similarly, snake $a_5(5)$, by S_4, S_5, and S_{22}, and $a_6(5)$, by S_3, S_5, and S_9, are excluded. Then $T(5)$ of Figure 4 is unique.

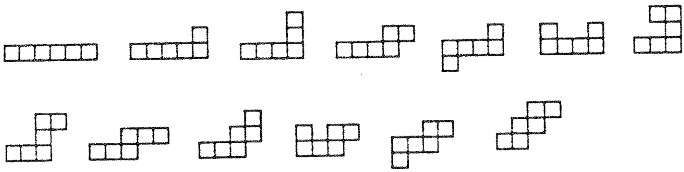

Figure 11. $S(6) = \{a_1(6), \ldots, a_{13}(6)\}$.

Snakes $a_1(6)$, $a_8(6)$, and $a_{13}(6)$ belong to any cut-set $C(6)$ because of S_1, S_3, and S_2, respectively. If snake $a_9(6)$ is not in $C(6)$ then $a_{11}(6)$, $a_{12}(6)$, and $a_4(6)$ or $a_5(6)$ must occur in C(6) because of S_4, S_9, S_{10}. In both cases, S_{23} for snake $a_4(6)$ and S_{25} for snake $a_5(6)$ force the contradiction $|C(6)| \geq 7 > f(6)$. Thus $a_9(6)$ is in $T(6)$. Now similarly, S_5, S_{24}, and S_{19} determine snake $a_3(6)$, and then S_{19} and S_{25} also snake $a_{10}(6)$ as members of $T(6)$, which then is unique.

For $n = 7$ the minimum cut-set $T(7)$ is conjectured to be unique, too. However, snake $s_7(8)$ of $T(8)$ may be replaced by the snake of Figure 12 to give a different minimum cut-set, so that uniqueness no longer holds.

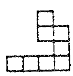

Figure 12.

Theorem 3. *The smallest number $f(n)$ of n-ominoes such that any sufficiently large polyomino has one of them as partial polyomino is such that*

$$2.205\ldots \leq \lim_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1+\sqrt{2} = 2.414\ldots \quad .$$

Proof. The upper bound follows as in [5] from $f(n) \leq s(n) \leq g(n)$, where $g(n)$ denotes the number of sequences $x_1, \ldots, x_n$ with $x_i \in \{-1, 0, 1\}$ such that neither -1,-1 nor 1,1 appear as consecutive terms x_i, x_{i+1}, and the recurrence relation $g(n) = 2g(n) + g(n-2)$ together with $g(1) = 3$ and $g(2) = 7$ yields $g(n) = \frac{1}{2}((1+\sqrt{2})^{n+1} + ((1-\sqrt{2})^{n+1})$.

In calculating the lower bound, the first step is the construction of $h(k)$ polyominoes with k squares in the following way: a first square is fixed for example by a cross. Then the k-th square ($k \geq 2$) is added to the preceding square only to the right, upwards, and downwards, if possible, and never to the left. Since addition of the k-th square to the right, and either upwards or downwards always is possible, it follows that $h(k) \geq 2h(k-1)$. Only to those $h(k-3)$ snakes with $k-1$ cells which end by two steps to the right the k-th square may be added in all three directions. Thus $h(k) = 2h(k-1) + h(k-3)$, and together with $h(1) = 1$, $h(2) = 3$, and $h(3) = 7$ (see Figure 13), it follows that $h(k)^{\frac{1}{k}} \geq \lambda = 2.205\ldots$ for large k, since $x^3 - 2x^2 - 1 = 0$ has λ as its largest real root.

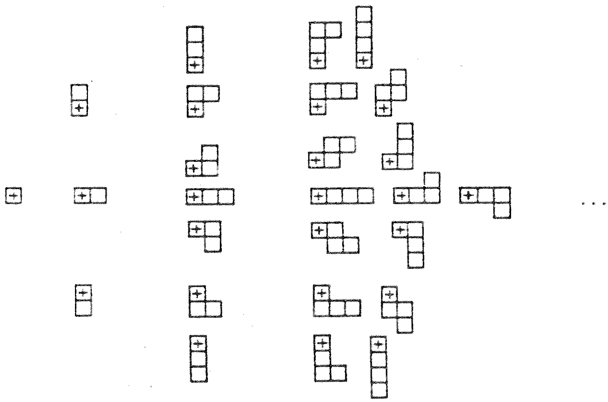

Figure 13.

If, for all $h(k)$ snakes of Figure 13, two squares are added to the left of the first square, and two squares are added to the right of the last square, and then a third square upwards, then these $h(k)$ snakes, which now have $k+5$ cells, are pairwise nonisomorphic (see Figure 14).

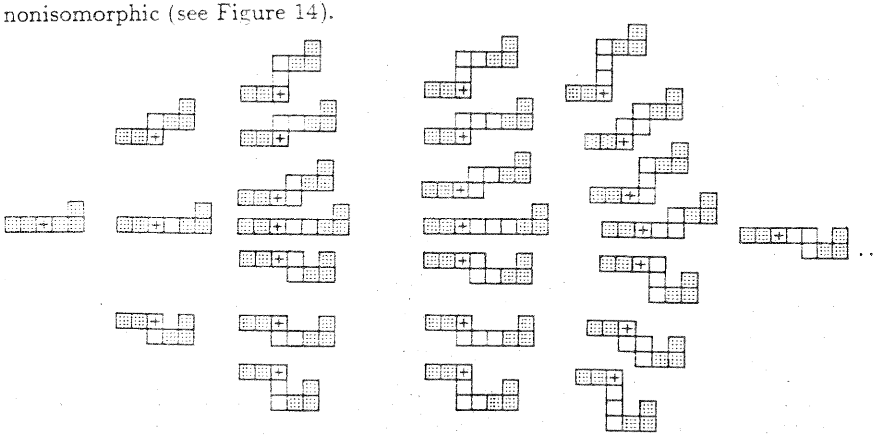

Figure 14.

Note that for any isomorphism, the first and the last three squares would yield a congruence under translation such that the crossed squares of the corresponding snakes of Figure 13 coincide. Now each of the $h(k)$ snakes with $k+5$ squares is completed by an omega-snake with 15 squares (see Figure 15) to determine the period of an infinite periodic snake as in Figure 15. No partial snake with $k+5+2\cdot 15-1=k+34$ squares occurs in two of these infinite periodic snakes since one period contains exactly one omega-snake, which proceeds in all four directions, whereas the remaining snakes with $k+5$ cells by construction use only three directions, and they are nonisomorphic for different infinite snakes. Thus at least one $(k+34)$-snake of each of the $h(k)$ infinite snakes has to belong to the cut-set $T(n)=T(k+34)$, and

$$f(n)=f(k+34)\geq h(k)>C_1\lambda^k=C_2\lambda^n,\quad C_2=C_1\lambda^{-34}=constant,$$

is equivalent to the lower bound of Theorem 3 .

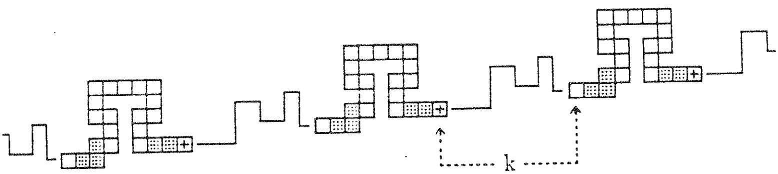

Figure 15.

Many questions remain unanswered. To find a general formula for f(n) seems to be hopeless. It can be remarked that the corresponding problem for g-adic number sequences is known in the context of deBruijn sequences. Here the minimal cut-set of n-digit blocks such that every sufficient large g-adic sequence has at least one partial n-digit block of this cut-set, has cardinality

$$Z_g(n)=\frac{1}{n}\Sigma_{d|n}\varphi(d)g^{\frac{n}{d}}$$

(see [3]). The first author thanks R.B. Eggleton for interesting discussions on these g-adic sequences.

REFERENCES

[1] S.W. Golomb: Polyominoes. Scribner's, New York, 1965

[2] W.F. Lunnon: Counting polyominoes. In: Computers in Number Theory, eds. A.O.L. Atkin an B.J. Birch, Acad. Press, London, 1971, 347-372.

[3] J. Mykkeltveit: A proof of Golomb's conjecture for the deBruijn graph. J. Combinatorial Theory (B) 13 (1972), 40-45.

[4] N.J.A. Sloane: A Handbook of Integer Sequences. Acad. Press, New York and London 1973, 69-70.

[5] R.G. Stanton, R.C. Tilley and D.D. Cowan: The cell growth problem for filaments. In: Combinatorics, Graph Theory and Computing, eds. R.C. Mullin, K.B. Reid, D.P. Roselle, Baton Rouge (1970), 310-339.

Heiko Harborth
Diskrete Mathematik

Hartmut Weiss
Institut für Algebra

Technische Universität Braunschweig
3300 Braunschweig
Germany