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Maximal sets of hamilton cycles in complete multipartite graphs II

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Abstract

A set S of edge-disjoint hamilton cycles in a graph T is said to be maximal if the hamilton cycles in S form a subgraph of T such that $T - E(S)$ has no hamilton cycle. The set of integers m for which a graph T contains a maximal set of m edge-disjoint hamilton cycles has previously been determined whenever T is a complete graph, a complete bipartite graph, and in many cases when T is a complete multipartite graph. In this paper we solve all but one of the remaining cases when T is a complete multipartite graph. The proof technique could also be used to simplify the proofs of previous results.

1 Introduction

A *hamilton cycle* in a graph T is a spanning cycle of T. If S is a set of edge-disjoint hamilton cycles in T and if $E(S)$ is the set of edges occurring in the hamilton cycles in S, then S is said to be *maximal* if $T - E(S)$ has no hamilton cycle.

In 1993, Hoffman, Rodger, and Rosa [7] showed that there exists a maximal set S of m edge-disjoint hamilton cycles in K_n if and only if $m \in \{\lfloor\frac{n+3}{4}\rfloor, \lfloor\frac{n+3}{4}\rfloor + 1, \ldots, \lfloor\frac{n-1}{2}\rfloor\}$. Using amalgamation techniques, Bryant, El-Zanti, and Rodger [1] showed that there exists a maximal set of m edge-disjoint hamilton cycles in the complete bipartite graph $K_{n,n}$ if and only if $n/4 < m \leq n/2$. Later, Daven, MacDougall, and Rodger [2] extended the use of amalgamation techniques by showing for $n \geq 3$ and $p \geq 3$, there exists a maximal set of m hamilton cycles in the complete multipartite graph K_n^p (p parts, each of size n) if and only if $\lceil (n(p-1))/4 \rceil \leq m \leq \lfloor (n(p-1))/2 \rfloor$, and $m > (n(p-1))/4$ if n is odd and $p \equiv 1 \pmod 4$, except possibly if n is odd and $m \leq ((n+1)(p-1)-2)/4$. Fu, Logan, and Rodger [5] later showed that if $\lceil (p-1)/2 \rceil \leq m \leq p-1$ then there exists a maximal set of m hamilton cycles in $T = K_{2p} - F$, where F is a 1-factor of K_{2p}.

In these results, if $T = K_n$ or $T = K_{n,n}$, then in every case the set S of m hamilton cycles is maximal because $T - E(S)$ is disconnected. However if $T = K_n^p$, then the constructions usually result in $T - E(S)$ being disconnected, but in some cases it has edge-connectivity of 1.

These results together prove the following result (see [2, 5]).

Theorem 1.1 ([1, 2, 5, 7]) *There exists a maximal set of m hamilton cycles in K_n^p (p parts of size n) if and only if*

1. $\lceil \frac{n(p-1)}{4} \rceil \leq m \leq \lfloor \frac{n(p-1)}{2} \rfloor$ *and*
2. $m > \frac{n(p-1)}{4}$ *if*
   (a) *n is odd and $p \equiv 1 \pmod 4$, or*
   (b) $p = 2$, $n = 1$

except possibly for the undecided case when $n \geq 3$ is odd, p is odd, and $m \leq ((n+1)(p-1)-2)/4$.

In this paper, we extend this result by removing all but one of the possible exceptions when $n \geq 3$ is odd, p is odd, and $m \leq \frac{(n+1)(p-1)-2}{4}$; the one case remaining to be settled is when $n = 3$ and m is the lowest possible value. We will consider the values $n = 3$ and $n \geq 5$ separately. In each case, the set S of hamilton cycles is produced so that there exists a partition $\{V', W'\}$ of the vertex set of K_n^p such that each edge joining a vertex in V' to one in W' occurs in a hamilton cycle in S; so clearly S is maximal because $K_n^p - E(S)$ is disconnected. Graphs will often contain multiple edges, so we assume that all sets may in fact contain repeated elements (ie. are multi-sets). So in particular $|C \cup D| = |C| \cup |D|$.

In this paper we make use of the proof technique of amalgamations. The idea behind the method, informally speaking, is as follows. An *amalgamation* of a graph T is the graph U defined by a homomorphism $g : V(T) \rightarrow V(U)$. Each vertex u in U can be considered to "contain" $f(u) = |g^{-1}(u)|$ vertices of T; f is called the *amalgamation function* of (T, U). In the following section, we begin with a graph U together with an *associated* function f that could conceivably be the amalgamation of $(T = K_n^p, U)$. We then utilize a theorem in [2] that shows f is indeed this amalgamation function and each vertex u could be disentangled into $f(u)$ vertices one by one.

2 The case $n \geq 5$

Throughout this paper it will help to think of the vertices in each part as being arranged in a vertical column. With this in mind, in this section, we will eventually define a set S of hamilton cycles in such a way that $K_n^p - E(S)$ has no edges joining vertices in the "top half" of any part to vertices in the "bottom half" of any other part. So $K_n^p - E(S)$ is disconnected and thus S is maximal. More formally, throughout this section we will assume the following. Let $p \geq 3$ be odd and $n \geq 5$ be odd. Let $\lceil \frac{n(p-1)}{4} \rceil \leq m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ if $p \equiv 3 \pmod 4$ and $\frac{n(p-1)}{4} < m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ if $p \equiv 1 \pmod 4$. Let $\alpha_1 = \alpha_2 = \lfloor \frac{p}{2} \rfloor$ and $\alpha_3 = \alpha_4 = \lceil \frac{p}{2} \rceil$. Let $V = A_1 \cup A_3$ and $W = A_2 \cup A_4$, where $A_i = \{a_{i,1}, \ldots, a_{i,\alpha_i}\}$. Let $f_1 = f_4 = \lceil \frac{n}{2} \rceil$ and $f_2 = f_3 = \lfloor \frac{n}{2} \rfloor$.

As will become clear in the proof of Theorem 2.4 we will define an amalgamation G on $2p$ vertices in which for $1 \leq z \leq 4$ there are α_z vertices $u \in A_z$ that each have the amalgamation number $f(u) = f_z$. In proving Theorem 2.4 we will require Theorem 5.1 of [2], restricted for our purposes to the following result.

Theorem 2.1 *Suppose there exists an m-edge-colored loopless multi-graph G on the vertex set $V \cup W$ satisfying:*

1. *the number of edges joining $a_{i,k}$ and $a_{j,k}$ is 0 if $(i,j) \in \{(1,2),(3,4)\}$ and $1 \leq k \leq \alpha_i$, and is at most $f_i f_j$ otherwise, with*

   (a) equality if one vertex is in V and the other is in W,

2. *each vertex in A_i is incident with $2f_i$ edges of each color, and*

3. *each color class is connected.*

Then there exists a maximal set S of m hamilton cycles in K_n^p.

In the notation above, it is shown in [2] that conditions (1–3) ensure that G is an amalgamation of a subgraph of K_n^p that has a hamilton decomposition. Condition (1a) ensures that the complement of G in the appropriate amalgamation of K_n^p is disconnected; this means that if V' and W' are the sets of vertices in K_n^p that are disentangled from the vertices in V and W respectively, and if v' and w' are vertices in V' and W' respectively, then the edge $\{v', w'\}$ occurs in a hamilton cycle in S.

The following result of Hilton [6] based on a notion developed by de Werra [3], can be used to vastly simplify the proof technique used in [2]. A graph is *Eulerian* if it is connected and if all vertices have even degree. An edge-coloring is *evenly-equitable* [3] if it partitions $E(G)$ into color classes $K_1, K_2, \ldots, K_m$ such that
(i) $k_i(v)$ is even for each $v \in V(G)$ and each i, where $1 \leq i \leq m$, and
(ii)$|k_i(v) - k_j(v)| = 0$ or 2 for each $v \in V(G)$ and each i,j with $1 \leq i < j \leq m$,
where $k_i(v)$ denotes the number of edges of color i incident with the vertex v.

Theorem 2.2 [6] *For each $m \geq 1$, each finite Eulerian graph has an evenly-equitable edge-coloring with m colors.*

The following lemma is well known.

Lemma 2.3 *Let $r, \mu, n \geq 1$. There exists an r-regular multi-graph G on n vertices with multiplicity at most μ iff*

1. *$r \leq \mu(n-1)$, and*
2. *if n is odd then r is even.*

Proof: If n is even then K_n has a 1-factorization, so form G with r of the 1-factors, using each 1-factor at most μ times. If n is odd then K_n has a hamilton decomposition, so form G with $r/2$ hamilton cycles, using each hamilton cycle at most μ times. ∎

Theorem 2.4 *If p is odd, $n \geq 5$ is odd, and if $\lceil \frac{n(p-1)}{4} \rceil \leq m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$, with a strict inequality in lower bound on m when $p \equiv 1 \pmod 4$, then there exists a maximal set S of m hamilton cycles in $T = K_n^p$.*

Proof: Clearly we can assume that $p \geq 3$ since if $p = 1$ there are no values of m to consider (and T has no edges anyway!). We prove the result by defining a graph G which satisfies conditions (1–3) of Theorem 2.1. Recall that V is the set $A_1 \cup A_3$ and W is the set $A_2 \cup A_4$, where $A_i = \{a_{i,1}, \ldots, a_{i,\alpha_i}\}$. The edges of G are defined as follows. There are no edges joining $a_{i,k}$ to $a_{j,k}$ for $(i,j) \in \{(1,2),(3,4)\}$ and $1 \leq k \leq \alpha_i$; otherwise, if $x \in V$ and $y \in W$, then join x and y by exactly $f(x)f(y)$ edges (recall that if $u \in A_z$ then $f(u) = f_z$). For each z satisfying $1 \leq z \leq 4$, all vertices in A_z have the same degree in the graph G' defined so far, namely if $u \in A_z$ then $d_{G'}(u) = D_z$ where

$D_1 = f_1 f_2(\lfloor \frac{p}{2} \rfloor - 1) + f_1 f_4 \lceil \frac{p}{2} \rceil$,
$D_2 = f_2 f_1(\lfloor \frac{p}{2} \rfloor - 1) + f_2 f_3 \lceil \frac{p}{2} \rceil$,
$D_3 = f_3 f_2 \lfloor \frac{p}{2} \rfloor + f_3 f_4(\lceil \frac{p}{2} \rceil - 1)$, and
$D_4 = f_4 f_1 \lfloor \frac{p}{2} \rfloor + f_4 f_3(\lceil \frac{p}{2} \rceil - 1)$.

In order to be able to satisfy condition (2) of Theorem 2.1, edges still need to be added to G' in order to boost each vertex degree in A_z up to degree $2mf_z$ for $1 \leq z \leq 4$. Form the graph G from G' as follows: for $1 \leq z \leq 4$, place a graph G_z of multiplicity at most f_z^2 and regular of degree $2mf_z - D_z$ on the vertices of A_z. To see this is possible, we will apply Lemma 2.3 with $n = |A_z|$, $r = 2mf_z - D_z$ and $\mu = f_z^2$ for $1 \leq z \leq 4$, so we need to check that

1. $0 \leq 2mf_z - D_z \leq f_z^2(|A_z| - 1)$, and
2. for $1 \leq z \leq 4$, if $|A_z|$ is odd then $2mf_z - D_z$ is even.

(1) Because n and p are both odd we can write $n = 4x + 2\mathcal{E}_1 + 1$ where $\mathcal{E}_1 \in \{0,1\}$ and $p = 4y + 2\mathcal{E}_2 + 1$ where $\mathcal{E}_2 \in \{0,1\}$. Clearly $x > 0$ since $n \geq 5$ and y and $\mathcal{E}_2$ cannot both equal zero since $p \geq 3$. Then $f_1 = f_4 = \lceil \frac{n}{2} \rceil = 2x + \mathcal{E}_1 + 1$, $f_2 = f_3 = \lfloor \frac{n}{2} \rfloor = 2x + \mathcal{E}_1$, $|A_1| = |A_2| = \lfloor \frac{p}{2} \rfloor = 2y + \mathcal{E}_2$, and $|A_3| = |A_4| = \lceil \frac{p}{2} \rceil = 2y + \mathcal{E}_2 + 1$. First we need to check the lower bounds on $2mf_z - D_z$. We will check the cases $z = 1, 2, 3$ and 4 in turn.

If $z = 1$ then $2mf_1 - D_1$

$$
\begin{aligned}
&= 2mf_1 - f_1 f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_1 f_4 \lceil \tfrac{p}{2} \rceil \\
&= f_1(2m - f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_4 \lceil \tfrac{p}{2} \rceil) \qquad (1) \\
&= f_1(2m - (2x + \mathcal{E}_1)(2y + \mathcal{E}_2 - 1) - (2x + \mathcal{E}_1 + 1)(2y + \mathcal{E}_2 + 1)) \\
&\geq \begin{cases} f_1(2\lceil \frac{n(p-1)}{4} \rceil - (2x + \mathcal{E}_1)(2y) - (2x + \mathcal{E}_1 + 1)(2y + 2)), \\ \qquad\qquad \text{if } p \equiv 3 \pmod 4 \\ f_1(2(\frac{n(p-1)}{4} + 1) - (2x + \mathcal{E}_1)(2y - 1) - (2x + \mathcal{E}_1 + 1)(2y + 1)), \\ \qquad\qquad \text{if } p \equiv 1 \pmod 4 \end{cases} \\
&= \begin{cases} f_1(2(4xy + 2x + 2\mathcal{E}_1 y + \mathcal{E}_1 + y + 1) - (2x + \mathcal{E}_1)(2y) \\ \qquad\qquad -(2x + \mathcal{E}_1 + 1)(2y + 2)), \quad \text{if } p \equiv 3 \pmod 4 \\ f_1(2(4xy + 2\mathcal{E}_1 y + y + 1) - (2x + \mathcal{E}_1)(2y - 1) - (2x + \mathcal{E}_1 + 1)(2y + 1)), \\ \qquad\qquad \text{if } p \equiv 1 \pmod 4 \end{cases} \\
&= \begin{cases} 0, & \text{if } p \equiv 3 \pmod 4 \\ f_1, & \text{if } p \equiv 1 \pmod 4 \end{cases} \\
&\geq 0.
\end{aligned}
$$

If $z = 2$ then $2mf_2 - D_2$

$$
\begin{aligned}
&= 2mf_2 - f_2 f_1(\lfloor \tfrac{p}{2} \rfloor - 1) - f_2 f_3 \lceil \tfrac{p}{2} \rceil \\
&= f_2(2m - f_1(\lfloor \tfrac{p}{2} \rfloor - 1) - f_3 \lceil \tfrac{p}{2} \rceil) \\
&= f_2(2m - f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_4 \lceil \tfrac{p}{2} \rceil - (\lfloor \tfrac{p}{2} \rfloor - 1) + \lceil \tfrac{p}{2} \rceil) \\
&= f_2(2m - f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_4(\lceil \tfrac{p}{2} \rceil) + 2) \\
&\geq 0 \text{ (compare to (1))}.
\end{aligned}
$$

If $z = 3$ then $2mf_3 - D_3$

$$
\begin{aligned}
&= 2mf_3 - f_3 f_2(\lfloor \tfrac{p}{2} \rfloor) - f_3 f_4(\lceil \tfrac{p}{2} \rceil - 1) \\
&= f_3(2m - f_2(\lfloor \tfrac{p}{2} \rfloor) - f_4(\lceil \tfrac{p}{2} \rceil - 1)) \\
&= f_3(2m - f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_4(\lceil \tfrac{p}{2} \rceil) + f_4 - f_2) \qquad (2) \\
&= f_3(2m - f_2(\lfloor \tfrac{p}{2} \rfloor - 1) - f_4(\lceil \tfrac{p}{2} \rceil) + 1) \\
&\geq 0 \text{ (compare to (1))}.
\end{aligned}
$$

If $z = 4$ then $2mf_4 - D_4$

$$
\begin{aligned}
&= 2mf_4 - f_4 f_1(\lfloor \tfrac{p}{2} \rfloor) - f_4 f_3(\lceil \tfrac{p}{2} \rceil - 1) \\
&= f_4(2m - f_1(\lfloor \tfrac{p}{2} \rfloor) - f_3(\lceil \tfrac{p}{2} \rceil - 1)) \\
&= f_4(2m - f_2\lfloor p/2 \rfloor - f_4(\lceil p/2 \rceil - 1) - \lfloor p/2 \rfloor + (\lceil p/2 \rceil - 1)) \\
&= f_4(2m - f_2(\lfloor p/2 \rfloor - 1) - f_4(\lceil p/2 \rceil)) \\
&\geq 0 \text{ (compare to (2))}.
\end{aligned}
$$

Next, we will check the upper bounds on $2mf_z - D_z$. Again, we will check the cases $z = 1, 2, 3$ and 4 in turn.

If $z=1$ then

$$\begin{aligned}
& f_1^2(|A_1|-1)-(2mf_1-D_1) \\
\geq\ & (\lceil\frac{n}{2}\rceil)^2(\lfloor\frac{p}{2}\rfloor-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor\lceil\frac{n}{2}\rceil-\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor(\lfloor\frac{p}{2}\rfloor-1) \\
& \qquad -(\lceil\frac{n}{2}\rceil)^2\lceil\frac{p}{2}\rceil) \\
>\ & \lceil\frac{n}{2}\rceil(\lfloor\frac{p}{2}\rfloor-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor-\lfloor\frac{n}{2}\rfloor(\lfloor\frac{p}{2}\rfloor-1)-\lceil\frac{n}{2}\rceil\lceil\frac{p}{2}\rceil) \\
=\ & (2x+\mathcal{E}_1+1)(2y+\mathcal{E}_2-1)-((8xy+4x\mathcal{E}_2+4y\mathcal{E}_1+2\mathcal{E}_1\mathcal{E}_2+4y+2\mathcal{E}_2-2) \\
& -(2x+\mathcal{E}_1)(2y+\mathcal{E}_2-1)-(2x+\mathcal{E}_1+1)(2y+\mathcal{E}_2+1)) \\
=\ & 2x\mathcal{E}_2+2y\mathcal{E}_2+4xy+\mathcal{E}_1\mathcal{E}_2+2-(\mathcal{E}_1+2x) \\
\geq\ & 0,
\end{aligned}$$

since $2\geq\mathcal{E}_1$, and since $4xy\geq 2x$ if $y>0$ and $2x\mathcal{E}_2\geq 2x$ if $y=0$ (recall that y and $\mathcal{E}_2$ cannot both equal zero).

If $z=2$ then

$$\begin{aligned}
& f_2^2(|A_2|-1)-(2mf_2-D_2) \\
\geq\ & (\lfloor\frac{n}{2}\rfloor)^2(\lfloor\frac{p}{2}\rfloor-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil(\lfloor\frac{p}{2}\rfloor-1) \\
& \qquad -(\lfloor\frac{n}{2}\rfloor)^2\lceil\frac{p}{2}\rceil)) \\
>\ & \lfloor\frac{n}{2}\rfloor(\lfloor\frac{p}{2}\rfloor-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor-\lceil\frac{n}{2}\rceil(\lfloor\frac{p}{2}\rfloor-1)-\lfloor\frac{n}{2}\rfloor\lceil\frac{p}{2}\rceil)) \\
=\ & (2x+\mathcal{E}_1)(2y+\mathcal{E}_2-1)-((8xy+4x\mathcal{E}_2+4y\mathcal{E}_1+2\mathcal{E}_1\mathcal{E}_2+4y+2\mathcal{E}_2-2) \\
& -(2x+\mathcal{E}_1+1)(2y+\mathcal{E}_2-1)-(2x+\mathcal{E}_1)(2y+\mathcal{E}_2+1)) \\
=\ & 4xy+2x\mathcal{E}_2+2y\mathcal{E}_1+\mathcal{E}_1\mathcal{E}_2+1-(2x+2y+\mathcal{E}_1+\mathcal{E}_2) \\
\geq\ & 0,
\end{aligned}$$

for the following reasons. If $y=0$ then $\mathcal{E}_2\neq 0$ so $2x\mathcal{E}_2\geq 2x$, $\mathcal{E}_1\mathcal{E}_2\geq\mathcal{E}_1$, and $1\geq\mathcal{E}_2$. If $y>0$ then $4xy\geq 2x+2y$, $2x\mathcal{E}_2\geq\mathcal{E}_2$, and $2y\mathcal{E}_1\geq\mathcal{E}_1$ since $x>0$.

If $z=3$ then

$$\begin{aligned}
& f_3^2(|A_3|-1)-(2mf_3-D_3) \\
\geq\ & (\lfloor\frac{n}{2}\rfloor)^2(\lceil\frac{p}{2}\rceil-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor\lfloor\frac{n}{2}\rfloor \\
& \qquad -(\lfloor\frac{n}{2}\rfloor)^2\lfloor\frac{p}{2}\rfloor-\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil(\lceil\frac{p}{2}\rceil-1)) \\
>\ & \lfloor\frac{n}{2}\rfloor(\lceil\frac{p}{2}\rceil-1)-(2\lfloor\frac{(n+1)(p-1)-2}{4}\rfloor-\lfloor\frac{n}{2}\rfloor\lfloor\frac{p}{2}\rfloor-\lceil\frac{n}{2}\rceil(\lceil\frac{p}{2}\rceil-1)) \\
=\ & (2x+\mathcal{E}_1)(2y+\mathcal{E}_2)-((8xy+4x\mathcal{E}_2+4y\mathcal{E}_1+2\mathcal{E}_1\mathcal{E}_2+4y+2\mathcal{E}_2-2) \\
& -(2x+\mathcal{E}_1)(2y+\mathcal{E}_2)-(2x+\mathcal{E}_1+1)(2y+\mathcal{E}_2)) \\
=\ & 4xy+2x\mathcal{E}_2+2y\mathcal{E}_1+\mathcal{E}_1\mathcal{E}_2+2-(2y+\mathcal{E}_2) \\
\geq\ & 0,
\end{aligned}$$

since $2 \geq \mathcal{E}_2$, and since $4xy \geq 2y$ (recall that $x > 0$).

If $z = 4$ then

$$\begin{aligned}
& f_4^2(|A_4| - 1) - (2mf_4 - D_4) \\
\geq\ & (\lceil \frac{n}{2} \rceil)^2(\lceil \frac{p}{2} \rceil - 1) - (2\lfloor \frac{(n+1)(p-1)-2}{4} \rfloor \lceil \frac{n}{2} \rceil \\
& \quad -(\lceil \frac{n}{2} \rceil)^2 \lfloor \frac{p}{2} \rfloor - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor (\lceil \frac{p}{2} \rceil - 1)) \\
>\ & \lceil \frac{n}{2} \rceil (\lceil \frac{p}{2} \rceil - 1) - (2\lfloor \frac{(n+1)(p-1)-2}{4} \rfloor - \lceil \frac{n}{2} \rceil \lfloor \frac{p}{2} \rfloor - \lfloor \frac{n}{2} \rfloor (\lceil \frac{p}{2} \rceil - 1)) \\
=\ & (2x + \mathcal{E}_1 + 1)(2y + \mathcal{E}_2) - ((8xy + 4x\mathcal{E}_2 + 4y\mathcal{E}_1 + 2\mathcal{E}_1\mathcal{E}_2 + 4y + 2\mathcal{E}_2 - 2) \\
& -(2x + \mathcal{E}_1 + 1)(2y + \mathcal{E}_2) - (2x + \mathcal{E}_1)(2y + \mathcal{E}_2)) \\
=\ & 4xy + 2x\mathcal{E}_2 + 2y\mathcal{E}_1 + \mathcal{E}_1\mathcal{E}_2 + 2 \\
\geq\ & 0.
\end{aligned}$$

(2) To see that (2) is satisfied, we will consider each value of z in turn.

Suppose $z = 1$. Let $|A_1| = \lfloor \frac{p}{2} \rfloor = 2y + 1$. Then $r = 2m\lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor (\lfloor \frac{p}{2} \rfloor - 1) - (\lceil \frac{n}{2} \rceil)^2 \lceil \frac{p}{2} \rceil$. This is even because one of $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ must be even, and because $\lceil \frac{p}{2} \rceil = 2y + 2$ is even. Similarly, if $z = 2$ and if $|A_2| = \lfloor \frac{p}{2} \rfloor = 2y + 1$, then $r = 2m\lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil (\lfloor \frac{p}{2} \rfloor - 1) - (\lfloor \frac{n}{2} \rfloor)^2 \lceil \frac{p}{2} \rceil)$ is even. If $z = 3$ and if $|A_3| = \lceil \frac{p}{2} \rceil = 2y + 1$, then $r = 2m\lfloor \frac{n}{2} \rfloor - (\lfloor \frac{n}{2} \rfloor)^2 \lfloor \frac{p}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil (\lceil \frac{p}{2} \rceil - 1)$ is even since $\lfloor \frac{p}{2} \rfloor = \lceil \frac{p}{2} \rceil - 1 = 2y$ is even. Finally, for the same reason, if $z = 4$ and $|A_4| = \lceil \frac{p}{2} \rceil = 2y + 1$, then $r = 2m\lceil \frac{n}{2} \rceil - (\lceil \frac{n}{2} \rceil)^2 \lfloor \frac{p}{2} \rfloor - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor (\lceil \frac{p}{2} \rceil - 1)$ is even. Therefore, in each case, if $|A_z|$ is odd then r is even as required. Clearly G is a connected graph with each vertex having degree $2mf_z$ for $1 \leq z \leq 4$, so condition (1) of Theorem 2.1 is satisfied. We now proceed by defining an m-edge-coloring of G that satisfies conditions (2–3) of Theorem 2.1.

Let V and W also be named $V = \{v_1, ..., v_p\}$ and $W = \{w_1, ..., w_p\}$ with v_i and w_i being in the same part. For $1 \leq x \leq m$ let

$$B(x) = \{\{v_i, w_{i+2\lceil x/\lfloor n/2 \rfloor^2 \rceil - 1}\}, \{v_i, w_{i+2\lceil x/\lfloor n/2 \rfloor^2 \rceil}\} | 0 \leq i < p\}$$

reducing the subscripts modulo p. To see that $\cup_{x=1}^{m} B(x)$ is a subgraph of G, we check that $m \leq \lfloor \frac{n}{2} \rfloor^2 (p-1)/2$. If $n \geq 5$ let $n = 2x+1$, so $\lfloor n/2 \rfloor^2 = x^2 \geq x+1 = n+1/2$ (note that this is not true if $x = 1$, so the case $n = 3$ must be addressed separately); so using the upper bound on m in Theorem 2.4 $m \leq (n+1/2)(p-1/2) - 1 \leq \lfloor n/2 \rfloor^2 (p-1/2)$. Also $B(x)$ induces a hamilton cycle in G. Now color each hamilton cycle $B(x)$ with color x; so regardless of how the remaining edges of G are colored, each color class is connected already, thereby satisfying condition (3) of Theorem 2.1.

Let H be the subgraph of G induced by $\cup_{x=1}^{m} B(x)$. It is important to notice that every vertex in $G - E(H)$ has even degree $(2mf_z - 2m)$ since every vertex in H has degree $2m$. We now apply Theorem 2.2 and give $G - E(H)$ an evenly-equitable m-edge-coloring. For $1 \leq x \leq m$ let $C'(x)$ be the set of edges colored x in $G - E(H)$; then each color class induces a graph in which each vertex $v \in A_z$ has degree $2f_z - 2$ in $G - E(H)$ (it is easy to check that $G - E(H)$ is connected, but in any case, Theorem 2.2 could be applied to each component). Let $C(x) = B(x) \cup C'(x)$ for $1 \leq x \leq m$.

So now each vertex in A_z in G is incident with $2f_z$ edges of each color and each color class is connected since $B(x)$ is connected, thus condition (2) of Theorem 2.1 is satisfied.

Because conditions (1–3) of Theorem 2.1 are satisfied, it follows that there exists a maximal set of m hamilton cycles in K_n^p. ∎

3 The case $n = 3$, but m is not its smallest possible value

Compared to the approach in Section 2, we will now rotate our view ninety degrees. That is, S is defined here so that $K_n^p - E(S)$ has no edges joining vertices in the "first" $\lfloor \frac{p}{2} \rfloor$ parts to vertices in the "last" $\lceil \frac{p}{2} \rceil$ parts. Throughout this section, we will assume the following. Let p be odd and $n = 3$. Let $\lceil \frac{n(p-1)}{4} \rceil + 1 \leq m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ if $p \equiv 3 \pmod 4$ and $\frac{n(p-1)}{4} + 1 < m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ if $p \equiv 1 \pmod 4$. Let $|V| = \lfloor \frac{p}{2} \rfloor$ and $|W| = \lceil \frac{p}{2} \rceil$.

As will become clear in the proof of Theorem 3.2 we will define an amalgamation G on p vertices in which each vertex u in G has the amalgamation number $f(u) = 3$. In proving Theorem 3.2 we will require Theorem 5.1 of [2], restricted yet again for our purposes to the following result.

Theorem 3.1 *Suppose there exists an m-edge-colored multi-graph G on the vertex set $V \cup W$ of size p satisfying:*

1. *the number of edges joining any two vertices is at most* 9
    (a) *with equality if one vertex is in V and the other is in W,*
2. *each vertex is incident with* 6 *edges of each color, and*
3. *each color class is connected.*

Then there exists a maximal set S of m hamilton cycles in K_3^p.

As described in the previous section, conditions (1–3) ensure that G is an amalgamation of a subgraph of K_n^p that has a hamilton decomposition into a set S of hamilton cycles, and condition ($1a$) ensures that $K_n^p - E(S)$ is disconnected. We are now ready for the main result of this section which addresses the value $n = 3$. The parameter n is kept in the statement of the result to make clear how it relates to Theorem 2.4.

Theorem 3.2 *If p is odd, $n = 3$, $\lceil \frac{n(p-1)}{4} \rceil + 1 \leq m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ when $p \equiv 3 \pmod 4$, and $\lceil \frac{n(p-1)}{4} \rceil + 1 < m \leq \lfloor \frac{(n+1)(p-1)-2}{4} \rfloor$ when $p \equiv 1 \pmod 4$, then there exists a maximal set S of m hamilton cycles in $T = K_n^p$.*

Proof: Clearly we can assume that $p \geq 3$ since if $p = 1$ there are no values of m to consider (and K_n^p has no edges anyway!). We prove the result by defining a graph G which satisfies conditions (1–3) of Theorem 3.1. The edges of G are defined as follows. If $x \in V$ and $y \in W$, then join x and y by exactly 9 edges. All vertices have the same degree in the graph G' defined so far, namely if $x \in V$ then $d_{G'}(x) = D_1 = 9\lceil \frac{p}{2} \rceil$, and if $y \in W$ then $d_{G'}(y) = D_2 = 9\lfloor \frac{p}{2} \rfloor$.

In order to be able to satisfy condition (2) of Theorem 3.1, edges still need to be added to G' in order to boost each vertex degree up to $6m$. Form the graph G from G' as follows: for each $z \in \{1, 2\}$, place a graph G_z of multiplicity at most 9 and regular of degree $6m - D_z$ on the vertices of G'. To see this is possible, we will apply Lemma 2.3 with $n = |A_z|$, $r = 6m - D_z = 6m - D_z$ and $\mu = 9$ for $1 \le z \le 2$, so we need to check that

1. $0 \le 6m - D_z \le 9(|A_z| - 1)$, and

2. if $|A_z|$ is odd then $6m - D_z$ is even,

where $A_1 = V$ and $A_2 = W$.

(1) Because p is odd we can write $p = 4x + 2\mathcal{E} + 1$ where $\mathcal{E} \in \{0, 1\}$.
First, to see $6m - D_z \ge 0$, note that $6m - D_2 \ge 6m - D_1$. Also, $6m - D_1$

$$\begin{aligned}
&\ge 6(\lceil \frac{3(p-1)}{4} \rceil + 2 - \mathcal{E}) - 9(\lceil \frac{p}{2} \rceil) \\
&= 6(\lceil \frac{3(4x + 2\mathcal{E})}{4} \rceil + 2 - \mathcal{E}) - 9(2x + \mathcal{E} + 1) \\
&= 6(3x + 2\mathcal{E} + 2 - \mathcal{E}) - 18x - 9\mathcal{E} - 9 \\
&= -3\mathcal{E} + 3 \\
&\ge 0.
\end{aligned}$$

(Note that this lower bound is not met if m is the smallest value allowed in Theorem 1.1 (ie. less than the lower bound on m in this theorem))

Next, we check the cases $z = 1$ and $z = 2$ in turn. Clearly x and $\mathcal{E}$ cannot both equal 0 since $p \ge 3$. Recall that $|V| = \lfloor \frac{p}{2} \rfloor = 2x + \mathcal{E}$, and $|W| = \lceil \frac{p}{2} \rceil = 2x + \mathcal{E} + 1$.

If $z = 1$ then

$$\begin{aligned}
& 9(|V| - 1) - (6m - D_1) \\
&\ge 9(\lfloor \frac{p}{2} \rfloor - 1) - (6\lfloor \frac{4(p-1) - 2}{4} \rfloor - 9\lceil \frac{p}{2} \rceil) \\
&= 9(2x + \mathcal{E} - 1) - 6\lfloor \frac{4(4x + 2\mathcal{E}) - 2}{4} \rfloor + 9(2x + \mathcal{E} + 1) \\
&= 18x + 9\mathcal{E} - 9 - 6(4x + 2\mathcal{E} - 1) + 18x + 9\mathcal{E} + 9 \\
&= 12x + 6\mathcal{E} + 6 \\
&\ge 0.
\end{aligned}$$

If $z = 2$ then

$$\begin{aligned} & 9(|W| - 1) - (6m - D_2) \\ \geq\ & 9(\lceil \frac{p}{2} \rceil - 1) - (6\lfloor \frac{4(p-1)-2}{4} \rfloor - 9\lfloor \frac{p}{2} \rfloor) \\ =\ & 9(2x + \mathcal{E}) - 6\lfloor \frac{4(4x + 2\mathcal{E}) - 2}{4} \rfloor + 9(2x + \mathcal{E}) \\ =\ & 18x + 9\mathcal{E} - 6(4x + 2\mathcal{E} - 1) + 18x + 9\mathcal{E} \\ =\ & 12x + 6\mathcal{E} + 6 \\ \geq\ & 0. \end{aligned}$$

(2) To see that (2) is satisfied, we will consider each value of z in turn.

Suppose $z = 1$. Let $|V| = \lfloor \frac{p}{2} \rfloor = 2y + 1$. Then $r = 6m - 9\lceil \frac{p}{2} \rceil$. This is even because $\lceil \frac{p}{2} \rceil = 2y + 2$ is even. Similarly, if $z = 2$ and if $|W| = \lceil \frac{p}{2} \rceil = 2y + 1$, then $r = 6m - 9\lfloor \frac{p}{2} \rfloor$ is even since $\lfloor \frac{p}{2} \rfloor = 2y + 2$ is even. Therefore, in each case, if $|A_z|$ is odd then r is even as required.

Clearly G is a connected graph with each vertex having degree $6m$, so condition (1) of Theorem 3.1 is satisfied. We now complete the proof by defining an m-edge-coloring of G that satisfies conditions (2–3) of Theorem 3.1.

Since G satisfies (1a) of Theorem 3.2, by ignoring one vertex w in W, it is easy to see that $8K_{\lfloor \frac{p}{2} \rfloor, \lfloor \frac{p}{2} \rfloor}$ is a subgraph of G. By [9], $8K_{\lfloor \frac{p}{2} \rfloor, \lfloor \frac{p}{2} \rfloor}$ has a hamilton decomposition into $8\lfloor \frac{p}{2} \rfloor/2$ hamilton cycles. Notice that $8\lfloor \frac{p}{2} \rfloor/2 \geq m$ since $4\lfloor \frac{p}{2} \rfloor - \lfloor \frac{4(p-1)-2}{4} \rfloor = 4\lfloor \frac{p}{2} \rfloor - (p-2) = 4\lfloor \frac{2y+1}{2} \rfloor - (2y+1) + 2 = 2y + 1 \geq 0$ (recall that p is odd). Therefore we can select m of these hamilton cycles, say $H_1, \ldots, H_m$, then color the edges of H_i with color i for $1 \leq i \leq m$.

Let H be the subgraph of G induced by $\cup_{i=1}^m E(H_i)$. It is important to notice that every vertex in $G - E(H)$ has even degree since every vertex in H has degree $2m$ (specifically each vertex in $G - E(H)$ has degree $6m - 2m = 4m$ except for w which has degree $6m$). We now apply Theorem 2.2 and give $G - E(H)$ an evenly-equitable m-edge-coloring (it is easy to check that $G - E(H)$ is connected, but in any case, Theorem 2.2 could be applied to each component). If $z \in V \cup W - \{w\}$ then it is incident with $4m/m = 4$ edges of each color in $G - E(H)$, and is incident with $2m/m = 2$ edges of each color in H. So, since w is incident with $6m/m = 6$ edges of each color in $G - E(H)$, each vertex in G is incident with 6 edges of each color, thus condition (2) of Theorem 3.1 is satisfied. Each color class in G is also connected since for $1 \leq i \leq m$, w is incident with an edge colored i (in fact 6 such edges) that join it to H_i, and H_i is a spanning connected subgraph of $G - \{w\}$.

So condition (3) of Theorem 3.1 is satisfied. Because conditions (1–3) of Theorem 3.1 are satisfied, it follows that there exists a maximal set of m hamilton cycles in K_n^p. ∎

References

[1] D.E. Bryant, S. El-Zanati and C.A. Rodger, Maximal sets of hamilton cycles in $K_{n,n}$, *J. Graph Theory* **33** (2000), 25–31.

[2] M.D. Daven, J.A. MacDougall, and C.A. Rodger, Maximal sets of hamilton cycles in complete multipartite graphs, *J. Graph Theory* **43** (2003), 49–66.

[3] D. de Werra, A few remarks on chromatic scheduling, *Combinatorial Programming: Methods and Applications* (B. Roy (ed.)), D. Reidel Pub. Co. (1975), 337–342.

[4] D. de Werra, Equitable colorations of graphs, *Rev. Fran. Rech. Oper.* **5(3)** (1971), 3–8.

[5] H.L. Fu, S.L. Logan, and C.A. Rodger, Maximal sets of hamilton cycles in $K_{2p} - F$, submitted.

[6] A.J.W. Hilton, Canonical edge-colourings of locally finite graphs, *Combinatorica* **2** (1982), 37–51.

[7] D.G. Hoffman, C.A. Rodger, and A. Rosa, Maximal sets of 2-factors and hamilton cycles, *J. Combin. Theory Ser. B* **57** (1993), 69–76.

[8] C.D. Leach and C.A. Rodger, Nondisconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs, *J. Combin. Designs* **9** (2001), 460–467.

[9] T.W. Tillson, A Hamilton decomposition of K^*_{2m}, $2m \geq 8$, *J. Combin. Theory Ser. B* **29** (1980), 68–74.

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