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The circular k-partite crossing number of $K_{m,n}$

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Abstract

We define a new kind of crossing number which generalizes both the bipartite crossing number and the outerplanar crossing number. We calculate exact values of this crossing number for many complete bipartite graphs and also give a lower bound.

1 Preliminaries

The *bipartite crossing number* of a bipartite graph G was defined by Watkins in [7] to be the minimum number of crossings over all bipartite drawings of G. A *bipartite drawing* of bipartite G is one in which the vertices of the parts V_1 and V_2 are placed respectively on two distinct parallel lines and then the edges of G are drawn as straight line segments joining appropriate pairs of vertices. The calculation and estimation of this number are of interest to those who study VLSI design, graph drawing algorithms, and/or topological graph theory. See [4] for a bibliography on the topic as well as some of the few known exact results.

The *outerplanar crossing number* of a graph G, also known as the *circular* or *convex crossing number* of G, was defined by Kainen in [3] to be the minimum number of crossings taken over all plane drawings of G where the vertices lie on a circle and the edges are chords of that circle. The calculation and estimation of this number are of interest to the same audience as the bipartite crossing number. See [1] and [2] for an introduction and bibliography, and [2] and [5] for the few known exact results.

In this paper we introduce a notion, the *circular k-partite crossing number* of a k-partite graph G, which generalizes both of these definitions. A *circular k-partite drawing* of k-partite G is constructed as follows: partition a circle into k segments of arc. Place the vertices of the i^{th} part into the i^{th} segment of arc and then add the edges as chords of the circle with the usual proviso that no more than two edges should meet at a crossing. The *circular k-partite crossing number* of G, denoted by $\mathrm{cpr}_k(G)$, is the minimum number of crossings taken over all circular k-partite drawings, all possible assignments of vertices to parts, and all numberings of the parts of G. Note that if G has p vertices then G is p-partite, and $\mathrm{cpr}_p(G) = \nu_1(G)$, the familiar outerplanar crossing number. Furthermore, if G is bipartite then $\mathrm{cpr}_2(G) = \mathrm{bcr}(G)$, the familiar bipartite crossing number.

Note that we prepend the modifier "circular" to our epithet for $\mathrm{cpr}_k(G)$ to distinguish it from the quite different k-partite crossing number, also known as the k-layer crossing number. Finally, we will have occasion to mention the following:

Theorem 1. *If $m|n$ then the outerplanar crossing number of $K_{m,n}$ is*

$$\frac{1}{12}n(m-1)(2mn-3m-n).$$

A proof of this may be found in [5].

2 Results

Note that $K_{m,n}$ is k-partite for $2 \leq k \leq m+n$. Our goal is to determine $\mathrm{cpr}_k(K_{m,n})$ for each k in this range. To that end we observe that if $m \leq n$ then $\mathrm{cpr}_k(K_{m,n}) = \mathrm{cpr}_{2m}(K_{m,n})$ for all $2m \leq k \leq m+n$, and that $\mathrm{cpr}_{2k}(K_{m,n}) = \mathrm{cpr}_{2k+1}(K_{m,n})$ since every circular $(2k+1)$-partite drawing of $K_{m,n}$ is also a $2k$-partite drawing and vice-versa. Our result, which generalizes Theorem 1, is the following:

Theorem 2.
$$\mathrm{cpr}_{2k}(K_{m,n}) \geq \binom{m}{2}\binom{n}{2} - \frac{(k^2-1)m^2n^2}{12k^2},$$
with equality when $k|m$ and $k|n$.

Proof. Let D be a circular $2k$-partite drawing of $K_{m,n}$. We denote the number of crossings in D by $\mathrm{cpr}_{2k}(D)$. Let M and N be the two parts of $K_{m,n}$, with $|M| = m$ and $|N| = n$. We refer to the vertices in M as pink and to those in N as black. Let the $2k$ segments of arc be labeled consecutively $M_1, N_1, M_2, N_2, \ldots, M_k, N_k$, and let $|M_i| = m_i$ and $N_i = n_i$.

Now suppose that the $2k$-partite sets fail to alternate colors. In this case D is a circular $2j$-partite drawing of $K_{m,n}$ for some $j < k$ where the $2j$-partite sets do alternate colors. Since
$$\binom{m}{2}\binom{n}{2} - \frac{(k^2-1)m^2n^2}{12k^2}$$
is a strictly decreasing function of k for $k \geq 1$, if we prove the theorem for drawings where the partite sets do alternate colors we will have proved it also for drawings where they do not.

Let u_1 and u_2 be distinct pink vertices and let v_1 and v_2 be distinct black vertices. Let $u_j \in M_{i_j}$ and $v_j \in N_{i_j}$. Then the vertices u_1, u_2, v_1, and v_2 determine a crossing unless M_{i_1} and M_{i_2} separate N_{i_1} and N_{i_2} on the boundary of the circle. If $1 \le i < j \le m$, then M_i and M_j separate $(j-i)(k-(j-i))$ distinct pairs of black partite sets from one another. Explicitly, $N_i, \ldots, N_{j-1}$ are separated from $N_j, \ldots, N_{i-1}$ where the subscripts are, naturally, read modulo k. Suppose now that $u_1 \in M_i$ and $u_2 \in M_j$ with $1 \le i < j \le k$. Then there are

$$\sum_{\substack{i \le s \le j-1 \\ j \le t \le i-1}} m_i m_j n_s n_t$$

choices of v_1 and v_2 which do not determine crossings. Therefore

$$\mathrm{cpr}_{2k}(D) = \binom{m}{2}\binom{n}{2} - \sum_{\substack{1 \le i \le k-1 \\ i+1 \le j \le k}} \sum_{\substack{i \le s \le j-1 \\ j \le t \le i-1}} m_i m_j n_s n_t. \tag{1}$$

Clearly $m_i m_j n_s n_t \le \frac{m^2 n^2}{k^4}$, and the lower bound follows from this (well, this and a considerable amount of algebra). When $k|m$ and $k|n$, equality is obtained by distributing the vertices so that there are $\frac{m}{k}$ in each M_i and $\frac{n}{k}$ in each N_i. □

Note that we can obtain Theorem 1 from Theorem 2 by substituting $k = m$ in the case where $m|n$. Note also that the double sum in (1) is maximized when the values of m_i and n_i are as evenly distributed as possible, that is, when $m - k\left\lfloor \frac{m}{k} \right\rfloor$ of the m_i's are equal to $\left\lceil \frac{m}{k} \right\rceil$ and the other $k - m + k\left\lfloor \frac{m}{k} \right\rfloor$ of them are equal to $\left\lfloor \frac{m}{k} \right\rfloor$ and likewise for the n_i's. It is possible to use (1) to obtain an exact expression for the value of $\mathrm{cpr}_{2k}(K_{m,n})$ for particular values of k even when $k \nmid m$ or $k \nmid n$. Some care must be taken to arrange the different values as evenly as possible for values of $k \ge 3$. As a simple example, for $k = 2$ the exact result is:

$$\mathrm{cpr}_4 = \binom{m}{2}\binom{n}{2} - \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor.$$

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