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2-Regular leaves of partial 8-cycle systems

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Abstract

In this paper, we find necessary and sufficient conditions for the existence of an 8-cycle system of $K_n - E(R)$ where R is any 2-regular not necessarily spanning subgraph R of K_n.

1 Introduction

An *H-decomposition* of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In the case where H is an m-cycle, such a decomposition is referred to as an *m-cycle system* of G. An m-cycle system will be formally described as an ordered pair (V,B) where V is the vertex set of G and B is the set of m-cycles.

Results in this area date back to the 1800s, but have received a lot of attention over the past 40 years. There have been many results found on H-decompositions of G for various graphs H and G, but mainly on H-decompositions of K_n. The graphs H that have been of most interest are paths [16], m-stars [15], m-cycles [10, 7, 9],

m-wheels [4] and m-nestings [4, 11] (which are two decompositions of K_n, one into m-stars and the other into m-cycles, so that each m-cycle can be paired with an m-star to form a wheel). Recently a paper by Alspach and Gavlas [1], based on a paper by Hoffman, Lindner, and Rodger [7], and another by Šajna [13] settled the problem of finding the values of n for which there exists an m-cycle system of K_n and of $K_n - I$, where I is a one-factor. This can alternatively be viewed as a *partial* m-cycle system in which the set of edges not in any m-cycle is either $\emptyset$ or induces a one-factor respectively. These edges not in any m-cycle (or the subgraph they induce) are called the *leave* R.

Using such disparate techniques as amalgamations, difference methods, and induction, various authors have considered m-cycle decompositions of other graphs which are close to being complete, each having a leave that is 2-regular. For example, the existence of m-cycle systems of $K_n - E(R)$, for any 2-regular graph R has been settled when $m = 3$ [5], $m = 4$ [6], and $m = 6$ [2], and $m = n$ [3] (i.e. hamilton decompositions). Hamilton decompositions of the regular complete multipartite graph with any 2-factor removed has also been settled [8, 12].

In this paper, we extend these results by finding necessary and sufficient conditions for the existence of a 8-cycle system of $K_n - E(R)$, for every 2-regular not necessarily spanning subgraph R of K_n; see Theorem 5.1. The proof of Theorem 5.1 uses two tools extensively. One tool, Lemma 3.2, requires that we solve the cases for $n \leq 15$. This initially appears to be quite a bit of work; however, if the 8-cycles are chosen wisely one m-cycle system can solve all possible leaves, see Figure 2. The second tool, Lemma 3.3, requires that we solve the cases for $n \leq 23$. Additionally, we can use Lemma 3.2 to solve many of the cases for $17 \leq n \leq 23$. Lemma 3.2 along with choosing 8-cycles wisely greatly minimizes the m-cycle systems we must construct. Once we have Lemmas 3.2 and 3.3, we use them to exhaust all of the remaining cases in Theorem 5.1.

2 The Small Cases

Necessary conditions limiting the number of edges in R are given in Lemma 2.1 and Table 1. Alternatively, since R is 2-regular, these conditions can be described in terms of the number of vertices in K_n that occur in no cycle in R; such vertices are called *isolated vertices of* R. The set of isolated vertices of R is denoted by $I(R)$. This is also shown in Lemma 2.1 and Table 1.

Lemma 2.1 *Let R be a 2-regular subgraph of the complete graph K_n. If there exists an 8-cycle system of $K_n - E(R)$, then n is odd and the number of edges in $K_n - E(R)$ is divisible by 8, and these hold if and only if n, $|E(R)|$, and $|I(R)|$ are related as in Table 1.*

Proof: Clearly once the edges in R are removed each vertex must have even degree in order for $K_n - E(R)$ to have a 8-cycle decomposition, so n is odd, and clearly 8 must divide $|E(K_n - E(R))|$.

n	$16k+1$	$16k+3$	$16k+5$	$16k+7$	$16k+9$	$16k+11$	$16k+13$	$16k+15$
$\vert E(R)\vert$	0	3	2	5	4	7	6	1
$\vert I(R)\vert$	1	0	3	2	5	4	7	6

Table 1: The number of edges in R and the number of isolated vertices of R required in order that 8 divides $|E(K_n - E(R))|$ when n is odd.
(Values $|E(R)|$ and $|I(R)|$ are modulo 8.)

Suppose n is odd and $|E(K_n - E(R))|$ is divisible by 8. Then $|E(R)| \equiv \binom{n}{2}$ (mod 8), thereby giving the second line of Table 1. Also, $|I(R)| = n - |E(R)|$, gives the third line of Table 1. A proof of the converse statement follows similarly.

Consider a 2-regular subgraph R of K_n for some n. By Table 1, we know that $|I(R)| \equiv 0, 1, 2, 3, 4, 5, 6, 7$ (mod 8) and n is odd. Thus if $|I(R)| \geq 8$, we add 8-cycles to R until the resulting 2-regular graph R' satisfies $|I(R')| \equiv |I(R)|$ (mod 8) and $|I(R')| \leq 7$. Once a set of 8-cycles B with leave R' is found, the required 8-cycle system with leave R is $B \cup (R' \setminus R)$. Therefore, throughout the rest of this paper, we may assume that $|I(R')| \leq 7$.

Lemma 2.2 *Let $n \in \{1, 3, 9, 11, 13, 15\}$ and let R be a 2-regular subgraph in K_n. If n is odd and $|E(K_n - E(R))|$ is divisible by 8, then there exists a 8-cycle system of $K_n - E(R)$.*

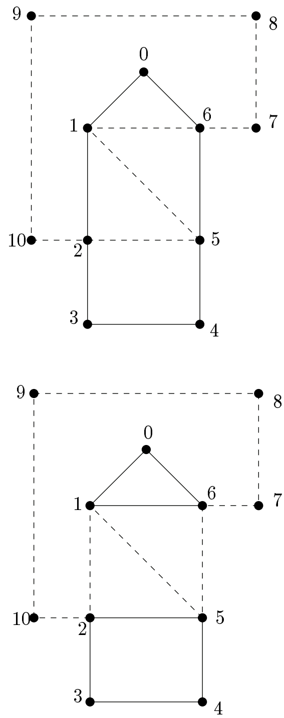
Proof: We consider each value of n in turn. In each case, we construct an 8-cycle system $(\mathbb{Z}_n, B)$. Using Table 1, it is easy to check that the given conditions require that if $n = 1$ then $E(R) = \emptyset$ and $B = \emptyset$, and if $n = 3$ then R is a 3-cycle, and $B = \emptyset$.

$\boldsymbol{n = 9}$: R is a 4-cycle. $R = \{(0, 1, 2, 3)\}$ and $B = \{(1, 3, 5, 6, 7, 8, 0, 4), (2, 4, 7, 0, 5, 8, 1, 6), (3, 4, 5, 1, 7, 2, 8, 6), (2, 5, 7, 3, 8, 4, 6, 0)\}$.

$\boldsymbol{n = 11}$:If R_1 is a 7-cycle, then $R_1 = (0, 1, 2, 3, 4, 5, 6)$ and $B_1 = \{(10, 2, 5, 1, 6, 7, 8, 9), (0, 4, 6, 9, 3, 8, 10, 7), (2, 4, 10, 0, 9, 7, 5, 8), (1, 10, 3, 0, 5, 9, 2, 7), (1, 9, 4, 8, 6, 10, 5, 3), (0, 2, 6, 3, 7, 4, 1, 8)\}$; if R_2 is a 3-cycle and 4-cycle, then $R_2 = \{(0, 1, 6), (2, 3, 4, 5)\}$ and $B_2 = \{(10, 2, 1, 5, 6, 7, 8, 9), (0, 4, 6, 9, 3, 8, 10, 7), (2, 4, 10, 0, 9, 7, 5, 8), (1, 10, 3, 0, 5, 9, 2, 7), (1, 9, 4, 8, 6, 10, 5, 3), (0, 2, 6, 3, 7, 4, 1, 8)\}$.

Note: The sets B_1 and B_2 of 8-cycles are identical except for the first 8-cycle listed in each set. The first cycle was picked carefully so that by swapping two of its edges with two edges in the leave, the leave would be transformed from one possible leave to another. (See Figure 1.) This method will be used in for $n \in \{13, 15\}$ to generate partial 8-cycle systems with all possible leaves.

$\boldsymbol{n = 13}$: If R_1 is a 6-cycle, then $R_1 = \{(0, 1, 2, 10, 11, 12)\}$ and $B_1 = \{(12, 1, 11, 2, 3, 5, 7, 10), (0, 11, 6, 9, 1, 4, 8, 3), (10, 8, 2, 4, 7, 6, 0, 5), (9, 4, 3, 6, 8, 7, 0, 2), (1, 7, 12, 3, 9, 5, 2, 6), (0, 9, 11, 3, 7, 2, 12, 8), (0, 10, 3, 1, 5, 12, 6, 4), (12, 9, 8, 11, 5, 6, 10, 4), (1, 10, 9, 7, 11, 4, 5, 8)\}$; if R_2 is two 3-cycles, then the switching process gives $R_2 = \{(0, 1, 12),$



We denote this switch by:

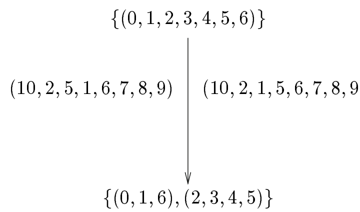


Figure 1: Combining and separating leaves.

$(2,10,11)\}$ and $B_2 = \{(12,11,1,2,3,5,7,10),\ (0,11,6,9,1,4,8,3),\ (10,8,2,4,7,6,0,5),\ (9,4,3,6,8,7,0,2),\ (1,7,12,3,9,5,2,6),\ (0,9,11,3,7,2,12,8),\ (0,10,3,1,5,12,6,4),\ (12,9,8,11,5,6,10,4),\ (1,10,9,7,11,4,5,8)\}$;

$\boldsymbol{n = 15}$: If R_1 is a 3-cycle and 6-cycle, then $R_1 = \{(0,1,8),(2,3,4,5,6,7)\}$ and $B_1 = \{(10,1,2,8,7,14,13,11),\ (9,3,0,4,8,12,13,10),\ (11,3,7,4,6,14,9,12),\ (0,9,6,11,1,5,12,7),\ (0,6,3,10,14,4,2,11),\ (0,10,6,1,9,11,5,2),\ (1,3,5,13,6,8,10,12),\ (0,12,14,8,13,2,10,5),\ (1,7,9,13,3,12,2,14),\ (2,6,12,4,13,7,5,9),\ (3,8,11,4,1,13,0,14),\ (4,9,8,5,14,11,7,10)\}$. The remaining partial 8-cycle decompositions are found using the same method shown in Figure 1. This is shown as a flowchart in Figure 2.

3 Some Tools

One tool that we will need is the following, a particular case of a theorem by Sotteau [14]:

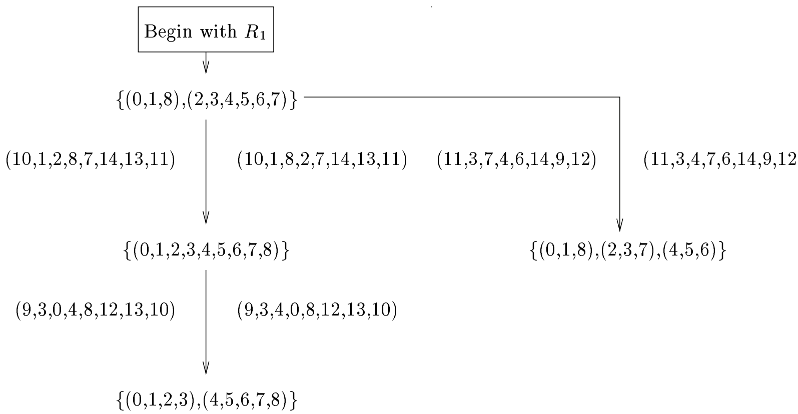


Figure 2: All possible leaves when $n = 15$.

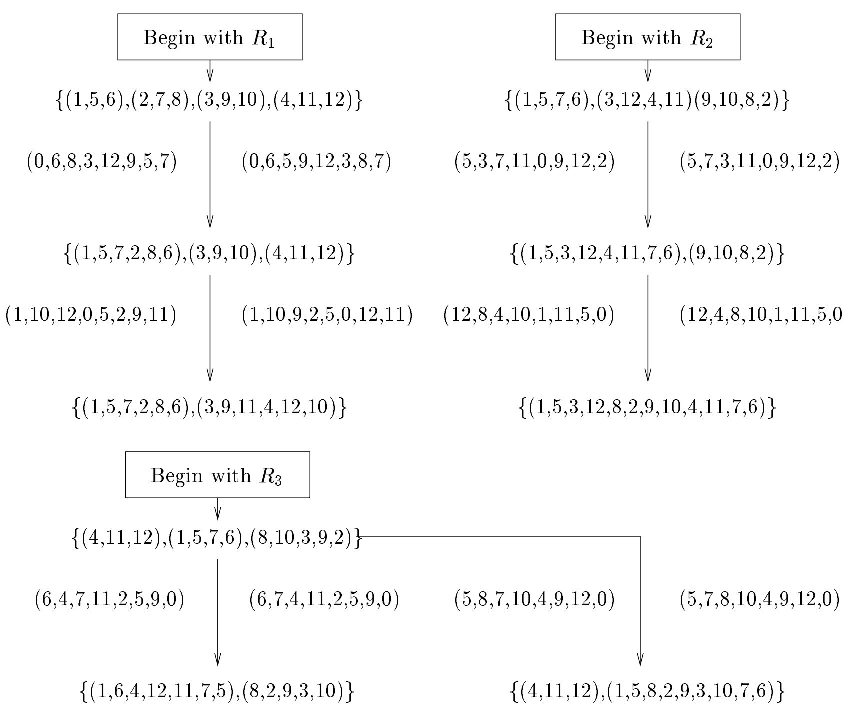


Figure 3: The remaining leaves in Lemma 3.1.

Theorem 3.1 *There exists an 8-cycle system of $K_{a,b}$ if and only if:*
1) a and b are even,
2) 4 divides a or b, and
3) $\min\{a,b\} \geq 4$.

Let G^c denote the complement of a graph G. Also let $G \vee H$ denote the *join* of two vertex disjoint graphs G and H (so $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$). We now show the existence of an 8-cycle system of $K_8 \vee K_5^c - E(R_i)$ where R_i is any 2-regular subgraph of $K_8 \vee K_5^c$ satisfying the following:

1. R_i has 12 edges,
2. exactly eight edges of R_i have one end in the set $\{1,2,3,4\}$ and one end in $\{5,6,7,8,9,10,11,12\}$,
3. the remaining four edges have both ends in $\{5,6,7,8,9,10,11,12\}$.

These edges of R_i are chosen carefully because they will be used later in the proof of Lemma 3.2.

Lemma 3.1 *Define three graphs on the vertex set $\mathbb{Z}_{13}$ as follows:*
Let G_i be $K_8 \vee K_5^c - E(R_i)$ for $i \in \{1,2,3\}$ where $R_1 = \{(1,5,6),(2,7,8),(3,9,10),(4,11,12)\}$, $R_2 = \{(1,5,7,6),(3,12,4,11),(9,10,8,2)\}$, and $R_3 = \{(4,11,12),(1,5,7,6),(8,10,3,9,2)\}$, with $V(K_5^c) = \{0,1,2,3,4\}$ and $V(K_8) = \{5,6,7,8,9,10,11,12\}$ for each case. Then there exists an 8-cycle system of G_i for $1 \leq i \leq 3$.

Proof: For $1 \leq i \leq 3$, the required 8-cycle system $(\mathbb{Z}_{13}, B_i)$ of G_i is defined by
$B_1 = \{(0,6,8,3,12,9,5,7),\ (1,10,12,0,5,2,9,11),\ (5,4,9,0,10,2,12,8),\ (1,12,5,11,6,4,10,7),\ (1,9,6,3,11,7,4,8),\ (0,11,10,6,12,7,9,8),\ (5,3,7,6,2,11,8,10)\}$,
$B_2 = \{(5,3,7,11,0,9,12,2),\ (12,8,4,10,1,11,5,0),\ (2,11,6,9,5,10,0,7),\ (3,8,9,4,7,12,10,6),\ (4,6,8,7,10,11,12,5),\ (5,6,2,10,3,9,1,8),\ (9,11,8,0,6,12,1,7)\}$,
$B_3 = \{(5,8,7,10,4,9,12,0),\ (6,4,7,11,2,5,9,0),\ (1,10,12,3,6,5,11,8),\ (3,8,6,10,11,9,1,7),\ (4,8,9,7,2,6,12,5),\ (11,3,5,10,0,8,12,1),\ (12,7,0,11,6,9,10,2)\}$.

For each remaining leave R', the 8-cycle decomposition of $K_8 \vee K_5^c - E(R')$ satisfying (1-3) are obtained using cycle switches as outlined in Figure 3.

A 2-regular subgraph R of K_n is said to be *n-admissible* if 8 divides $|E(K_n) - E(R)|$ and n is odd.

Lemma 3.2 *Suppose that R is n-admissible and that R contains cycles $c_1, \ldots, c_\alpha$ of lengths $\ell_1, \ldots, \ell_\alpha$ respectively. Suppose further that $\ell_i = a_i + b_i$ for $1 \leq i \leq \alpha$ and $x \in \{1,2,3,4\}$ such that*

1. *for $1 \leq i \leq x$, $a_i \geq 3$ and either $b_i = 0$ or $b_i \geq 3$; for $x+1 \leq i \leq \alpha$, $a_i = 0$ and $b_i = \ell_i \geq 3$,*

2. $\sum_{i=1}^{x} a_i = 12$, *and*

3. $n \geq 13 + 4x$.

   Suppose also that

4. *for every* $(n-8)$*-admissible* 2*-regular graph* R' *there exists an* 8*-cycle system of* $K_{n-8} - E(R')$.

Then there exists an 8*-cycle system of* $K_n - E(R)$.

Proof: Let the vertex set of K_n be $\mathbb{Z}_n$. By condition (1) we can form R_1 from R as follows: Begin by forming a 2-regular graph R_1 from R by replacing the cycle c_i with cycles $c_{i,1}$ and $c_{i,2}$ of lengths a_i and b_i respectively for $1 \leq i \leq x$. By (2) we can name the vertices so that

(a) $\cup_{i=1}^{x} V(c_{i,1}) = \mathbb{Z}_4 \cup (\mathbb{Z}_n \setminus \mathbb{Z}_{n-8})$,

(b) for $1 \leq i \leq x$, $\{z_{i,1}, z_{i,2}\} \in E(c_{i,1})$ for some vertices $z_{i,1}, z_{i,2}$ in $\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}$ (this is possible since $a_i \geq 3$), and

(c) for $1 \leq i \leq x$, $\{w_{i,1}, w_{i,2}\} \in E(c_{i,2})$ for some vertices $w_{i,1}, w_{i,2}$ in $\mathbb{Z}_{n-8} \setminus \mathbb{Z}_5$.

Also, for $1 \leq i \leq x$, by (3) we can define sets $S_1, \ldots, S_x$ satisfying

(d) $\{S_i, \ldots, S_x\}$ is a partition of $\mathbb{Z}_{n-8} \setminus \mathbb{Z}_5$, where $|S_i| \geq 4$ is even, and

(e) $\{w_{i,1}, w_{i,2}\} \subseteq S_i$ for $1 \leq i \leq x$.

By Lemma 3.1 there exists an 8-cycle system $(\mathbb{Z}_5 \cup (\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}), B_1)$ of $K_{13} - (E(c_{1,1}) \cup E(c_{2,1}) \cup \cdots \cup E(c_{x,1}) \cup E(K_5))$.

Form R_2 from R_1 by removing $c_{1,1}, \ldots, c_{x,1}$. Since $|E(R_2)| = |E(R_1)| - 12$, (by (2)), and since $K_n - E(R)$ is n-admissible, by Table 1, $K_{n-8} - E(R_2)$ is $(n-8)$-admissible. Therefore by (4), there exists an 8-cycle system $(\mathbb{Z}_{n-8}, B_2)$ of $K_{n-8} - E(R_2)$.

By Sotteau's Theorem, for $1 \leq i \leq x$ there exists an 8-cycle system $((S_i, \mathbb{Z}_n \setminus \mathbb{Z}_{n-8}), B_{3,i})$ of $K_{|S_i|,8}$. Name one of the 8-cycles in $B_{3,i}$ so that one of them begins with $c_{i'} = (w_{i,1}, z_{i,1}, w_{i,2}, z_{i,2}, \ldots)$.

Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup (\cup_{i=1}^{x} B_{3,i}))$ is an 8-cycle system of $K_n - E(R_1)$. The required 8-cycle system of $K_n - E(R)$ can now be made by swapping the edges $\{w_{i,1}, z_{i,1}\}$ and $\{w_{i,2}, z_{i,2}\}$ in $B_{3,i}$ with the edges $\{w_{i,1}, w_{i,2}\}$ and $\{z_{i,1}, z_{i,2}\}$ in R_1 for $1 \leq i \leq x$.

Lemma 3.3 *Let* $n \geq 25$ *be odd. Suppose* R *is an* n*-admissible* 2*-regular graph which contains cycles of lengths* $\ell_1, \ldots, \ell_\alpha$*, named so that for some* $x \geq 0$

$$\sum_{i=1}^{x-1} \ell_i \leq 16 \quad \textit{and} \quad \sum_{i=1}^{x} \ell_i \geq 19.$$

Suppose also that if R' is an $(n-16)$-admissible 2-regular graph then there exists an 8-cycle system of $K_{n-16} - E(R')$. Then there exists an 8-cycle system of $K_n - E(R)$.

Proof: If $x = 0$ then let $\ell = 0$, and otherwise let $\ell = \sum_{i=1}^{x-1} \ell_i$. Let $L = \sum_{i=1}^{x} \ell_i$. We consider several cases depending on the values of ℓ and L. In each case we define graphs R_1 and R_2 on the vertex sets $\mathbb{Z}_{19}$ and $\mathbb{Z}_n \backslash \mathbb{Z}_{16}$ respectively, the edges in each such graph inducing a 2-regular graph.

If $\ell = 16$ then let R_1 consist of cycles of lengths $\ell_1, \ldots, \ell_{x-1}$ and 3, the cycle of length 3 being $(16, 17, 18)$. R_1 is 19-admissible. Let R_2 consist of cycles of lengths $\ell_x, \ldots, \ell_\alpha$. R_2 is $(n-16)$-admissible. By Lemma 4.1 there exists an 8-cycle system $(\mathbb{Z}_{19}, B_1)$ of $K_{19} - E(R_1)$. By supposition, since $n \geq 25$, there exists an 8-cycle system $(\mathbb{Z}_{n-16}, B_2)$ of $K_{n-16} - E(R_2)$. By Theorem 3.1 there exists an 8-cycle system $(\mathbb{Z}_{16}, \mathbb{Z}_n \backslash \mathbb{Z}_{19}, B_3)$ of $K_{16,n-19}$ since $n \geq 25$ is odd. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is an 8-cycle system of K_n.

If $L = 19$ then proceed similarly by defining R_1 to consist of cycles of lengths $\ell_1, \ldots, \ell_x$, and by defining R_2 to consist of cycles of lengths $3, \ell_{x+1}, \ell_{x+2}, \ldots, \ell_\alpha$, the cycle of length 3 in R_2 being $(16, 17, 18)$. R_2 is $(n-16)$-admissible.

Finally, suppose $\ell < 16$ and $L > 19$. Let R_1 consist of cycles of lengths $\ell_1, \ldots, \ell_{x-1}$ and $19 - \ell$ (≥ 4), the cycle of length $19 - \ell$ being $c = (\ell, \ell+1, \ldots, 16, 18, 17)$. R_1 is 19-admissible. Let R_2 consist of cycles of lengths $\ell_{x+1}, \ell_{x+2}, \ldots, \ell_\alpha$ and $L - 16$ (≥ 4), the cycle of length $L-16$ being $(16, 17, 18, \ldots, L-1)$. R_2 is $(n-16)$-admissible. By Lemma 4.1 there exists an 8-cycle system $(\mathbb{Z}_{19}, B_1)$ of $K_{19} - E(R_1)$. By supposition there exists an 8-cycle system $(\mathbb{Z}_n \backslash \mathbb{Z}_{16}, B_2)$ of $K_{n-16} - E(R_2)$. By Theorem 3.1 there exists an 8-cycle system $(\mathbb{Z}_{16}, \mathbb{Z}_n \backslash \mathbb{Z}_{19}, B_3)$ of $K_{16,n-19}$ since $n \geq 23$ is odd. So now consider the set of 8-cycles $B_1 \cup B_2 \cup B_3$, and in particular consider the edges in 8-cycles that join vertices in $\{16, 17, 18\}$. The edge $\{16, 17\}$ is not an edge in R_1 but is an edge in R_2, so it is in an 8-cycle in B_1 (and is not in one in B_2). Similarly $\{16, 18\}$ occurs in an 8-cycle in B_2 (and not in one in B_1). However $\{17, 18\}$ occurs in both R_1 and R_2, so $\{17, 18\}$ occurs in *no* 8-cycle in $B_1 \cup B_2 \cup B_3$. So the edges in the cycle $(\ell, \ell+1, \ldots, 16, L-1, L-2, \ldots, 18, 17)$ of length $L - \ell = \ell_x$ occur in no 8-cycle in $B_1 \cup B_2 \cup B_3$. Therefore $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is an 8-cycle system of $K_n - E(R)$ as required.

4 Additional Small Cases

Since Lemma 3.3 cannot be used unless $n \geq 25$, the cases $n = 17, 19, 21$, and 23 must be treated as additional small cases. Daunting as this may sound, it is made simple by the repeated use of Lemma 3.2.

Lemma 4.1 *Let $n \in \{17, 19, 21, 23\}$ and let R be a 2-regular subgraph in K_n. If n is odd and $|E(K_n - E(R))|$ is divisible by 8, then there exists a 8-cycle system of $K_n - E(R)$.*

Proof: We consider each value of n in turn. In each case, we construct an 8-cycle system $(\mathbb{Z}_n, B)$.

$\boldsymbol{n = 17}$: If R_1 is a 6-cycle and two 5-cycles, then

$$R_1 = \{(0,1,2,3,4,5),(10,12,14,11,16),(6,8,13,7,15)\}$$

and

$$\begin{aligned} B_1 \;=\; & \{(1,5,2,4,6,7,8,9),(10,2,16,13,11,4,1,14),(10,9,5,3,1,16,15,13),\\ & (9,6,13,2,12,0,15,3),(11,7,16,14,8,4,10,0),(0,3,13,12,4,9,2,14),\\ & (0,9,15,11,1,12,6,16),(1,10,3,6,5,11,12,15),(1,6,11,9,14,3,16,8),\\ & (3,7,9,16,12,8,10,11),(15,2,11,8,3,12,7,14),(12,9,13,1,7,4,16,5),\\ & (13,0,8,2,6,10,15,5),(13,14,5,10,7,2,0,4),(15,8,5,7,0,6,14,4)\}. \end{aligned}$$

The partial 8-cycle decomposition in which R_2 is two 3-cycles and two 5-cycles, can be formed from R_1 by replacing $(1,5,2,4,6,7,8,9)$ in B_1 with $(1,2,5,4,6,7,8,9)$ and R_2 becomes $\{(0,1,5),(2,3,4),(10,12,14,11,16),(6,8,13,7,15)\}$

The remaining partial 8-cycle decompositions can be constructed using Lemma 3.2. For each case, we list $(l_1,\ldots,l_\alpha;a_1,\ldots,a_x)$.
$(16;12)$, $(13,3;9,3)$, $(12,4;12)$, $(11,5;7,5)$, $(10,6;6,6)$, $(10,3,3;6,3,3)$, $(9,7;5,7)$, $(9,3,4;9,3)$, $(8,8;8,4)$, $(7,6,3;3,6,3)$, $(7,5,4;7,5)$, $(7,3,3,3;3,3,3,3)$, $(6,6,4;6,6)$, $(6,3,3,4;6,3,3)$, and $(5,3,4,4;5,3,4)$

$\boldsymbol{n = 19}$: If R_1 is a 9-cycle and two 5-cycles, then

$$R_1 = \{(0,1,2,3,4,5,6,7,8),(9,10,11,12,13),(14,15,16,17,18)\}$$

and

$$\begin{aligned} B_1 \;=\; & \{(1,8,2,7,13,10,18,16),(7,3,6,4,12,17,14,1),(1,6,2,5,17,15,9,18),\\ & (1,9,8,13,6,14,16,3),(13,1,12,9,7,4,15,11),(2,4,13,18,6,10,7,16),\\ & (0,2,10,17,11,5,3,14),(15,2,13,16,8,12,6,0),(14,2,17,7,5,9,11,4),\\ & (3,10,0,4,18,15,6,17),(11,3,12,10,5,16,9,0),(5,8,11,16,0,7,14,13),\\ & (1,5,12,18,3,15,8,10),(4,8,18,0,17,9,14,10),(11,1,4,16,10,15,12,2),\\ & (1,15,5,14,11,6,8,17),(2,18,7,12,14,8,3,9),(11,7,15,13,3,0,5,18),\\ & (0,13,17,4,9,6,16,12)\}. \end{aligned}$$

R_2, R_3, R_4, R_5, and R_6 are formed using the switching method, with the switches shown in Figure 4.

The remaining partial 8-cycle decompositions can be constructed using Lemma 3.2. For each case, we list $(l_1,\ldots,l_\alpha;a_1,\ldots,a_x)$.
$(l_1 \geq 10,\ l_2,\ldots,l_\alpha; l_1 - 7, l_2,\ldots,l_\alpha)$, $(6,3,3,3,4;6,3,3)$, $(5,4,3,4,3;5,4,3)$, and $(3,3,3,3,3,4;3,3,3,3)$.

$\boldsymbol{n = 21}$: If R_1 is two 4-cycles and two 5-cycles, then

$$R_1 = \{(0,1,2,3),(4,5,6,7),(16,17,18,19,20),(8,10,14,9,13)\}$$

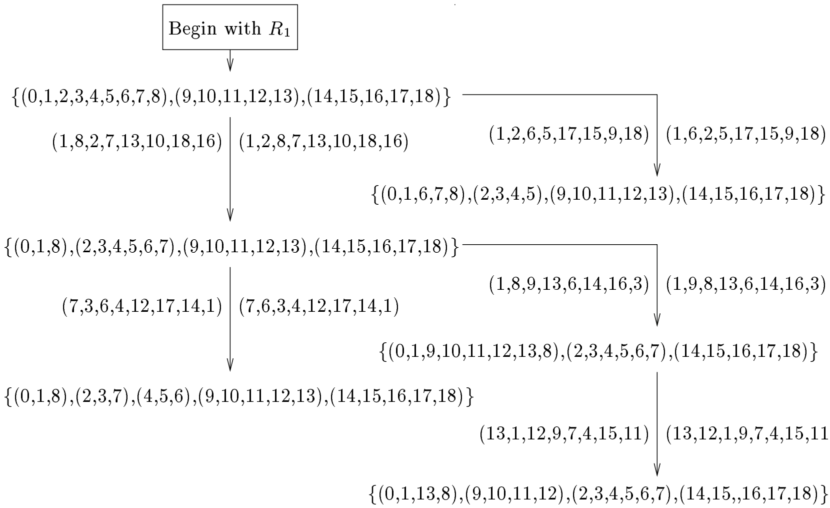


Figure 4: Leaves for $n = 19$.

and

$$\begin{aligned} B_1 \;=\; \{&(1,4,2,7,8,9,10,11),(2,0,7,1,12,13,14,15),(0,20,6,16,8,15,5,18),\\ &(20,5,14,12,8,19,15,7),(4,16,11,5,17,7,14,19),(20,3,10,12,17,6,13,18),\\ &(3,14,6,15,0,19,9,17),(1,9,16,18,2,8,11,15),(0,6,9,5,16,19,7,13),\\ &(4,11,20,14,8,5,12,15),(16,1,10,5,13,20,17,15),(0,12,9,11,6,3,7,16),\\ &(2,10,17,4,12,18,11,19),(0,8,3,18,10,13,2,14),(2,9,18,6,4,13,3,16),\\ &(7,10,4,0,11,14,1,18),(1,5,2,20,9,3,19,13),(3,15,13,17,19,12,7,5),\\ &(4,9,7,11,13,16,10,20),(2,12,16,14,4,8,17,11),(0,5,19,1,3,12,6,10),\\ &(1,8,18,4,3,11,12,20),(0,9,15,20,8,6,1,17),(2,6,19,10,15,18,14,17). \end{aligned}$$

The partial 8-cycle decomposition in which R_2 is an 8-cycle and two 5-cycles, can be formed from R_1 by replacing $(1,4,2,7,8,9,10,11)$ in B_1 with $(1,2,4,7,8,9,10,11)$ and R_2 becomes $\{(0,1,4,5,6,7,2,3),(16,17,18,19,20),(8,10,14,9,13)\}$.

The partial 8-cycle decomposition in which R_3 is a 3-cycle and three 5-cycles, can be formed from R_2 by replacing $(2,0,7,1,12,13,14,15)$ in B_2 with $(2,7,0,1,12,13,14,15)$ and R_3 becomes $\{(0,2,3),(1,4,5,6,7),(16,17,18,19,20),(8,10,14,9,13)\}$.

The remaining partial 8-cycle decompositions can be constructed using Lemma 3.2. For each case, we list $(l_1,\ldots,l_\alpha;a_1,\ldots,a_x)$.
$(l_1 \geq 9, l_2,\ldots,l_\alpha; l_1-6, l_2,\ldots,l_\alpha)$, $(8,7,3;8,4)$, $(8,4,6;8,4)$, $(8,4,3,3;8,4)$, $(7,7,4;4,4,3)$, $(7,5,6;7,5)$, $(7,4,4,3;4,4,4)$, $(l_1,\ldots,l_\alpha-1,6;l_1,\ldots,l_\alpha-1)$, and $(5,4,3,3,3;5,4,3)$.

$\boldsymbol{n = 23}$: If R_1 is three 7-cycles, then

$$R_1 = \{(14, 15, 17, 20, 19, 18, 16), (0, 1, 2, 3, 4, 5, 6), (7, 8, 9, 10, 11, 12, 13)\}$$

and

$$\begin{aligned} B_1 = \{ &(16, 15, 18, 17, 6, 22, 13, 4), (0, 12, 8, 3, 19, 5, 9, 14), (1, 3, 11, 4, 14, 20, 2, 10), \\ &(13, 8, 11, 0, 21, 7, 19, 6), (11, 17, 2, 5, 10, 12, 15, 21), (13, 9, 7, 4, 15, 22, 1, 20), \\ &(2, 12, 5, 18, 6, 3, 16, 22), (8, 6, 4, 19, 22, 12, 3, 21), (17, 0, 2, 14, 8, 20, 3, 13), \\ &(15, 9, 22, 14, 3, 10, 20, 7), (4, 8, 16, 2, 6, 11, 15, 0), (11, 1, 17, 3, 22, 5, 0, 19), \\ &(4, 21, 9, 6, 16, 13, 10, 17), (5, 20, 12, 7, 11, 22, 0, 13), (18, 8, 10, 22, 20, 15, 2, 21), \\ &(12, 18, 0, 7, 1, 4, 22, 17), (19, 14, 5, 7, 10, 4, 2, 13), (5, 21, 16, 9, 1, 14, 10, 15), \\ &(19, 12, 6, 20, 11, 13, 18, 2), (19, 10, 21, 14, 7, 18, 11, 9), (1, 16, 12, 9, 2, 11, 14, 6), \\ &(8, 2, 7, 17, 19, 1, 13, 15), (16, 11, 5, 17, 21, 1, 8, 0), (12, 14, 13, 21, 19, 8, 5, 1), \\ &(17, 14, 18, 1, 15, 3, 7, 16), (0, 3, 5, 16, 20, 18, 4, 9), (0, 10, 16, 19, 15, 6, 21, 20), \\ &(3, 9, 20, 4, 12, 21, 22, 18), (6, 7, 22, 8, 17, 9, 18, 10). \end{aligned}$$

The partial 8-cycle decomposition in which R_2 is a 3-cycle, 4-cycle, and two 7-cycles, can be formed from R_1 by replacing $(16, 15, 18, 17, 6, 22, 13, 4)$ in B_1 with $(16, 18, 15, 17, 6, 22, 13, 4)$ and R_2 becomes

$$\{(14, 15, 16), (17, 18, 19, 20), (0, 1, 2, 3, 4, 5, 6), (7, 8, 9, 10, 11, 12, 13)\}.$$

If R_3 is a 6-cycle and three 5-cycles (R_3 is not formed from the switching process), then $R_3 = \{(15, 14, 16, 18, 19, 17), (0, 1, 2, 3, 4), (5, 6, 7, 8, 20), (9, 10, 11, 12, 13)\}$ and

$$\begin{aligned} B_3 = \{ &(16, 15, 18, 17, 6, 22, 13, 4), (22, 0, 12, 7, 21, 3, 10, 14), (19, 21, 1, 5, 8, 11, 2, 20), \\ &(11, 1, 19, 3, 14, 9, 6, 4), (16, 0, 2, 15, 5, 7, 13, 1), (17, 21, 12, 10, 0, 15, 4, 8), \\ &(19, 7, 22, 5, 3, 13, 8, 16), (11, 22, 20, 17, 10, 7, 2, 13), (5, 11, 15, 3, 0, 20, 16, 2), \\ &(0, 6, 13, 5, 14, 2, 10, 18), (22, 15, 7, 16, 9, 1, 8, 18), (1, 4, 18, 9, 3, 17, 12, 22), \\ &(13, 10, 20, 1, 12, 19, 6, 18), (11, 6, 3, 16, 10, 8, 14, 0), (11, 7, 1, 18, 20, 21, 4, 19), \\ &(12, 9, 20, 6, 21, 2, 17, 5), (15, 12, 3, 22, 19, 14, 13, 20), (17, 14, 21, 9, 2, 18, 12, 16), \\ &(19, 0, 7, 4, 22, 8, 15, 9), (17, 9, 5, 10, 21, 8, 3, 1), (19, 8, 6, 2, 22, 16, 21, 5), \\ &(14, 11, 20, 4, 2, 8, 9, 7), (13, 21, 15, 19, 2, 12, 8, 0), (6, 16, 11, 17, 13, 15, 1, 10), \\ &(13, 19, 10, 4, 17, 0, 5, 16), (0, 9, 22, 17, 7, 3, 11, 21), (1, 6, 15, 10, 22, 21, 18, 14), \\ &(3, 18, 5, 4, 14, 6, 12, 20), (4, 9, 11, 18, 7, 20, 14, 12). \end{aligned}$$

The partial 8-cycle decomposition in which R_4 is two 3-cycles and three 5-cycles, can be formed from R_3 by replacing $(16, 15, 18, 17, 6, 22, 13, 4)$ in B_3 with $(16, 18, 15, 17, 6, 22, 13, 4)$ and R_2 becomes

$$\{(14, 15, 16), (17, 18, 19), (0, 1, 2, 3, 4), (5, 6, 7, 8, 20), (9, 10, 11, 12, 13)\}.$$

The remaining partial 8-cycle decompositions can be constructed using Lemma 3.2. For each case, we list $(l_1, \ldots, l_\alpha; a_1, \ldots, a_x)$.
$(l_1 \geq 15, l_2, \ldots, l_\alpha; 12), (14,7;5,7), (14,4,3;5,4,3), (13,8;4,8), (13,5,3;4,5,3)$, $(13,4,4;4,4,4)$, $(12, l_2, \ldots, l_\alpha; 12)$, $(11,10;6,6)$, $(11,7,3;5,7)$, $(11,6,4;6,6)$, $(11,5,5;7,5)$, $(11,3,3,4;6,3,3)$, $(10,8,3;4,8)$, $(10,7,4;5,7)$, $(10,6,5;6,6)$, $(10,4,3,4;6,3,3)$, $(10,3,3,5;6,3,3)$, $(l_1, \ldots, l_\alpha - 1; l_1, \ldots, l_\alpha - 1)$, $(8,8,5;8,4)$, $(8,7,6;5,7)$, $(8,7,3,3;5,7)$, $(8,4,3,3,3;8,4)$, $(8,4,6,3;8,4)$, $(8,5,3,5;4,5,3)$, $(8,4,4,5;8,4)$, $(8,4,l_2,\ldots,l_\alpha;8,4), (7,5,6,3;7,5), (7,4,4,6;4,4,4), (7,5,5,4;7,5)$, $(7,4,4,3,3;4,4,4)$, $(6,6,l_2,\ldots,l_\alpha;6,6)$, $(4,4,4,6,3;4,4,4)$, $(6,3,3,3,3,3;6,3,3)$, $(5,4,3,5,4;5,4,3), (4,4,4,4,5;4,4,4), (5,4,3,3,3;5,4,3), (4,4,4,3,3,3;4,4,4)$, and $(3,3,3,3,3,3,3;3,3,3,3)$.

5 The Main Result

Theorem 5.1 *Let R be a 2-regular subgraph in K_n that consists of α cycles of lengths $l_1, \ldots, l_\alpha$. There exists an 8-cycle system of $K_n - E(R)$ if and only if*

1. *n is odd,*
2. *$n \neq 5, 7$, and*
3. *$|E(K_n - E(R))|$ is divisible by 8.*

Proof:

At this point, we have found complete solutions for $n \leq 23$; see Lemma 2.2 and Lemma 4.1. So we can assume that $n \geq 25$.

If $l_i \geq 17$ for $1 \leq i \leq \alpha$, then Lemma 3.2 can easily be used. So we can assume all cycles have length at most 16.

Among all orderings of the cycle lengths of R, choose one such that

$$l \text{ is maximized} \tag{1}$$

where l is defined in Lemma 3.3. So $3 \leq l \leq 16$.

Using the ordering of the cycle lengths in Lemma 3.3, let $S_L = \{l_1, l_2, ..., l_{x-1}\}$ and $S_U = \{l_x, l_{x+1}, ..., l_\alpha\}$. By the maximality of l, S_U contains no cycles of length at most $16 - l$, and if S_U contains a cycle of length at least $19 - l$ then we can apply Lemma 3.3. So we can assume that if $l_i \in S_U$ then

$$l_i \in \{17 - l, 18 - l\} \tag{2}$$

We consider the possible values of l in turn.

If $l = 15$, then by (2) $l_i = 3$ for all $i \geq x$. So $E(R) = 15 + 3(\alpha - (x-1))$ is divisible by 3. A brief check of Table 1 shows that this forces $n \geq 41$. If $n \geq 41$, then R contains at least four 3-cycles (since $l_x, ..., l_\alpha$ are all 3-cycles and $l = 15$) and so Lemma 3.2 can be used, with $a_i = 3$ for $1 \leq i \leq 4$.

If $l = 14$, then $l_i = 3$ or 4 for $i \geq x$. We consider three cases in turn.

Suppose $l = 14$ and S_U contains only 3-cycles. Table 1 shows this happens when $n = 25, 27$, and when $n \geq 33$. If $n \geq 33$, S_U contains at least four 3-cycles and Lemma 3.2 can be used. If $n = 25$, S_U consists of two 3-cycles.

(a) Replacing a 4-cycle or a 5-cycle in S_L with the two 3-cycles in S_U would increase l to 16 or 15 respectively, contradicting (1); so S_L contains no 4-cycles and no 5-cycles.

(b) Otherwise,

$$(l_1, \ldots, l_\alpha) \in \{(14,3,3),(11,3,3,3),(8,3,3,3,3),(8,6,3,3),(7,7,3,3)\}$$

and in every case Lemma 3.2 can be used; choose the a_is to be $(6,3,3)$ in the first case, and to be $(3,3,3,3)$ in the remaining cases.

If $n = 27$, S_U consists of three 3-cycles. For the same reasons as in (a), S_L contains no 4-cycles and no 5-cycles. Therefore

$$(l_1, ..., l_\alpha) \in \{(14,3,3,3),(11,3,3,3,3),(8,6,3,3,3),(7,7,3,3,3)\}$$

and in every case Lemma 3.2 can be used; choose the a_is to be $(3,3,3,3)$ in each case.

Suppose $l = 14$ and S_U contains only 4-cycles. Table 1 shows this happens when $n = 29$ or when $n \geq 33$. If $n \geq 33$, R contains at least three 4-cycles and Lemma 3.2 can be used; choose the a_is to be (4,4,4) in each case. If $n = 29$, S_U consists of two 4-cycles. In this case:

(a) Replacing a 3-cycle in S_L with a 4-cycle from S_U would increase l to 15 which contradicts (1). Therefore, S_L contains no 3-cycle.

(b) Replacing a 6-cycle or a 7-cycle in S_L with the two 4-cycles in S_U would increase l to 16 or 15 respectively, contradicting (1); so S_L contains no 6-cycles and no 7-cycles.

(c) Otherwise, $(l_1, ..., l_\alpha) \in \{(14,4,4),(10,4,4,4),(5,5,4,4,4),(9,5,4,4)\}$ and in every case Lemma 3.2 can be used; choose the a_is to be $(4,4,4)$ in each case.

Suppose $l = 14$ and S_U contains both 3-cycles and 4-cycles. Table 1 shows this happens when $n \geq 33$. If $n \geq 33$, R contains at least four 3-cycles or contains at least three 4-cycles, so Lemma 3.2 can be used.

If $l = 13$, then by (2) if $l_i \in S_U$ then $l_i \in \{4,5\}$. Replacing a 3-cycle in S_L with a cycle in S_U would contradict (1) so $3 \notin S_L$. If $l_i = 4$ for $x \leq i \leq \alpha$ then S_U contains three 4-cycles so Lemma 3.2 can be used. Otherwise, $5 \in S_U$, so by (2) $4 \notin S_L$. So it remains to check the cases where $S_L \in \{\{5,8\},\{6,7\},\{13\}\}$. $S_L = \{5,8\}$ contradicts the maximality of l, since swapping the 8-cycle in S_L with two cycles from S_U, at least one being a 5-cycle, increases l to 14 or 15. Lemma 3.2 can be used with a 5-cycle in S_U together with either a 7-cycle or a 13-cycle in S_L using $a_1 = 5$ and $a_2 = 7$ in Lemma 3.2 to settle the remaining cases.

If $l = 12$, use the cycles in S_L in Lemma 3.2.

If $l = 11$, then by (2) $l_i \in \{6, 7\}$ for $x \leq i \leq \alpha$. So replacing a cycle of length $3, 4$, or 5 in S_L with any cycle from S_U would increase l to at most 15, thus contradicting (1). So $S_L = \{11\}$. So use Lemma 3.2 with the 11-cycle and any cycle in S_U.

If $l = 10$, then by (2) $l_i \in \{7, 8\}$ for $x \leq i \leq \alpha$. So replacing any cycle of length at most 6 in S_L with any cycle in S_U contradicts the maximality of L. So $S_L = \{10\}$ and Lemma 3.2 can be used with $\alpha = 2$ to prove the result.

If $l = 9$, $l_i \in \{8, 9\}$. Pick any two cycles in S_U and use Lemma 3.2.

If $l \leq 8$, then by (2) $15 \geq l_i > 9 > l$ for $x \leq i \leq \alpha$, contradicting the maximality of l.

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