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Changing and unchanging acyclic domination: edge addition

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Abstract

A subset A of vertices in a graph G is acyclic if the subgraph it induces contains no cycles. The acyclic domination number $\gamma_a(G)$ of a graph G is the minimum cardinality of an acyclic dominating set of G. An acyclic dominating set A of a graph G with $|A| = \gamma_a(G)$ is called a γ_a-set of G. A vertex x of a graph G is called: (i) γ_a-good if x belongs to some γ_a-set, (ii) γ_a-fixed if x belongs to every γ_a-set, (iii) γ_a-free if x belongs to some γ_a-set but not to all γ_a-sets, (iv) γ_a-bad if x belongs to no γ_a-set. In this paper we deal with γ_a-good/bad/fixed/free vertices and present results on changing and unchanging of the acyclic domination number when a graph is modified by adding an edge.

1 Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [3]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. The complement of a graph G is denoted by $\overline{G}$. For a vertex x of G, $N(x, G)$ denotes the set of all neighbors of x in G, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of x is $\deg(x, G) = |N(x, G)|$. The maximum degree in the graph G is denoted by $\Delta(G)$. For a graph G, let $x \in X \subseteq V(G)$. The *private neighbor set of x with respect to X* is $\mathrm{pn}[x, X] = \{y \in V(G) : N[y, G] \cap X = \{x\}\}$.

A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. A subset of vertices A in a graph G is said to be *acyclic* if $\langle A, G\rangle$ contains no cycles. The *acyclic domination number* $\gamma_a(G)$ of a graph G is the minimum cardinality of an acyclic dominating set of G. The concept of acyclic domination in graphs was introduced by Hedetniemi et al. [5].

A vertex v of a graph G is γ_a-*critical* if $\gamma_a(G-v) \neq \gamma_a(G)$. A vertex v of a graph G is γ_a^+-*critical* (γ_a^--*critical*, respectively) if $\gamma_a(G-v) > \gamma_a(G)$ ($\gamma_a(G-v) < \gamma_a(G)$, respectively).

Let $\mu(G)$ be a numerical invariant of a graph G defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ with a given property P. A set with the property P and with $\mu(G)$ vertices in G is called a μ-set of G.

Fricke et al. [2] defined a vertex v to be

(i) μ-*good*, if v belongs to some μ-set of G and

(ii) μ-*bad*, if v belongs to no μ-set of G.

Sampathkumar and Neerlagi [10] defined a vertex v to be:

(iii) μ-*fixed* if v belongs to every μ-set;

(iv) μ-*free* if v belongs to some μ-set but not to all μ-sets.

For a graph G we define:

$\mathbf{G}_a(G) = \{x \in V(G) : x \text{ is } \gamma_a\text{-good}\}$;
$\mathbf{B}_a(G) = \{x \in V(G) : x \text{ is } \gamma_a\text{-bad}\}$;
$\mathbf{Fi}_a(G) = \{x \in V(G) : x \text{ is } \gamma_a\text{-fixed}\}$;
$\mathbf{Fr}_a(G) = \{x \in V(G) : x \text{ is } \gamma_a\text{-free}\}$;
$\mathbf{V}_a^0(G) = \{x \in V(G) : \gamma_a(G-x) = \gamma_a(G)\}$;
$\mathbf{V}_a^-(G) = \{x \in V(G) : \gamma_a(G-x) < \gamma_a(G)\}$;
$\mathbf{V}_a^+(G) = \{x \in V(G) : \gamma_a(G-x) > \gamma_a(G)\}$.

By a *partition of a set* S we mean an unordered family $\{S_1, S_2, \ldots, S_n\}$ of pairwise disjoint subsets of S with $\cup_{i=1}^n S_i = S$. Note that some of the S_i's may be empty.

Clearly, $\{\mathbf{V}_a^-(G), \mathbf{V}_a^0(G), \mathbf{V}_a^+(G)\}$ and $\{\mathbf{G}_a(G), \mathbf{B}_a(G)\}$ are partitions of $V(G)$, and $\{\mathbf{Fi}_a(G), \mathbf{Fr}_a(G)\}$ is a partition of $\mathbf{G}_a(G)$.

Much has been written about the effects on domination related parameters when a graph is modified by deleting a vertex or adding an edge. For surveys see [3, Chapter 5], [4, Chapter 16], [1], [6] and [11]. In this paper we deal with γ_a-good/bad/fixed/free vertices and present results on changing and unchanging of the acyclic domination number when an edge is added.

We need the following results.

Theorem 1.1. *Let G be a graph of order $n \geq 2$ and $u, v \in V(G)$.*

(i) *Let $\gamma_a(G-v) < \gamma_a(G)$.*

(i.1) [8] *If* $uv \in E(G)$ *then* u *is a* γ_a*-bad vertex of* $G - v$*;*

(i.2) [9] *If* M *is a* γ_a*-set of* $G - v$ *then* $M \cup \{v\}$ *is a* γ_a*-set of* G*;*

(i.3) [8] $\gamma_a(G - v) = \gamma_a(G) - 1$*;*

(ii) [9] *Let* $v \in \mathbf{V}_a^+(G)$*. Then* v *is a* γ_a*-fixed vertex of* G*;*

(iii) [9] *If* $v \in \mathbf{V}_a^-(G)$ *and* $u \in \mathbf{V}_a^+(G)$ *then* $uv \notin E(G)$*;*

(iv) [9] *If* v *is a* γ_a*-bad vertex of* G *then* $\gamma_a(G - v) = \gamma_a(G)$*.*

For the sake of completeness, we repeat the proof.

Proof. (i): (i.1): Let $uv \in E(G)$ and M be a γ_a-set of $G - v$. If $u \in M$ then M will be an acyclic dominating set of G with $|M| < \gamma_a(G)$ — a contradiction.

(i.2) and (i.3): If M is a γ_a-set of $G - v$ then (i.1) implies that $M_1 = M \cup \{v\}$ is an acyclic dominating set of G with $|M_1| = \gamma_a(G - v) + 1 \leq \gamma_a(G)$. Hence M_1 is a γ_a-set of G and $\gamma_a(G - v) = \gamma_a(G) - 1$.

(ii) If $M$is a γ_a-set of G and $v \notin M$ then M is an acyclic dominating set of $G - v$. But then $\gamma_a(G) = |M| \geq \gamma_a(G - v) > \gamma_a(G)$ and the result follows.

(iii) Let $\gamma_a(G - v) < \gamma_a(G)$ and M be a γ_a-set of $G - v$. Then by (i.2), $M \cup \{v\}$ is a γ_a-set of G. Let $\gamma_a(G - u) > \gamma_a(G)$. Now (ii) implies that $u \in M$ and by (i.1), $uv \notin E(G)$.

(iv) By (ii), $\gamma_a(G - v) \leq \gamma_a(G)$. Assume $\gamma_a(G - v) < \gamma_a(G)$. It follows from (i.2) that $M \cup \{v\}$ is a γ_a-set of G, where M is a γ_a-set of $G - v$ — a contradiction. ■

Since for every $v \in V(G)$ we clearly have $\gamma_a(G - v) \leq |V(G)| - 1$ and because of Theorem 1.1 it follows that $\gamma_a(G-v) = \gamma_a(G)+p$, where $p \in \{-1, 0, 1, \dots, |V(G)| - 2\}$. This motivated us to define for a graph G:
$\mathbf{Fr}_a^-(G) = \{x \in \mathbf{Fr}_a(G) : \gamma_a(G - x) = \gamma_a(G) - 1\}$;
$\mathbf{Fr}_a^0(G) = \{x \in \mathbf{Fr}_a(G) : \gamma_a(G - x) = \gamma_a(G)\}$;
$\mathbf{Fi}_a^p(G) = \{x \in \mathbf{Fi}_a(G) : \gamma_a(G - x) = \gamma_a(G) + p\}$, $p \in \{-1, 0, 1, \dots, |V(G)| - 2\}$.

Now, by Theorem 1.1 we have:

Corollary 1.2. *Let* G *be a graph of order* $n \geq 2$*.*

(i) $\{\mathbf{Fr}_a^-(G), \mathbf{Fr}_a^0(G)\}$ *is a partition of* $\mathbf{Fr}_a(G)$*;*

(ii) $\{\mathbf{Fi}_a^{-1}(G), \mathbf{Fi}_a^0(G), \dots, \mathbf{Fi}_a^{n-2}(G)\}$ *is a partition of* $\mathbf{Fi}_a(G)$*;*

(iii) $\{\mathbf{Fi}_a^{-1}(G), \mathbf{Fr}_a^-(G)\}$ *is a partition of* $\mathbf{V}_a^-(G)$*;*

(iv) $\{\mathbf{Fi}_a^0(G), \mathbf{Fr}_a^0(G), \mathbf{B}_a(G)\}$ *is a partition of* $\mathbf{V}_a^0(G)$*;*

(v) $\{\mathbf{Fi}_a^1(G), \mathbf{Fi}_a^2(G), \dots, \mathbf{Fi}_a^{n-2}(G)\}$ *is a partition of* $\mathbf{V}_a^+(G)$*.*

Corollary 1.2 will be used in the sequel without specific reference.

As an immediate result of Theorem 1.1 we also have:

Corollary 1.3. *Let G be a graph of order at least two and $x \in \mathbf{V}_a^-(G)$. Then:*

(i) $\mathbf{B}_a(G) \cup N(x, G) \subseteq \mathbf{B}_a(G - x)$;

(ii) $\mathbf{Fi}_a(G) - \{x\} \subseteq \mathbf{Fi}_a(G - x)$.

We will refine the definitions of the $\gamma_a(G)$-free vertex and the $\gamma_a(G)$-fixed vertex as follows. Let x be a vertex of a graph G.

(i) x is called γ_a^0-*free* if $x \in \mathbf{Fr}_a^0(G)$;

(ii) x is called $\gamma_a^-(G)$-*free* if $x \in \mathbf{Fr}_a^-(G)$ and

(iii) x is called $\gamma_a^q(G)$-*fixed* if $x \in \mathbf{Fi}_a^q(G)$, where $q \in \{-1, 0, 1, \dots, |V(G)| - 2\}$.

We conclude this section with the following useful lemma:

Lemma 1.4. *Let x be a γ_a^0-fixed vertex of a graph G. Then $N(x, G) \subseteq \mathbf{B}_a(G - x) \cap (\mathbf{V}_a^0(G) \cup \mathbf{Fi}_a^1(G))$.*

Proof. Let M be a γ_a-set of $G - x$ and $y \in N(x, G)$. If $y \in M$ then M will be an acyclic dominating set of G of cardinality $|M| = \gamma_a(G - x) = \gamma_a(G)$ — a contradiction with $x \in \mathbf{Fi}_a(G)$. Thus $N(x, G) \subseteq \mathbf{B}_a(G - x)$. If y is a γ_a^--critical vertex of G, then by Theorem 1.1 there will exist a γ_a-set M_1 of G with $x \notin M_1$ — again a contradiction with $x \in \mathbf{Fi}_a(G)$. Assume $y \in \mathbf{Fi}_a^p(G)$ for some $p \geq 2$. It follows from $M \cap N(x, G) = \emptyset$ that $M_2 = M \cup \{x\}$ is an acyclic dominating set of G with $|M_2| = \gamma_a(G - x) + 1 = \gamma_a(G) + 1$. But $y \notin M$ and then $|M_2| \geq \gamma_a(G) + p$. Thus we have a contradiction. ∎

2 Edge Addition

It is often of interest to know how the value of a graphical parameter is affected when a small change is made in a graph. In this connection, we now consider this question in the case of $\gamma_a(G)$ when an edge is added on G.

Theorem 2.1. *Let x and y be two nonadjacent vertices in a graph G. If $\gamma_a(G+xy) < \gamma_a(G)$ then $\gamma_a(G + xy) = \gamma_a(G) - 1$. Moreover, $\gamma_a(G + xy) = \gamma_a(G) - 1$ if and only if at least one of the following holds:*

(i) *x is a γ_a^--critical vertex of G and y is a γ_a-good vertex of $G - x$;*

(ii) *x is a γ_a-good vertex of $G - y$ and y is a γ_a^--critical vertex of G.*

Proof. Let $\gamma_a(G+xy) < \gamma_a(G)$ and M be a γ_a-set of $G+xy$. Then $|\{x,y\} \cap M| = 1$, otherwise M will be an acyclic dominating set of G which is a contradiction. Let, without loss of generality, $x \notin M$ and $y \in M$. Since M is no dominating set of G, then $M \cap N(x,G) = \emptyset$. Hence $M_1 = M \cup \{x\}$ is an acyclic dominating set of G with $|M_1| = \gamma_a(G+xy)+1$ which implies $\gamma_a(G) = \gamma_a(G+xy)+1$. Since M is an acyclic dominating set of $G-x$, $\gamma_a(G-x) \leq \gamma_a(G+xy)$. Hence $\gamma_a(G) \geq \gamma_a(G-x)+1$ and by Theorem 1.1 it follows that $\gamma_a(G) = \gamma_a(G-x)+1$. Thus x is a γ_a^--critical vertex of G and M is a γ_a-set of $G-x$. Since $y \in M$, it follows that y is a γ_a-good vertex of $G-x$.

For the converse, without loss of generality suppose (i) holds. Then there is a γ_a-set M of $G-x$ with $y \in M$. Certainly M is an acyclic dominating set of $G+xy$ and then $\gamma_a(G+xy) \leq |M| = \gamma_a(G-x) = \gamma_a(G)-1 \leq \gamma_a(G+xy)$. ■

Corollary 2.2. *Let x and y be two nonadjacent vertices in a graph G and $x \in \mathbf{V}_a^-(G)$. Then $\gamma_a(G)-1 \leq \gamma_a(G+xy) \leq \gamma_a(G)$.*

Proof. Let M be a γ_a-set of $G-x$. By Theorem 1.1, $M_1 = M \cup \{x\}$ is a γ_a-set of G and $M_1 \cap N(x,G) = \emptyset$. Hence M_1 is an acyclic dominating set of $G+xy$ and $\gamma_a(G+xy) \leq |M_1| = \gamma_a(G-x)+1 = \gamma_a(G)$. The rest follows by Theorem 2.1. ■

It is well known fact that for any edge $e \in \overline{G}$, $\gamma(G+e) \leq \gamma(G)$. In general, for the acyclic domination number this is not valid.

Theorem 2.3. *Let x and y be two nonadjacent vertices in a graph G. Then $\gamma_a(G+xy) > \gamma_a(G)$ if and only if every γ_a-set of G is not an acyclic set of $G+xy$ and one of the following holds:*

(i) *x is a γ_a^p-fixed vertex of G and y is a γ_a^q-fixed vertex of G for some $p,q \geq 1$;*

(ii) *$x \in \mathbf{Fi}_a^0(G)$ and $y \in \mathbf{Fi}_a^1(G) \cap \mathbf{B}_a(G-x)$;*

(iii) *$x \in \mathbf{Fi}_a^1(G) \cap \mathbf{B}_a(G-y)$ and $y \in \mathbf{Fi}_a^0(G)$;*

(iv) *x and y are γ_a^0-fixed vertices of G, x is a γ_a-bad vertex of $G-y$ and y is a γ_a-bad vertex of $G-x$;*

Proof. Let $\gamma_a(G+xy) > \gamma_a(G)$. By Corollary 2.2, $x,y \in \mathbf{V}_a^0(G) \cup \mathbf{V}_a^+(G)$. Assume to the contrary, that (without loss of generality) x is no γ_a-fixed vertex of G. Hence there is a γ_a-set M of G with $x \notin M$. But then M will be an acyclic dominating set of $G+xy$ and $|M| = \gamma_a(G) < \gamma_a(G+xy)$, a contradiction. Thus x and y are both γ_a-fixed vertices of G. This implies that each γ_a-set M of G is a dominating set of $G+xy$ and is no acyclic set of $G+xy$.

Let x be γ_a^p-fixed, y be γ_a^q-fixed, and without loss of generality, $q \geq p \geq 0$. Assume (i) does not hold. Hence $p = 0$. Let M_1 be a γ_a-set of $G-x$. Then $|M_1| = \gamma_a(G-x) = \gamma_a(G) < \gamma_a(G+xy)$ and we have that y is a γ_a-bad vertex of $G-x$. By Lemma 1.4, $N(x,G) \cap M_1 = \emptyset$. Then $M_1 \cup \{x\}$ is an acyclic dominating set of $G+xy$ which implies $\gamma_a(G+xy) = \gamma_a(G)+1$. Since $y \notin M_1 \cup \{x\}$, then $M_1 \cup \{x\}$ is an acyclic dominating set of $G-y$ and then $\gamma_a(G)+1=|M_1\cup\{x\}|\geq\gamma_a(G-y)=\gamma_a(G)+q$. Thus if $q\geq 2$ then we have a contradiction. So $q\in\{0,1\}$. If $q=1$ then (ii) holds. If $q=0$ then by symmetry, it follows that x is a γ_a-bad vertex of $G-y$ and hence (iv) holds.

For the converse, let every γ_a-set of G be a non acyclic set of $G+xy$ and let one of the conditions (i), (ii), (iii) or (iv) hold. Assume to the contrary, that $\gamma_a(G+xy)\leq\gamma_a(G)$. By Theorem 2.1, $\gamma_a(G+xy)=\gamma_a(G)$. Let M_2 be a γ_a-set of $G+xy$. Hence $|M_2\cap\{x,y\}|=1$ — otherwise M_2 will be a γ_a-set of G. Let, without loss of generality, $x\notin M_2$. Then M_2 is an acyclic dominating set of $G-x$ which implies $\gamma_a(G-x)\leq|M_2|=\gamma_a(G+xy)=\gamma_a(G)$. Thus $\gamma_a(G-x)=\gamma_a(G+xy)=\gamma_a(G)$ and then M_2 is a γ_a-set of $G-x$. Hence x is a γ_a^0-fixed vertex of G and y is a γ_a-good vertex of $G-x$, which is a contradiction with some of (ii), (iii), (iv). ■

By Theorem 2.1 and Theorem 2.3 we immediately have:

Theorem 2.4. *Let x and y be two nonadjacent vertices in a graph G. Then $\gamma_a(G+xy)=\gamma_a(G)$ if and only if at least one of the following holds:*

(i) $x\in\mathbf{V}_a^-(G)\cap\mathbf{B}_a(G-y)$ *and* $y\in\mathbf{V}_a^-(G)\cap\mathbf{B}_a(G-x)$;

(ii) $x\in\mathbf{V}_a^-(G)$ *and* $y\in\mathbf{B}_a(G-x)-\mathbf{V}_a^-(G)$;

(iii) $x\in\mathbf{B}_a(G-y)-\mathbf{V}_a^-(G)$ *and* $y\in\mathbf{V}_a^-(G)$;

(iv) $x,y\notin\mathbf{V}_a^-(G)$ *and* $|\{x,y\}\cap\mathbf{Fi}_a(G)|\leq 1$;

(v) $x\in\mathbf{Fi}_a^0(G)$ *and* $y\in\mathbf{Fi}_a^s(G)\cap\mathbf{G}_a(G-x)$ *for some* $s\in\{0,1\}$;

(vi) $x\in\mathbf{Fi}_a^s(G)\cap\mathbf{G}_a(G-y)$ *and* $y\in\mathbf{Fi}_a^0(G)$ *for some* $s\in\{0,1\}$;

(vii) $x\in\mathbf{Fi}_a^0(G)$ *and* $y\in\mathbf{Fi}_a^q(G)$ *for some* $q\geq 2$;

(viii) $x\in\mathbf{Fi}_a^q(G)$ *and* $y\in\mathbf{Fi}_a^0(G)$ *for some* $q\geq 2$;

(ix) *there is a γ_a-set of G which is an acyclic set of $G+xy$ and one of the (i), (ii), (iii) and (iv) of Theorem* 2.3 *holds.*

Corollary 2.5. *Let x and y be two nonadjacent vertices in a graph G. If $x\in\mathbf{B}_a(G)$ then $\gamma_a(G+xy)=\gamma_a(G)$.*

Proof. If $y\notin\mathbf{V}_a^-(G)$ then the result follows by Theorem 2.4 (iv). If $y\in\mathbf{V}_a^-(G)$ then by Corollary 1.3, $x\in\mathbf{B}_a(G-y)$ and the result now follows by Theorem 2.4 (iii). ■

Sumner and Blitch [12] defined a graph to be edge-domination critical if $\gamma(G+e)\neq\gamma(G)$ for every edge e missing from G. Analogously, we define a graph G to be *edge-γ_a-critical* if $\gamma_a(G+e)\neq\gamma_a(G)$ for every edge e of the complement of G. Relating edge addition to vertex removal, Sumner and Blitch [12] showed that $\mathbf{V}^+(G)=\{x\in V(G):\gamma(G-x)>\gamma(G)\}$ is empty for edge-domination critical graphs. For edge-γ_a-critical graphs the following holds.

Theorem 2.6. *Let G be an edge-γ_a-critical graph. Then*

(i) $V(G) = \mathbf{Fi}_a^{-1}(G) \cup \mathbf{Fr}_a(G)$;

(ii) $\gamma_a(G+e) < \gamma_a(G)$ *for every edge e missing from G;*

(iii) *If* $\mathbf{Fr}_a^0(G) \neq \emptyset$ *then* $\langle \mathbf{Fr}_a^0(G), G\rangle$ *is complete;*

(iv) $\mathbf{Fi}_a^{-1}(G) = \{x \in V(G) : \deg(x, G) = 0\}$.

Proof. (iii) Let $x, y \in \mathbf{Fr}_a^0(G)$ and $xy \notin E(G)$. Then by Theorem 2.4 follows $\gamma_a(G+xy) = \gamma_a(G)$.

(i) By Corollary 2.5, $\mathbf{B}_a(G) = \emptyset$. Assume $x \in \mathbf{Fi}_a^q(G)$ for some $q \geq 0$. Let M be any γ_a-set of G. Hence there is $y \in \mathrm{pn}[x, M] - \{x\}$ — otherwise $\mathrm{pn}[x, M] = \{x\}$ which implies $x \in \mathbf{V}_a^-(G)$. Since $\mathrm{pn}[x, G] \cap \mathbf{V}_a^-(G) = \emptyset$ (by Theorem 1.1 when $q \geq 1$ and Lemma 1.4 when $q = 0$), $\mathbf{B}_a(G) = \emptyset$ and $y \notin M$ then $y \in \mathbf{Fr}_a^0(G)$. Let M_1 be a γ_a-set of G and $y \in M_1$. Then there is $z \in (\mathrm{pn}[x, M_1] - \{x\}) \cap \mathbf{Fr}_a^0(G)$. Hence $y, z \in \mathbf{Fr}_a^0(G)$ and $yz \notin E(G)$ — a contradiction with (iii). Thus $\mathbf{Fi}_a(G) = \mathbf{Fi}_a^{-1}(G)$ and the result follows.

(ii) This immediately follows by (i) and Theorem 2.3.

(iv) Let $x \in \mathbf{Fi}_a^{-1}(G)$. Assume $N(x, G) \neq \emptyset$ and let $y \in N(x, G)$. By Corollary 1.3, $y \notin \mathbf{V}_a^-(G)$. So $y \in \mathbf{Fr}_a^0(G)$ because of (i). Thus $N(x, G) \subseteq \mathbf{Fr}_a^0(G)$. Now let M be a γ_a-set of G with $y \in M$. By (iii), $\mathbf{Fr}_a^0(G) \subseteq N[y, G]$ and then $N[x, G] \subseteq N[y, G]$ which implies that $M - \{x\}$ is an acyclic dominating set of G — a contradiction with the choice of M. ■

Kok and Mynhardt [7] defined the reinforcement number $r(G)$ to be the smallest number of edges which must be added to G to decrease the domination number. Similarly we define the *acyclic reinforcement number* $r_a(G)$ of a graph G to be the smallest number of edges which must be added to G to decrease the acyclic domination number. If $\gamma_a(G) = 1$, then define $r_a(G) = 0$. For any graph G, [7] $\gamma(G) \leq |V(G)| - \Delta(G) - r(G) + 1$. For $r_a(G)$ the following holds.

Theorem 2.7. *For any graph G*:

(i) $r_a(G) \leq |V(G)| - \Delta(G) - 1$;

(ii) $\gamma_a(G) \leq |V(G)| - \Delta(G) - r_a(G) + 1$.

Proof. If $\Delta(G) = |V(G)| - 1$ then $\gamma_a(G) = 1$ and the results are trivial. So, let $\Delta(G) < |V(G)| - 1$, $x \in V(G)$, $\deg(x, G) = \Delta(G)$ and $G_1 = G + \{xy_1, \ldots, xy_s\}$ where $\{y_1, \ldots, y_s\} = N(x, \overline{G})$. Clearly $\deg(x, G_1) = \Delta(G_1) = |V(G_1)| - 1$ and $\gamma_a(G_1) = 1 < \gamma_a(G)$. Hence $r_a(G) \leq |N(x, \overline{G})| = |V(G)| - \Delta(G) - 1$. Now, let $G_2 = G + \{xy_1, \ldots, xy_m\}$ where $m = r_a(G) - 1 \leq s - 1$. Then $\gamma_a(G) = \gamma_a(G_2) \leq 1 + \gamma_a(G_2 - N[x, G_2]) \leq 1 + (|V(G_2)| - \Delta(G_2) - 1) = |V(G)| - (\Delta(G) + r_a(G) - 1)$. ■

Remark 2.8.

(a) *It follows by the proof of Theorem 2.7 that the bounds in Theorem 2.7 (i) and (ii) are sharp for all graphs* G *with* $\gamma_a(G) = 2$.

(b) *For each graph* G *with* $\gamma_a(G) \geq 3$ *and* $|V(G)| = \Delta(G) + \gamma_a(G)$, *the bound in Theorem 2.7 (ii) is also sharp. Note for example that such a graph is the corona* $H \circ K_1$ *where* H *is any graph of order* $n \geq 3$ *with* $\Delta(H) = n - 1$.

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