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On upper bounds and connectivity of cages

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Abstract

In this paper we give an upper bound for the order of (k, g)-cages when $k - 1$ is not a prime power and $g \in \{6, 8, 12\}$. As an application we obtain new upper bounds for the order of cages when $g = 11$ and $g = 12$ and $k - 1$ is not a prime power. We also confirm a conjecture of Fu, Huang and Rodger on the k-connectivity of (k, g)-cages for $g = 12$, and for $g = 7, 11$ when $k - 1$ is a prime power.

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1 Introduction

Given two integers $k \geq 2$ and $g \geq 3$, a (k,g)-*graph* is a k-regular graph G with girth $g(G) = g$. A (k,g)-graph of minimum order is called a (k,g)-*cage*. For references on cages see for instance the survey of Wong [9] or the website of Royle [7], which contains the current best known bounds for the order of (k,g) cages.

We denote as $\nu(k,g)$ the order of the (k,g)-cages. The problem of determining the value of $\nu(k,g)$ is still wide open for most pairs (k,g). By counting the vertices emerging from a vertex or from an edge the following lower bounds are easily obtained:

$$\nu(k,g) \geq \nu_l(k,g) = \begin{cases} 2\sum\limits_{i=0}^{\frac{g-2}{2}}(k-1)^i = \frac{2(k-1)^{(g/2)}-2}{k-2} & \text{if } g \text{ is even,} \\ 1 + \sum\limits_{i=1}^{\frac{g-1}{2}} k(k-1)^{i-1} = \frac{k(k-1)^{(g-1)/2}-2}{k-2} & \text{if } g \text{ is odd.} \end{cases} \quad (1)$$

Improving on previous results of Sauer [8], Lazebnik, Ustimenko and Woldar [5] recently obtained the following upper bounds

$$\nu(k,g) \leq 2kq^{\frac{3g}{4}-a}, \quad (2)$$

where q is the smallest odd prime power satisfying $k \leq q$ and $a = 4, \frac{11}{4}, \frac{7}{2}, \frac{13}{4}$ for $g \equiv 0,1,2,3 \pmod 4$ respectively.

Fu, Huang and Rodger [3] established that $\nu(k,\cdot)$ is a monotone function of the girth. More precisely, they showed that

$$\text{For } 3 \leq k \leq g_1 \leq g_2 \text{ we have } \nu(k,g_1) < \nu(k,g_2). \quad (3)$$

In the special case when $\nu(k,g) = \nu_l(k,g)$, the (k,g)-cages are called *Moore graphs* when g is odd and *generalized polygons* if g is even. It is known that Moore graphs exist only for $k = 2$ (cycles), $g = 3$ (complete graphs) or $g = 5$ and $k = 3, 7$ or (possibly) 57; see [4]. On the other hand, for even girth, generalized polygons with $g = 4$ are the complete bipartite graphs; when $k-1$ is a prime power, known examples of (k,g) cages are the incidence graphs of projective planes for $g = 6$, of generalized quadrangles for $g = 8$ and of generalized hexagons for $g = 12$.

In this paper we prove the following result.

Theorem 1 *Let $g \in \{6,8,12\}$ and $k \geq 3$. Let q be the smallest prime power greater than or equal to k. Then*

$$\nu(k,g) \leq 2kq^{\frac{g-4}{2}}.$$

Moreover,

$$\nu(k,g) \leq \begin{cases} 2k(k-1)^{\frac{g-4}{2}}(\frac{7}{6})^{\frac{g-4}{2}} & \text{if } 3275 \geq k \geq 7, \\ 2k(k-1)^{\frac{g-4}{2}}(1+\frac{1}{2ln^2(k)})^{\frac{g-4}{2}} & \text{if } k \geq 3276. \end{cases}$$

Theorem 1, combined with the monotonocity of $\nu(k,\cdot)$, improves the upper bound (2) given by Lazebnik, Ustimenko and Woldar when $g = 11$ and $g = 12$.

It is worth mentioning that, by using similar arguments as in [1], the construction in the proof of Theorem 1 for graphs of girth $g = 6$ can be extended to degrees k close to q still keeping their girth exactly six.

Concerning the structure of cages, an interesting problem is that of their vertex-connectivity. Fu, Huang and Rodger [3] conjectured that every (k,g)-cage is k-connected and they proved the statement for $k = 3$. Marcote, Balbuena and Pelayo [6] show that every (k,g)-cage is k-connected when $g = 6$ or $g = 8$ and also showed that some $(k,5)$-cages have connectivity k, including the cases when $k-1$ is a prime power. To prove their results these authors prove that a connected graph G with minimum degree $\delta \geq 3$, girth g and order n is k-connected for $2 \leq k \leq \delta$ if $n \leq 2\nu_l(k,g) - k$; see Theorem 1 of [6]. The next proposition is a direct consequence of this result, and for the convenience of the reader we include a short proof.

Proposition 2 *Let G be a graph with minimum degree $\delta(G) = k \geq 3$, girth $g(G) = g$ and vertex-connectivity $\kappa(G) \leq k-1$. Then $|V(G)| \geq 2\nu_l(k,g) - \kappa(G)$.*

We also prove the following result.

Theorem 3 *Let G be a (k,g)-cage. If*

i) $g = 12$, or

ii) $g \in \{7, 11\}$ and $k-1$ is a prime power,

then G is k-connected.

Therefore we confirm the conjecture of Fu, Huang and Rodger for the value $g = 12$ and for infinitely many values of k when $g = 7$ or 11.

2 Notation

In what follows G denotes an undirected graph with no loops and no multiple edges. For each vertex $x \in V(G)$, $N_G(x)$ and $d_G(x)$ denote the set of neighbors and the degree of x in G respectively. The minimum degree of G will be denoted as $\delta(G)$. Given a pair of vertices $x, y \in V(G)$, by an (xy)*-path* we mean a sequence $x_0, \ldots, x_r$ of vertices of G such that $x_0 = x$, $x_r = y$ and, for every $0 \leq i \leq r-1$, $x_i x_{i+1} \in E(G)$. If $x = y$, an (xy)-path is a *cycle*. The *length* of an (xy)-path (or cycle) $x_0, \ldots, x_r$ is r. Given two vertices $x, y \in V(G)$ the *distance* between x and y is the minimum length of an (xy)-path in G, and will be denoted by $D_G(x,y)$. A *matching* is a 1-regular graph.

For a subset $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S. A subset $S \subset V(G)$ will be said to be *independent* if no two vertices in S are adjacent in G. Given $x \in V(G)$, an (x,S)-path is an (xy)-path where $y \in S$ and we denote $D_G(x,S) = \min\{D_G(x,y) : y \in S\}$. For simplicity, if $e = wy$ is an edge of G and $x \in V(G)$, we will write (e,x)-path and $D_G(e,x)$ instead of $(\{w,y\},x)$-path and $D_G(\{w,y\},x)$, respectively.

Let D be a directed graph. The *out-degree* of x is the cardinality of the set $\{y \in V(D) : (x,y) \in A(D)\}$. An *oriented cycle* $\mathcal{C}$ of D is a sequence $x_0, x_1, \ldots, x_{r-1}, x_r = x_0$ of vertices such that, for each $0 \le i \le r-1$, $(x_i, x_{i+1}) \in A(D)$.

We shall use the following result on the existence of prime powers in short intervals of integers; see e.g. Dusart [2].

$$\text{If } k \ge 3275 \text{ then the interval } [k, k(1 + \frac{1}{2ln^2(k)})] \text{ contains a prime number.} \quad (4)$$

$$\text{If } 6 \le k \le 3276 \text{ then the interval } [k, \frac{7k}{6}] \text{ contains a prime power.} \quad (5)$$

3 Proofs of the Theorems

Proof of Theorem 1. Let $k \ge 3$. If $k-1$ is a prime power, $\nu_l(k,g) = \nu(k,g)$ and the result follows. Let us suppose that $k-1$ is not a prime power. Let q be the smallest prime power greater than or equal to k and let $g \in \{6, 8, 12\}$. Let $G = (V,E)$ be a $(q+1,g)$-cage, that is, a $(q+1,g)$-graph of minimum order.

For a given edge $e = xy$ of G and for each $i \ge 0$, let

$$\mathcal{N}_i(e) = \{z \in V(G) : D_G(e,z) = i\}.$$

The subgraph spanned by the vertices within distance $l \le \frac{g-2}{2}$ from $\{x,y\}$ is a tree as shown in Figure 1. Therefore, for $0 \le i \le \frac{g-2}{2}$, we have

$$|\mathcal{N}_i(e)| = 2q^i.$$

Since q is a prime power and G is a $(q+1,g)$-cage,

$$V(G) = \bigcup_{0 \le i \le \frac{g-2}{2}} \mathcal{N}_i(e).$$

This implies that the subgraph of G induced by $\mathcal{N}_{\frac{g-2}{2}}(e)$ is a q-regular graph (see an illustration in Figure 1.)

Let $N_G(x) \setminus y = \{x_1, \ldots, x_q\}$ and $N_G(y) \setminus x = \{y_1, \ldots, y_q\}$, and, for each $1 \le i \le q$, let

$$\mathcal{B}(x_i) = \{z \in \mathcal{N}_{\frac{g-2}{2}}(e) : D_G(x_i, z) = \frac{g-4}{2}\}$$

and

$$\mathcal{B}(y_i) = \{z \in \mathcal{N}_{\frac{g-2}{2}}(e) : D_G(y_i, z) = \frac{g-4}{2}\}.$$

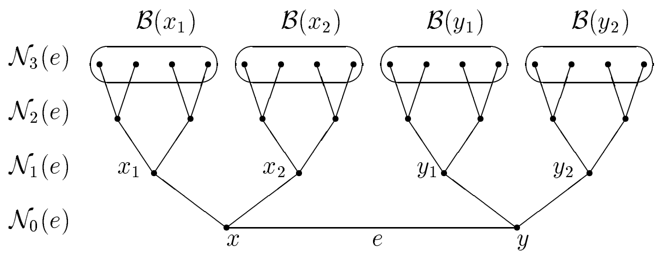

Figure 1: Tree of neighbors emerging from an edge for $k = 3$ and $g = 8$.

For each vertex $z \in \mathcal{N}_{\frac{g-2}{2}}(e)$ there is exactly one (e, z)-path of length $\frac{g-2}{2}$ in G. It follows that $\{B(x_1), \ldots, B(x_q), B(y_1), \ldots, B(y_q)\}$ is a partition of $\mathcal{N}_{\frac{g-2}{2}}(e)$. In particular,

$$|\mathcal{B}(x_i)| = |\mathcal{B}(y_i)| = q^{\frac{g-4}{2}},\ i = 1, \ldots, q.$$

Note that the set $U = \mathcal{B}(x_1) \cup \ldots \cup \mathcal{B}(x_q)$ is an independent set, and similarly $W = \mathcal{B}(y_1) \cup \ldots \cup \mathcal{B}(y_q)$ is also an independent set. Furthermore, for $1 \leq i, j \leq q$, a vertex in $B(x_i)$ can be joined by an edge to at most one vertex in $B(y_j)$, since otherwise the graph would contain a cycle of length at most $g-1$. Since the subgraph of G induced by $U \cup W = \mathcal{N}_{\frac{g-2}{2}}(e)$ is q-regular, the subgraph $G[B(x_i), B(y_j)]$ induced by $B(x_i) \cup B(y_j)$ is a matching for each i, j with $1 \leq i, j \leq q$.

For each $1 \leq k \leq q$ let G_k be the subgraph induced by $\bigcup_{i=1}^{k}(\mathcal{B}(x_i) \cup \mathcal{B}(y_i))$. By the above remarks, G_k is k-regular, has order $2kq^{(g-4)/2}$ and, being a subgraph of G, it has girth $g_k \geq g$.

In particular we have $\nu(k, g_k) \leq 2kq^{(g-4)/2}$ for each $k = 3, \ldots, q$. Since $g \leq g_k$, by (3) we have that

$$\nu(k, g) \leq 2kq^{(g-4)/2}. \tag{6}$$

Finally, since $k-1$ is not a prime power, by (4) and (5) we see that, for $k - 1 \geq 3275$, there is a prime in the interval $[k, (k-1)(1 + \frac{1}{2ln^2(k)})]$ and, for $3275 \geq k - 1 \geq 6$, there is a prime power in the interval $[k, \frac{(k-1)7}{6}]$. Therefore,

$$q \leq \begin{cases} \frac{(k-1)7}{6} & \text{if } 7 \leq k \leq 3276; \\ (k-1)(1 + \frac{1}{2ln^2(k)}) & \text{if } k \geq 3276. \end{cases}$$

Substitution of the above inequalities in (6) gives the second inequality in Theorem 1. □

Proof of Proposition 2. Let G be a graph with girth $g(G) = g$, minimum degree $\delta(G) = k \geq 3$ and connectivity $\kappa(G) \leq k - 1$.

For each vertex $x \in V(G)$ define

$$\mathcal{B}(x) = \{z \in V(G) : D_G(x,z) \leq \lfloor \frac{g-1}{2} \rfloor\},$$

and, for each $y \in N_G(x)$, let $\mathcal{B}_x(y)$ be the set of vertices z of G such that there is a (zy)-path in G of length at most $\lfloor \frac{g-1}{2} \rfloor - 1$ which does not use the edge xy. Since $g(G) = g$, the set $\{\mathcal{B}_x(y) : y \in N_G(x)\}$ is a partition of $\mathcal{B}(x) \setminus x$ for each $x \in V(G)$ (see an ilustration in Figure 2).

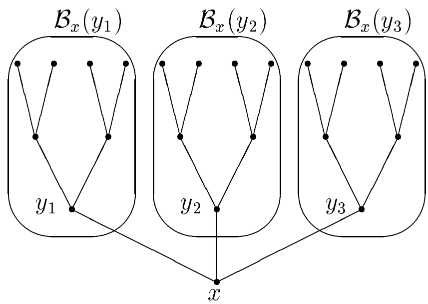

Figure 2: Tree emerging from a vertex x for $\lfloor \frac{g-1}{2} \rfloor = 3$ and $k = 3$.

Let $S \subseteq V(G)$ be a minimum cutset of G and let H be a connected component of $G \setminus S$. Since $\delta(G) = k > \kappa(G) = |S|$, each vertex $x \in V(G)$ has a neighbor $y \in N_G(x)$ such that $\mathcal{B}_x(y) \cap S = \emptyset$. In particular, for each $x \in V(H)$, there is $y \in N_G(x) \cap V(H)$ such that $\mathcal{B}_x(y) \subseteq V(H)$.

Let D be a digraph defined as follows: the vertex set of D is $V(D) = V(H)$ and (x,y) is an arc of D if and only if $\mathcal{B}_x(y) \subseteq V(H)$. By the above remark, the minimum out-degree of D is at least 1. It follows that D contains an oriented cycle $\mathcal{C} = (x_1, \ldots, x_t)$, $t \geq 2$.

We claim that

$$\min\{D_G(S, x_i),\ i = 1, \ldots, t\} \geq \lfloor \frac{g-1}{2} \rfloor. \tag{7}$$

Suppose on the contrary that $m = \min\{D_G(S, x_i),\ i = 1, \ldots, t\} \leq \lfloor \frac{g-1}{2} \rfloor - 1$; say $D_G(S, x_{i+1}) = m$. Let P be a (S, x_{i+1})-path in G of length m. Since (x_i, x_{i+1}) is an arc of D, we have $\mathcal{B}_{x_i}(x_{i+1}) \subseteq V(H)$. Therefore P must use the edge $x_i x_{i+1}$, which implies $D_G(x_i, S) = m - 1$, a contradiction. This proves (7).

By (7), each connected component H of $G \setminus S$ contains at least two adjacent vertices x, y such that

$$\min\{D_G(S,x), D_G(S,y)\} \geq \lfloor \frac{g-1}{2} \rfloor.$$

It follows that $\mathcal{B}(x) \cup \mathcal{B}(y) \subseteq V(H) \cup S$.

Suppose that g is even. Then $\mathcal{B}(x) \cup \mathcal{B}(y) = \{z \in V(G) : D_G(xy, z) \leq \lfloor \frac{g-1}{2} \rfloor\}$. By comparing with (1) we have $|\mathcal{B}(x) \cup \mathcal{B}(y)| \geq \nu_l(k, g)$. This implies that each connected component of $G \setminus S$ has order at least $\nu_l(k, g) - |S|$.

Suppose now that g is odd. Then it is clear that $|V(H)| \geq |\mathcal{B}(x)| - |S|$ and again $|\mathcal{B}(x)| \geq \nu_l(k, g)$, which implies that each connected component of $G \setminus S$ has order at least $\nu_l(k, g) - |S|$.

In both cases we get

$$|V(G)| \geq 2\nu_l(k, g) - |S|.$$

□

Proof of Theorem 3. Let G be a (k, g)-cage. The statement will follow from Proposition 2 if we show that $|V(G)| < 2\nu_l(k, g) - (k-1)$ whenever $g = 12$, or whenever $g \in \{7, 11\}$ and $k - 1$ is a prime power.

i) Let $g = 12$, and let q be the smallest prime power greater than or equal to $k-1$. If $q = k-1$ then G is a minimal $(k, 12)$-cage and $|V(G)| = \nu_l(k, 12) < 2\nu_l(k, 12) - (k-1)$. Suppose that $k-1$ is not a prime power. Since $1 + \frac{1}{2ln^2(k)} \leq \frac{7}{6}$ whenever $k \geq 3276$, by Theorem 1 we may assume that $|V(G)| \leq 2k(k-1)^4(\frac{7}{6})^4$ and so $|V(G)| < 4k(k-1)^4$. Using (1) we have

$$\begin{aligned} |V(G)| &< 4k(k-1)^4 = 4((k-1)^5 + (k-1)^4) \\ &\leq 4((k-1)^5 + (k-1)^4 + (k-1)^3) - (k-1) \\ &\leq 2\nu_l(k, 12) - (k-1). \end{aligned}$$

ii) Let $g \in \{7, 11\}$ and suppose that $k-1$ is a prime power. By (3) we have $|V(G)| < \nu(k, g+1)$. Since $k-1$ is a prime power we have

$$|V(G)| < \nu_l(k, g+1) = \frac{2(k-1)^{\frac{g+1}{2}} - 2}{k-2}.$$

On the other hand

$$\begin{aligned} 2\nu_l(k, g) - (k-1) &= \frac{2k(k-1)^{\frac{g-1}{2}} - 4}{k-2} - (k-1) \\ &= \frac{2(k-1)^{\frac{g+1}{2}} + 2(k-1)^{\frac{g-1}{2}} - 4}{k-2} - (k-1) \\ &= \nu_l(k, g+1) + \frac{2(k-1)^{\frac{g-1}{2}} - 2}{k-2} - (k-1), \end{aligned}$$

and since

$$\begin{aligned} \frac{2(k-1)^{\frac{g-1}{2}} - 2}{k-2} &= \frac{2(k-2)(k-1)^{\frac{g-3}{2}} + 2(k-1)^{\frac{g-3}{2}} - 2}{k-2} \\ &= 2(k-1)^{\frac{g-3}{2}} + \frac{2(k-1)^{\frac{g-3}{2}} - 2}{k-2} \\ &> (k-1) \end{aligned}$$

it follows that $\nu_l(k, g+1) < 2\nu_l(k, g) - (k-1)$ and so $|V(G)| < 2\nu_l(k, g) - (k-1)$. □

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