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Factorization of complete graphs into three factors with the smallest diameter equal to 3 or 4

Tomáš Vetrík

Department of Mathematics, SvF
Slovak University of Technology
Bratislava
Slovakia
vetrik@math.sk

Abstract

We determine the set of all positive integers n for which the complete graph of order n decomposes into three factors of given finite diameters $d_1 \leq d_2 \leq d_3$ in the case when d_1 is equal to 3 or 4.

1 Introduction

Let u, v be two vertices of maximum distance in a graph G. Then any path between u and v is called a diameter path of G. A factor of a graph G is a subgraph of G containing all the vertices of G. We say that G has a factorization (a decomposition into factors) if there exists a collection $F_1, F_2, \ldots, F_m$ of factors such that each edge of G belongs to exactly one of the factors. Graph factorizations have been extensively studied for many years. The study of decompositions of complete graphs into factors with given diameters was initiated by Bosák, Rosa and Znám [2] who proved the following fundamental result: If the complete graph K_n is decomposable into m factors $F_1, F_2, \ldots, F_m$ with prescribed diameters $d_1, d_2, \ldots, d_m$, then for every $n' > n$ the complete graph $K_{n'}$ is also decomposable into such factors. Therefore there exists the smallest positive integer $F(d_1, d_2, \ldots, d_m)$ such that for any $n \geq F(d_1, d_2, \ldots, d_m)$ the complete graph K_n admits a decomposition into m factors with prescribed diameters For information about the rich literature on the subject we recommend [1–3].

In what follows we focus on the special case of decomposition into three factors. Let us assume that the diameters d_1, d_2, d_3 satisfy $d_1 \leq d_2 \leq d_3 < \infty$. A construction of Bosák, Rosa and Znám [2] of decompositions of $K_{d_1+d_2+d_3-8}$ into three factors with the smallest diameter $d_1 \geq 5$ yields

$$F(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8.$$

Substantial progress was achieved by Hrnčiar [4] who proved that for $d_1 > 65$ the graph $K_{d_1+d_2+d_3-9}$ is not decomposable into three factors, which gives

$$F(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8.$$

Bosák, Rosa and Znám also began investigating decompositions of complete graphs into three factors with the smallest diameter $d_1 = 2$. In [2] they derived a solution of this problem except for the value of $F(2, 2, 2)$ for which they proved the inequality $12 \leq F(2, 2, 2) \leq 13$. The exact value was found by Stacho and Urland with the help of a computer search. In [7] they proved that K_{12} cannot be decomposed into three factors with diameters equal to 2, and therefore $F(2, 2, 2) = 13$.

In this paper we present a complete solution to the problem of decomposing a complete graph into three factors with the smallest diameter equal to 3 or 4.

2 An auxiliary result

We state an auxiliary result about the number of common neighbours of a pair of vertices under certain hypotheses. In the proof we will use the following Lemma A, proved in [2].

Lemma A Let G be a graph with n vertices and diameter d $(1 < d < n)$. Then the degree of every vertex in G is at most $n - d + 1$.

Lemma 1 *Let n and d be integers satisfying $n \geq 5$ and $2 \leq d \leq n - 3$, and let u and v be two distinct vertices in a graph G with n vertices and diameter d. Let $\deg(u) = a$ and $\deg(v) = b$, and let A_1 be a fixed diameter path of G. Suppose that $a + b > n - d + 3$ and that there are t vertices not in A_1 which are adjacent to neither u nor v in G.*

I. If $uv \in E(G)$, then there exist at least $(a + b) - (n - d + 3) + t$ vertices adjacent to both u and v in G.

II. If $uv \notin E(G)$, then there exist at least $(a + b) - (n - d + 2) + t$ vertices adjacent to both u and v in G.

Proof Let $a + b = n - d + 4 + k$ where k is a nonnegative integer. The set of all vertices adjacent to the vertex u (v) will be denoted by A (B). Note that the number of vertices in the fixed diameter path A_1 is $d + 1$ and the number of vertices not in A_1 is $n - d - 1$. If an arbitrary vertex not in A_1 is adjacent to three vertices in A_1, then one of these vertices must be adjacent in A_1 to both of the others. Suppose that $y_1, y_2, \ldots, y_t$, $0 \leq t \leq n - d - 3$, are the vertices not in A_1 adjacent neither to u nor to v in G.

I. Let $uv \in E(G)$. We first prove that $|A \cup B| \leq (n - d + 3) - t$. We consider three cases:

a) Let $u, v \in A_1$. Each of u and v can be adjacent to at most two vertices in A_1 and there are at most $(n - d - 1) - t$ vertices not in A_1 which are adjacent to u or v. Thus $|A \cup B| \leq 2 + 2 + (n - d - 1) - t = (n - d + 3) - t$.

b) Let $u \in A_1$, $v \notin A_1$. Except for u, the vertex v is adjacent to at most two vertices $v_1, v_2 \in A_1$ (possibly $v_1 = v_2$). The vertex u is adjacent to at most two vertices $u_1, u_2 \in A_1$. If $v_1 \neq v_2$ (as $uv \in E(G)$), at least one of these vertices coincides with some vertex u_i, $i \in \{1, 2\}$. Therefore there exist at most four vertices in A_1 and $(n-d-1)-t$ vertices not in A_1 contained in $A \cup B$. It follows that $|A \cup B| \leq (n-d+3)-t$.

c) Let $u \notin A_1$, $v \notin A_1$. The vertex u (v) is adjacent to at most three vertices $u_i, 1 \leq i \leq 3$ $(v_j, 1 \leq j \leq 3)$, in A_1. Again, we prove that there exist at most four distinct vertices in A_1 contained in $A \cup B$. If there exist at least five such vertices in A_1, then there would exist vertices u_i, v_j, where $i, j \in \{1, 2, 3\}$ such that in A_1 we have $\rho(u_i, v_j) = 4$. On the other hand, we have the path $u_i u v v_j$ of length 3 in G, which contradicts the equality $\rho(u_i, v_j) = 4$.

From the fact that $|A \cup B| \leq (n-d+3)-t$ it follows that $|A \cap B| = |A| + |B| - |A \cup B| \geq (a+b)-(n-d+3)+t$.

II. Let $uv \notin E(G)$. We first prove that $|A \cup B| \leq (n-d+2)-t$. Again, we distinguish three cases:

a) Let u, $v \in A_1$. Denote by $u_i, 1 \leq i \leq 2$ $(v_j, 1 \leq j \leq 2)$, the vertices in A_1 adjacent to u (v). Since $a+b = n-d+4+k$, there are at least $n-d$ edges in G joining u or v with the vertices not in A_1. Because the number of vertices not in A_1 adjacent to u or v is at least $(n-d-1)-t$, obviously there exists at least one vertex $\notin A_1$ adjacent to both u and v in G. Then we have $\rho(u, v) = 2$ in A_1 and hence there exists an $i \in \{1, 2\}$ and a $j \in \{1, 2\}$ such that $u_i = v_j$. Therefore $|A \cup B| \leq 3 + (n-d-1) - t = (n-d+2)-t$.

b) Let $u \in A_1$, $v \notin A_1$. The vertex u is adjacent to at most two vertices u_i in A_1 $(1 \leq i \leq 2)$ and the vertex v is adjacent to at most three vertices v_j in A_1 $(1 \leq j \leq 3)$. Since $a+b = n-d+4+k$, the graph G contains at least $n-d-1$ edges joining u or v with the vertices not in A_1. Except for the vertex v, there are $n-d-2$ vertices not in A_1, therefore there exists at least one vertex $x \notin A_1$ adjacent to both u and v in G. Suppose that the vertices u_1, u_2, v_1, v_2, v_3 are distinct. Then $\rho(u, v_j) \geq 2$ for all $j \in \{1, 2, 3\}$ and hence there exists the vertex v_j such that $\rho(u, v_j) = 4$. At the same time, we have the path $uxvv_j$ of length 3 in G, a contradiction. Consequently, $|A \cup B| \leq 4 + (n-d-2) - t = (n-d+2)-t$.

c) Let $u \notin A_1$, $v \notin A_1$. The vertex u (v) is adjacent to at most three vertices $u_i, 1 \leq i \leq 3$ $(v_j, 1 \leq j \leq 3)$, from A_1. Since $a+b = n-d+4+k$, there are at least $n-d-2$ edges in G joining u or v with the vertices not in A_1. Except for u and v, the number of the vertices not in A_1 is $n-d-3$ and analogously as in II b) it can be easily shown that it is not possible to have six distinct vertices $u_1, u_2, u_3, v_1, v_2, v_3$ in A_1. It follows that $|A \cup B| \leq 5 + (n-d-3) - t = (n-d+2)-t$.

Since $|A \cup B| \leq (n-d+2)-t$, we get $|A \cap B| = |A| + |B| - |A \cup B| \geq (a+b)-(n-d+2)+t$. This concludes the proof. □

We note that Lemma 1 has been presented in a more general form than actually needed. In particular, in most cases we will use it in situations where we do not assume the existence of some vertices adjacent neither to u nor to v in G.

3 Results: the case $d_1 = 3$

For $d_2 \leq 8$ and any $d_3 \geq d_2$ the exact values of the function $F(3, d_2, d_3)$ were presented by Palumbíny [5]. Further, for $d_2 \geq 9$ he proved that

$$F(3, d_2, d_3) \leq d_2 + d_3 - 6. \tag{1}$$

Motivated by this result, we started with attempts to determine the values of $F(3, d_2, d_3)$ for any d_2 and d_3. Our investigation was successful, because we showed that the complete graph of order $d_2 + d_3 - 7$ is not decomposable into three factors with diameters 3, d_2 and d_3.

Theorem 1 *Let $d_2 \geq 9$. Then $F(3, d_2, d_3) = d_2 + d_3 - 6$.*

A part of the following proof was published in a lesser known journal [8]. To make this paper self-contained, we give a full proof here.

Proof By (1), $F(3, d_2, d_3) \leq d_2 + d_3 - 6$. We have to show that the graph $K_{d_2+d_3-7}$ is not decomposable into three factors F_1, F_2, F_3, with diameters 3, d_2 and d_3. Let us prove this by contradiction and assume that such a decomposition exists. Denote by $A_1 = u_1pqu_2$ a fixed diameter path of the factor F_1. Invoking Lemma A it follows that the degree of every vertex in F_2 (F_3) is at most $(d_2 + d_3 - 7) - d_2 + 1 = d_3 - 6$ ($(d_2 + d_3 - 7) - d_3 + 1 = d_2 - 6$), thus every vertex of F_1 has degree greater than or equal to $(d_2 + d_3 - 8) - (d_2 - 6) - (d_3 - 6) = 4$. Let us denote the degree of the vertex u_1 (u_2) in F_1 by $\deg(u_1) = a$ ($\deg(u_2) = b$). Here $a \geq 4$, $b \geq 4$ and $a + b \leq d_2 + d_3 - 9$, because the number of vertices not in A_1 is $d_2 + d_3 - 11$. Since there are no vertices adjacent to both u_1 and u_2 in F_1, we have $(d_2 + d_3 - 9) - (a + b)$ vertices adjacent neither to u_1 nor to u_2 in F_1.

Without loss of generality we may suppose that the edge $u_1u_2 \in E(F_2)$. We denote the degree of $u_i, i = 1, 2$, in F_2 by $\deg(u_1) = c$ and $\deg(u_2) = d$. The degree of any vertex in F_3 is at most $d_2 - 6$ and the degree of u_1 (u_2) in F_1 is a (b), hence in F_2 we have $(d_2 + d_3 - 8) - (d_2 - 6) - a = d_3 - 2 - a \leq c \leq d_3 - 6$ ($(d_2 + d_3 - 8) - (d_2 - 6) - b = d_3 - 2 - b \leq d \leq d_3 - 6$). If $c + d > d_3 - 4 = (d_2 + d_3 - 7) - d_2 + 3$, then by part I of Lemma 1 there exist at least $c + d - (d_3 - 4)$ vertices adjacent to both u_1 and u_2 in F_2. If $c + d \leq d_3 - 4$, then $c + d - (d_3 - 4) \leq 0$ so trivially there exist at least $c + d - (d_3 - 4)$ vertices adjacent to both u_1 and u_2 in F_2.

Observe that the lower bound $c + d - (d_3 - 4)$ on the number of vertices adjacent to both u_1 and u_2 in F_2 must be less than or equal to the number $d_2 + d_3 - 9 - (a + b)$ of vertices adjacent neither to u_1 nor to u_2 in F_1. Let us denote the quantity $d_2 + d_3 - 9 - (a + b) - (c + d - d_3 + 4) = d_2 + 2d_3 - 13 - a - b - c - d$ by k, clearly $k \geq 0$. We show that the number of vertices adjacent to both u_1 and u_2 in F_3 exceeds k, which will give a contradiction. In F_3, $\deg(u_1) = d_2 + d_3 - 8 - a - c$ and

$\deg(u_2) = d_2+d_3-8-b-d$. Note that in F_3 we must have $\deg(u_1)+\deg(u_2) > d_2-4$ for otherwise we would get $2(d_2+d_3-8)-a-b-c-d \le d_2-4$ and, consequently, $d_2+2d_3-12-a-b-c-d = k+1 \le 0$.

It follows that $\deg(u_1)+\deg(u_2) > d_2-4 = (d_2+d_3-7)-d_3+3$ and by part II of Lemma 1, the number of vertices adjacent to both u_1 and u_2 in F_3 is at least $(d_2+d_3-8-a-c)+(d_2+d_3-8-b-d)-(d_2-5) = d_2+2d_3-11-a-b-c-d = k+2$, a contradiction. □

4 Results: the case $d_1 = 4$

We also studied the problem of decomposition into three factors with the smallest diameter 4. An early attempt to solve this problem is due to Říhová [6]. For d_2 equal to 4 and 5 she found the values of $F(4, d_2, d_3)$ and for $d_2 \ge 6$ she established the inequality

$$F(4, d_2, d_3) \le d_2 + d_3 - 4. \tag{2}$$

We improve the upper bound (2) by constructing of decomposition of the graph $K_{d_2+d_3-5}$ into such three factors, proving thereby the following result.

Theorem 2 *Let $d_2 \ge 6$. Then $F(4, d_2, d_3) \le d_2 + d_3 - 5$.*

Proof It suffices to decompose the graph $K_{d_2+d_3-5}$ into three factors F_1, F_2, F_3 with the diameters $d(F_1) = 4$, $d(F_2) = d_2$ and $d(F_3) = d_3$. Let us denote the vertices of $K_{d_2+d_3-5}$ by $u_1, u_2, \ldots, u_7, v_1, \ldots, v_{d_2-6}, w_1, \ldots, w_{d_3-6}$.

We first decompose K_7 with vertices $u_1, u_2, \ldots, u_7$ into the factors F_1', F_2', F_3' with the diameters $d(F_1') = 4$, $d(F_2') = 6$ and $d(F_3') = 6$. Define the sets of edges of F_1', F_2' and F_3' as follows:

$E(F_1') = \{u_1u_5, u_5u_7, u_7u_3, u_3u_2, u_6u_1, u_6u_5, u_6u_7, u_4u_3, u_4u_2\}$,

$E(F_2') = \{u_7u_4, u_4u_1, u_1u_2, u_2u_6, u_6u_3, u_3u_5\}$,

$E(F_3') = \{u_3u_1, u_1u_7, u_7u_2, u_2u_5, u_5u_4, u_4u_6\}$.

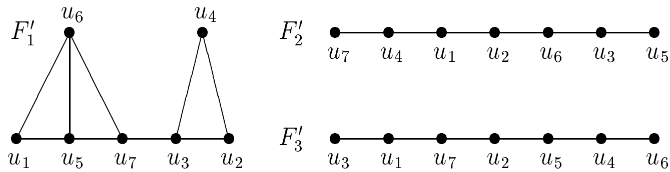

Figure 1

As the next step, we define the sets:

$X_2 = \emptyset$ if $d_3 = 6$,

$\{u_1w_i, u_2w_i, i = 1, \ldots, d_3-6\}$ if $d_3 \ge 7$;

$Y_2 = \emptyset$ if $d_2 = 6$,
$\{u_7v_1\}$, if $d_2 = 7$,
$\{u_7v_1, v_1v_2, \dots, v_{d_2-7}v_{d_2-6}\}$, if $d_2 \geq 8$;

$X_3 = \emptyset$ if $d_3 = 6$,
$\{u_3w_1\}$ if $d_3 = 7$,
$\{u_3w_1, w_1w_2, \dots, w_{d_3-7}w_{d_3-6}\}$ if $d_3 \geq 8$;

$Y_3 = \emptyset$ if $d_2 = 6$,
$\{u_1v_i, u_2v_i, i = 1, 2, \dots, d_2 - 6\}$ if $d_2 \geq 7$;

$X_1 = \{u_iv_j, u_iw_k, v_jw_k, v_xv_s, w_yw_t,\ i = 3, 4, 5, 6, 7;\ j, s, x = 1, 2, \dots, d_2 - 6,$ where $x \neq s;\ k, t, y = 1, 2, \dots, d_3 - 6,$ where $y \neq t\} - (Y_2 \cup X_3)$.

Then the sets of edges of the factors F_1, F_2, F_3 of decomposition of $K_{d_2+d_3-5}$ are:
$E(F_1) = E(F_1') \cup X_1$,
$E(F_2) = E(F_2') \cup X_2 \cup Y_2$,
$E(F_3) = E(F_3') \cup X_3 \cup Y_3$.
It is easy to check that $d(F_2) = d_2$ and $d(F_3) = d_3$. It remains to show that $d(F_1) = 4$. In F_1, the vertex u_1 is adjacent only to u_5 and u_6; and the vertex u_2 is adjacent only to u_3 and u_4. Note that no edge from the set $\{u_3u_5, u_3u_6, u_4u_5, u_4u_6\}$ belongs to $E(F_1)$, therefore $d(F_1) \geq 4$. Since every vertex v_j, $j = 1, 2, \dots, d_2 - 6$, and every vertex w_k, $k = 1, 2, \dots, d_3 - 6$, is adjacent to the vertices u_4, u_5, u_6, we have $d(F_1) = 4$. □

Finally, we show that for $d_2 \geq 6$ the graph $K_{d_2+d_3-6}$ can not be decomposed into three factors.

Theorem 3 *Let $d_2 \geq 6$. Then $F(4, d_2, d_3) = d_2 + d_3 - 5$.*

Proof By Theorem 2, we have $F(4, d_2, d_3) \leq d_2 + d_3 - 5$. If $d_2 = 6$, then $F(4, 6, d_3) \leq d_3 + 1$. It is evident that $F(4, 6, d_3) > d_3$, hence $F(4, 6, d_3) = d_3 + 1$.

Assume that $d_2 \geq 7$. We prove that the graph $K_{d_2+d_3-6}$ cannot be decomposed into three factors F_1, F_2 and F_3 with diameters 4, d_2 and d_3. Suppose the contrary and let $K_{d_2+d_3-6}$ be decomposable into such factors. Let $A_1 = u_1pqru_2$ be a fixed diameter path of F_1. According to Lemma A, the degree of each vertex in F_2 (F_3) is at most $d_3 - 5$ ($d_2 - 5$), therefore the degree of each vertex in F_1 is at least $(d_2 + d_3 - 7) - (d_2 - 5) - (d_3 - 5) = 3$.

Denote the degree of $u_i, i = 1, 2$, in F_1 by $\deg(u_1) = a$ and $\deg(u_2) = b$, where $a, b \geq 3$. Since the number of vertices not in A_1 is $d_2 + d_3 - 11$, we have $a + b \leq d_2 + d_3 - 9$. There are $d_2 + d_3 - 8 - (a + b)$ vertices adjacent neither to u_1 nor to u_2 in F_1.

We may assume without loss of generality that $u_1u_2 \in E(F_2)$. Let us denote the degree of u_1 and u_2 in F_2 by $\deg(u_1) = c$ and $\deg(u_2) = d$. Because every vertex of F_3 has degree at most $d_2 - 5$ and u_1 (u_2) is of degree a (b) in F_1, obviously

$(d_2+d_3-7)-(d_2-5)-a = d_3-2-a \le c \le d_3-5$ $((d_2+d_3-7)-(d_2-5)-b = d_3-2-b \le d \le d_3-5)$ in F_2. If $c+d > d_3-3 = (d_2+d_3-6)-d_2+3$, then by part I of Lemma 1 there exist at least $c+d-(d_3-3)$ vertices adjacent to both u_1 and u_2 in F_2. If $c+d \le d_3-3$, then $c+d-(d_3-3) \le 0$ so trivially there exist at least $c+d-(d_3-3)$ vertices adjacent to both u_1 and u_2 in F_2. It is evident that $c+d-(d_3-3)$ has to be $\le d_2+d_3-8-(a+b)$. We distinguish two cases:

a) Let $c+d-(d_3-3) < d_2+d_3-8-(a+b)$. Denote the number $d_2+d_3-8-(a+b)-(c+d-d_3+3) = d_2+2d_3-11-a-b-c-d$ by k, where $k>0$. In F_3 we have $\deg(u_1) = d_2+d_3-7-a-c$ and $\deg(u_2) = d_2+d_3-7-b-d$. If $\deg(u_1)+\deg(u_2) \le d_2-3$, then $2(d_2+d_3-7)-a-b-c-d \le d_2-3$ and consequently, $d_2+2d_3-11-a-b-c-d = k \le 0$. On the other hand, if $\deg(u_1)+\deg(u_2) > d_2-3 = (d_2+d_3-6)-d_3+3$, by part II of Lemma 1 it follows that the number of vertices adjacent to both u_1 and u_2 in F_3 is at least $(d_2+d_3-7-a-c)+(d_2+d_3-7-b-d)-(d_2-4) = d_2+2d_3-10-a-b-c-d = k+1$, which gives a contradiction.

b) Let $c+d-(d_3-3) = d_2+d_3-8-(a+b)$. It is easy to see that the vertices adjacent to both u_1 and u_2 in F_2 are the same as the vertices distinct from u_1 and u_2 which are adjacent neither to u_1 nor to u_2 in F_1. Denote by A_2 a fixed diameter path of F_2.

b1) Assume that the degrees of both end-vertices of A_2 are greater than 1. If each end-vertex has distance ≤ 2 from u_1 or distance ≤ 2 from u_2, then the diameter of F_2 cannot be d_2, because $u_1u_2 \in E(F_2)$.

If at least one end-vertex of A_2 has distance greater than 2 from both u_1 and u_2, then there exists the vertex $y \notin A_2$ adjacent to some end-vertex of A_2, but adjacent neither to u_1 nor to u_2 in F_2. Since $c+d-(d_3-3) = d_2+d_3-8-(a+b) \ge 1$, and thus $c+d > d_3-3 = (d_2+d_3-6)-d_2+3$, invoking part I of Lemma 1 it follows that the number of vertices adjacent to both u_1 and u_2 in F_2 is at least $c+d-((d_2+d_3-6)-d_2+3)+1 = c+d-(d_3-4)$. This is more than $d_2+d_3-8-(a+b)$, a contradiction.

b2) Suppose that the degree of some end-vertex of A_2 is equal to 1. We denote such a vertex by x. Note that there is at least one vertex adjacent to both u_1 and u_2 (since $d_2+d_3-8-(a+b) \ge 1$) and $u_1u_2 \in E(F_2)$, therefore $x \notin \{u_1,u_2\}$. Now we prove that in spite of the fact that x is in F_2 adjacent to at most one vertex u_i, $i=1$ or 2, the vertex x belongs to $d_2+d_3-8-(a+b)$ vertices adjacent neither to u_1 nor to u_2 in F_1, which will give a final contradiction. Let $a = \deg(u_1) \le \deg(u_2) = b$ in F_1. (The case $\deg(u_1) > \deg(u_2)$ can be handled similarly.)

A. If $\deg(x) < a$ in F_3, then $\deg(x) \ge d_2+d_3-7-a$ in F_1, which means that in F_1, the vertex x is not adjacent to at most a vertices. But observe that any vertex adjacent to u_2 (u_1) cannot be adjacent to the $a+1$ vertices (to the $b+1$ vertices) of distance ≤ 1 from u_1 (u_2) in F_1.

B. Let us assume that in F_3, $\deg(x) = a-1+z$, where $z \in N$. Then in F_1 we have $\deg(x) = (d_2+d_3-7)-1-(a-1+z) = d_2+d_3-7-a-z$. Since $(a-1+z)+(d_2+d_3-7-a-c) > d_2-3 = (d_2+d_3-6)-d_3+3$, by part II of Lemma

1, there exist at least $(a-1+z)+(d_2+d_3-7-a-c)-[(d_2+d_3-6)-d_3+2] = d_3-4-c+z$ vertices adjacent to both u_1 and x in F_3. Since $c \le d_3 - 5$, there are at least $z+1$ such vertices. Suppose that x is adjacent to u_2 (u_1) in F_1. Then x cannot be adjacent to the $a+1$ vertices (to the $b+1$ vertices) of distance ≤ 1 from u_1 (u_2) and to $z+1$ vertices adjacent to both u_1 and x in F_3. Because $b \ge a$ and $\deg(x) = d_2+d_3-7-a-z$ in F_1, we get a contradiction. This completes the proof. □

The problem of decomposing a complete graph into three factors with the smallest diameter d_1 equal to 3 or 4 is therefore solved. In view of [4], for a complete determination of the values of $F(d_1, d_2, d_3)$ for $d_1 \le d_2 \le d_3 < \infty$ it remains to determine the exact values for $5 \le d_1 \le 65$.

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