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Further results on antimagic graph labelings

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Abstract

For a graph $G = (V, E)$ with p vertices and q edges, a bijection f from $V(G) \cup E(G)$ onto $\{1, 2, \dots, p + q\}$ is called an (a, d)-edge-antimagic total labeling of G if the edge-weights $\{w(uv) : w(uv) = f(u) + f(v) + f(uv), uv \in E(G)\}$, form an arithmetic progression starting from a and having common difference d. We study graphs with no edge-antimagic labeling and show how to construct labelings for cycles with $d = 3$. We also show the relationship between the sequential graphs and the graphs having an (a, d)-edge-antimagic vertex labeling.

1 Introduction

All graphs in this paper will be finite, simple and undirected. The graph $G = G(V, E)$ has vertex-set $V = V(G)$ and edge-set $E = E(G)$, and we denote $|V(G)|$ and $|E(G)|$ by p and q respectively. We follow either Wallis [19] or West [20] for most of the graph theory terminology and notation used in this paper.

By a *labeling* we mean a one-to-one mapping that carries a set of graph elements onto a set of integers, called labels. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. In this paper we also deal with the case where the domain is $V \cup E$, and these are called *total labelings*. A general survey of graph labelings is found in [6].

Hartsfield and Ringel [12] introduced the concept of an antimagic graph. In their terminology, an *antimagic labeling* is an edge labeling of the graph with the integers $1, 2, \ldots, q$ such that all sums of the labels on the edges incident with each vertex v are distinct. Among the graphs known to be antimagic are paths, cycles, complete graphs and wheels (see [12]). Hartsfield and Ringel conjectured that all trees except K_2 are antimagic. Moreover, they conjectured that every connected graph other than K_2 is antimagic. These two conjectures are not yet settled. Recently, Alon et al. [1] validated that the last conjecture is true for all graphs with p vertices, $p > 4$, and minimum degree $\Omega(\log p)$.

In this paper we concentrate on the variations of antimagic labeling, including vertex-antimagic labeling and edge-antimagic labeling.

Simanjuntak et al. [18] defined an *(a, d)-edge-antimagic vertex labeling* for a graph $G(V, E)$ as an injective mapping f from $V(G)$ onto the set $\{1, 2, \ldots, p\}$ such that the set $\{f(u) + f(v) : uv \in E(G)\}$ is $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

Hegde in 1989 in his Ph.D. thesis (see [13]) introduced the concept of a *strongly (a, d)-indexable labeling* which is equivalent to (a, d)-edge-antimagic vertex labeling.

An *(a, d)-edge-antimagic total labeling* for graph $G(V, E)$ is defined in [18] as a one-to-one mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ so that the set of edge-weights $\{w(uv) : w(uv) = f(u) + f(v) + f(uv), uv \in E(G)\}$ of all edges uv in G is equal to $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, for two integers $a > 0$ and $d \geq 0$.

This labeling is a natural extension of the notion of edge-magic labeling which originally was introduced by Kotzig and Rosa in [14, 15], where an edge-magic labeling is called a *magic valuation*.

An (a, d)-edge-antimagic total labeling f is called *super* if it has the property that the vertex-labels are the integers $1, 2, \ldots, p$, the smallest possible labels. The notion of a super $(a, 0)$-edge-antimagic total labeling was defined by Enomoto et al. in [5], who called it a super edge-magic labeling.

A graph with an (a, d)-edge-antimagic total labeling or super (a, d)-edge-antimagic total labeling will be called (a, d)-edge-antimagic total or super (a, d)-edge-antimagic total, respectively.

Some properties of the (a, d)-edge-antimagic vertex labeling and the (a, d)-edge-antimagic total labeling are studied in [18]. The relationships between (a, d)-edge-antimagic total labeling and other labelings, namely, edge-antimagic vertex labeling and edge-magic total labeling are presented in [2].

2 Edge-antimagic vertex labelings

In this section we study properties of (a,d)-edge-antimagic vertex labelings and we also show the relationship between the sequential graphs and the graphs having an (a,d)-edge-antimagic vertex labeling.

Lemma 1. *Let G be a connected (p,q)-graph which is not a tree. If G has an (a,d)-edge-antimagic vertex labeling then $d=1$.*

Proof: Assume that G has an (a,d)-edge-antimagic vertex labeling $f : V(G) \to \{1,2,\ldots,p\}$ and $\{w(uv) : uv \in E(G)\} = \{a, a+d, a+2d, \ldots, a+(q-1)d\}$ is the set of edge-weights. The minimum possible edge-weight under the labeling f is at least 3. On the other hand, the maximum edge-weight is no more than $2p-1$.

Thus

$$a + (q-1)d \leq 2p - 1$$

and

$$d \leq \frac{2p-4}{q-1}\,. \tag{1}$$

Since we suppose that G is not a tree, i.e. $p \leq q$, then (1) gives $d < 2$.

It is not difficult to see that for every connected (p,q)-graph, $q \geq 2$, there is no $(a,0)$-edge-antimagic vertex labeling. □

Note that from inequality (1) it follows that if G is a connected tree then $d \leq 2$.

Lemma 2. *Let G be a connected (p,q)-graph which is not a tree, such that the degree of every vertex is odd, q is even, and $p+q \equiv 2 \pmod 4$. Then there is no (a,d)-edge-antimagic vertex labeling of G.*

Proof: Suppose G has an (a,d)-edge-antimagic vertex labeling $f : V(G) \to \{1,2,\ldots,p\}$ such that $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$ is the set of edge-weights. According to Lemma 1 we have that $d=1$. The sum of all the vertex labels used to calculate the edge-weights of G under the labeling f is equal to the sum of edge-weights. Thus the following equation holds

$$\sum_{v \in V(G)} \deg(v) \cdot f(v) = \frac{q}{2}(2a + q - 1)\,, \tag{2}$$

where $\deg(v)$ is the degree of vertex v. Let us distinguish two cases.

Case (i): If $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$ then the left hand side of equation (2) is odd but the right hand side is even.

Case (ii): If $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$ then the left hand side of (2) is even but the right hand side is odd.

It leads to a contradiction in both cases. □

Lemma 3. *Let G be an Eulerian (p,q)-graph with q even. If G has an (a,d)-edge-antimagic vertex labeling then $q \equiv 0 \pmod 4$.*

Proof: Assume G is an Eulerian (p,q)-graph, q is even, and $f : V(G) \to \{1,2,\ldots,p\}$ is an (a,d)-edge-antimagic vertex labeling. In light of Lemma 1 it follows that $d = 1$.

Since $\deg(v)$ is even for all $v \in V(G)$, the left hand side of equation (2) gives even integer. The right hand side of (2) is even only for $q \equiv 0 \pmod 4$. □

Chang et al. [4] define an injective labeling f of a graph G with q edges to be *strongly c-harmonious* if the vertex labels are from $\{0,1,\ldots,q-1\}$ and the edge labels induced by $f(x)+f(y)$ for each edge xy are $c, c+1, c+2, \ldots, c+q-1$. Grace [9] called such a labeling *sequential.*

Theorem 1. *Let G be a connected (p,q)-graph which is not a tree. If G has an (a,d)-edge-antimagic vertex labeling then $G+K_1$ is sequential.*

Proof: Let $f : V(G) \to \{1,2,\ldots,p\}$ be (a,d)-edge-antimagic vertex labeling of G. With respect to Lemma 1 the difference $d = 1$. Let $\{a, a+1, a+2, \ldots, a+q-1\}$ be the set of edge-weights under the labeling f. The maximum edge-weight is no more than $2p-1$. Consequently,

$$a + q - 1 \le 2p - 1\,. \tag{3}$$

Let us now construct a new function f_1 of $G+K_1$ as follows,

$$f_1(v) = \begin{cases} f(v) & \text{if } v \in V(G) \\ a+q-1 & \text{if } v \in V(K_1). \end{cases}$$

We observe that

$\{w_{f_1}(uv) : uv \in E(G)\} = \{a, a+1, a+2, \ldots, a+q-1\}$ and

$\{w_{f_1}(uv) : u \in V(G) \text{ and } v \in V(K_1)\} = \{a+q, a+q+1, a+q+2, \ldots, a+q+p-1\}$.

Since G is not a tree, i.e. $p \le q$, then from (3) we have

$$p < a + q - 1 \le 2p - 1 \le p + q - 1 = |E(G+K_1)| - 1\ .$$

This implies that the labeling f_1 is sequential. □

Let mC_n be the disjoint union of m copies of a cycle C_n.

Theorem 2. *mC_n has an (a,d)-edge-antimagic vertex labeling if and only if m and n are odd and $d = 1$.*

Proof: Consider the graph mC_n with $p = q = mn$. If mC_n has an (a,d)-edge-antimagic vertex labeling then in the light of the proof of Lemma 1 we have that $d = 1$.

Gnanajothi [7] (see [6]) has shown that mC_n is sequential if and only if m and n are odd. One can see that sequential labeling is equivalent to $(a,1)$-edge-antimagic vertex labeling for every (p,q)-graph having $p=q$.

Thus we arrive at the desired result. □

3 Edge-antimagic total labelings

Assume that a bijection $f: V(G)\cup E(G)\to\{1,2,\dots,p+q\}$ is (a,d)-edge-antimagic total and $W=\{w(uv): w(uv)=f(u)+f(v)+f(uv), uv\in E(G)\}=\{a,a+d,a+2d,\dots,a+(q-1)d\}$ is the set of edge-weights. In the computation of the edge-weights of G each edge label is used once and each label of vertex $v\in V(G)$ is used $\deg(v)$ times.

Thus the following equation holds

$$\sum_{v\in V(G)}\deg(v)\cdot f(v)+\sum_{e\in E(G)} f(e)=\sum_{e\in E(G)} w(e). \tag{4}$$

Consideration of the parity of the left hand side and the right hand side of equation (4) leads to the following results.

Theorem 3. *Let G be an odd degree (p,q)-graph.*

(i) If $p\equiv 2 \pmod 4$ and $q\equiv 0 \pmod 4$ then G is not (a,d)-edge-antimagic total for every d.

(ii) If $p\equiv 0 \pmod 4$ and $q\equiv 2 \pmod 4$ then G is not (a,d)-edge-antimagic total for every even d.

Proof: Suppose G is a (p,q)-graph and every vertex of G has odd degree. Let $f: V(G)\cup E(G)\to\{1,2,\dots,p+q\}$ be an (a,d)-edge-antimagic total labeling of G.

The sum of edge-weights in the set W is

$$\sum_{e\in E(G)} w(e)=\frac{q}{2}(2a+(q-1)d) \tag{5}$$

and the sum of all vertex labels and edge labels used to calculate the edge-weights is

$$\sum_{v\in V(G)}\deg(v)\cdot f(v)+\sum_{e\in E(G)} f(e)$$

$$\sum_{v\in V(G)}\deg(v)\cdot f(v)+\left((1+2+\cdots+p+q)-\sum_{v\in V(G)} f(v)\right). \tag{6}$$

In the light of equations (4), (5) and (6) we get

$$\sum_{v\in V(G)} (\deg(v)-1)\cdot f(v) = \frac{q}{2}\left[2a+(q-1)d\right] - \frac{(p+q)(p+q+1)}{2}. \tag{7}$$

If both conditions (i) and (ii) are satisfied then $(p+q)(p+q+1)/2$ is odd. Since every vertex of G has odd degree the left hand side of equation (7) is even. On the other hand, the right hand side of (7) under the conditions (i) and (ii) is odd and we have a contradiction. □

Theorem 4. *Let G be an odd degree (p,q)-graph. If $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 2$ then G is not super (a,d)-edge-antimagic total for every odd d.*

Proof: Assume that G is super (a,d)-edge-antimagic total and every vertex of G has odd degree.

From equation (4) we have

$$\sum_{v\in V(G)} \deg(v)\cdot f(v) = \sum_{e\in E(G)} w(e) - \left[(p+1)+(p+2)+\cdots+(p+q)\right], \tag{8}$$

so

$$\sum_{v\in V(G)} (\deg(v)-1)\cdot f(v) + \frac{p(p+1)}{2} = \frac{q}{2}\left[2(a-p-1)+(q-1)(d-1)\right]. \tag{9}$$

We can see that if $p \equiv 2$ (mod 4), q is even and d is odd and then the left hand side of equation (9) is odd but the right hand side is even which is a contradiction. □

Theorem 5. *Let G be an even degree (p,q)-graph. If $q \equiv 2 \pmod 4$ then G is not super (a,d)-edge-antimagic total for every even d.*

Proof: Assume that G is an even degree (p,q)-graph. Let G be super (a,d)-edge-antimagic total with a super (a,d)-edge-antimagic total labeling $f: V(G)\cup E(G) \to \{1,2,\ldots,p+q\}$.

From equation (4) we have

$$\sum_{v\in V(G)} \deg(v)\cdot f(v) + \left[(p+1)+(p+2)+\cdots+(p+q)\right] = \frac{q}{2}\left[2a+(q-1)d\right], \tag{10}$$

which is obviously equivalent to the equation

$$\sum_{v\in V(G)} \deg(v)\cdot f(v) = \frac{q}{2}\left[2(a-p-1)+(q-1)(d-1)\right]. \tag{11}$$

Since d is even and $q \equiv 2$ (mod 4), the right hand side of equation (11) gives an odd integer. However, the left hand side of equation (11) is even because every vertex of G has even degree. This leads to a contradiction. □

4 Edge-antimagic labeling of cycles

Let C_n be the cycle with $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.

Kotzig and Rosa [14] showed that the cycles C_n, $n \geq 3$, are edge-magic ($(a, 0)$-edge-antimagic total) with the common edge-weight $3n + 1$ (for n odd), $\frac{5n}{2} + 2$ (for $n \equiv 2$ (mod 4)) and $3n$ (for $n \equiv 0$ (mod 4)). An edge-magic labeling for C_n with the common edge-weight $\frac{5n+3}{2}$ (for n odd) and $\frac{5n}{2} + 2$ (for n even) was described by Godbold and Slater in [8].

In [2] it is shown that for every cycle C_n there is no (a, d)-edge-antimagic total labeling with $d > 5$. Simanjuntak et al. [18] constructed (a, d)-edge-antimagic total labelings with $d = 1$ for all cycles C_n, $d = 2$ for even cycles, and $d = 3$ for odd cycles. Also (a, d)-edge-antimagic total labelings for odd cycles and for $d \in \{2, 4\}$ are established in [2].

Now, we construct labelings with $d = 3$ for all cycles C_n.

Theorem 6. *Every cycle C_n, $n \geq 3$, has a $(2n + 2, 3)$-edge-antimagic total labeling and an $(n + 4, 3)$-edge-antimagic total labeling.*

Proof: Assign the vertices v_i of C_n such that

$f(v_i) = n + i - 1$ for $1 \leq i \leq n$.

Assign the edges $v_i v_{i+1}$ of C_n by

$f(v_i v_{i+1}) = i$ for $1 \leq i \leq n - 1$ and $f(v_n v_1) = 2n$.

It is easy to verify that the labeling f uses each integer from the set $V(C_n) \cup E(C_n) = \{1, 2, \ldots, 2n\}$ exactly once. The labeling f is total. Moreover, the edge-weights under the total labeling f are

$$w(v_i v_{i+1}) = f(v_i) + f(v_{i+1}) + f(v_i v_{i+1}) = 2n + 3i - 1 \text{ for } 1 \leq i \leq n - 1$$

and

$$w(v_n v_1) = f(v_n) + f(v_1) + f(v_n v_1) = 5n - 1.$$

Thus, the total labeling f is $(2n + 2, 3)$-edge-antimagic.

Next define a new labeling f' as follows

$f'(v_i) = 2n + 1 - f(v_i)$ for $1 \leq i \leq n$,

$f'(v_i v_{i+1}) = 2n + 1 - f(v_i v_{i+1})$ for $1 \leq i \leq n - 1$

and $f'(v_n v_1) = 2n + 1 - f(v_n v_1)$.

Clearly, the labeling f' is also a total labeling. The edge-weights under the total labeling f' are

$$\begin{aligned} w'(v_iv_{i+1}) &= f'(v_i) + f'(v_{i+1}) + f'(v_iv_{i+1}) \\ &= 2n+1-f(v_i)+2n+1-f(v_{i+1})+2n+1-f(v_iv_{i+1}) \\ &= 6n+3-w(v_iv_{i+1}) \ = \ 4n+4-3i \end{aligned}$$

for $i = 1, 2, \ldots, n$. This guarantees that f' is an $(n+4, 3)$-edge-antimagic total labeling. □

It remains an open problem to find (a, d)-edge-antimagic total labelings for even cycles with $d \in \{4, 5\}$ and for odd cycles with $d = 5$.

5 Vertex-antimagic total labelings

By the *vertex-weight* $wt(v)$ of a vertex $v \in V(G)$ under a total labeling $g : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ we mean the sum of values $g(e)$ assigned to all edges incident to a given vertex v together with the value on v itself.

A bijection $g : V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ is called an (a, d)-*vertex-antimagic total labeling* of G, if the set of vertex-weights of all vertices in G is $\{a, a+d, a+2d, \ldots, a+(p-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

The (a, d)-vertex-antimagic total labeling is a natural extension of the notion of vertex-magic total labeling ($(a, 0)$-vertex-antimagic total) which originally was introduced by MacDougall et al. in [17]. In [10] Gray et al. examined the existence of vertex-magic total labelings of trees, forests and galaxies. Lin and Miller [16] proved that all complete graphs of order divisible by 4 are vertex-magic total. It had been shown in [11] and [17] that all other complete graphs are vertex-magic total. The basic properties of (a, d)-vertex-antimagic total labelings and their relationships with several other previously studied graph labelings can be found in [3].

An (a, d)-vertex-antimagic total labeling g is called *super* if $g(V(G)) = \{1, 2, \ldots, p\}$ and $g(E(G)) = \{p+1, p+2, \ldots, p+q\}$.

Let a bijection $g : V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ be a super (a, d)-vertex-antimagic total labeling and $\{wt(v) : v \in V(G)\} = \{a, a+d, a+2d, \ldots, a+(p-1)d\}$ be the set of vertex-weights.

Summing the vertex-weights over all vertices in G is equal to summing all the values of vertices and edges where each vertex label is used once and each edge label is used twice.

So we get

$$\sum_{v \in V(G)} g(v) + 2 \sum_{e \in E(G)} g(e) = \sum_{v \in V(G)} wt(v). \tag{12}$$

Investigation of the parities of equation sides leads to the following results.

Theorem 7. *Let G be a (p,q)-graph.*

(i) If $p \equiv 2 \pmod 4$ then G is not super (a,d)-vertex-antimagic total for every even d.

(ii) If $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$ then G is not super (a,d)-vertex-antimagic total for every odd d.

(iii) If $p \equiv 0 \pmod 8$ and $q \equiv 2 \pmod 4$ then G is not super (a,d)-vertex-antimagic total for every d.

Proof: Suppose g is a super (a,d)-vertex-antimagic total labeling for a (p,q)-graph G. Consequently

$$\sum_{e\in E(G)} g(e) = \frac{q(2p+q+1)}{2}, \qquad \sum_{v\in V(G)} g(v) = \frac{p(p+1)}{2}$$

and

$$\sum_{v\in V(G)} wt(v) = ap + \frac{p(p-1)d}{2}.$$

Then from (12) we have the following equation:

$$(2p+q+1)q = \frac{p}{2}\left[(p-1)(d-1)+2(a-1)\right]. \qquad (13)$$

Case (i): If $p \equiv 2 \pmod 4$ and d is even, then the right hand side of (13) is odd. On the other hand, the left hand side is always even, which is a contradiction.

Case (ii): If $q \equiv 2 \pmod 4$ then $2p+q+1$ is odd and $(2p+q+1)q \equiv 2 \pmod 4$. If $p \equiv 0 \pmod 4$ and d is odd then $\frac{p}{2}\left[(p-1)(d-1)+2(a-1)\right] \equiv 0 \pmod 4$. This contradicts equation (13).

Case (iii): If $p \equiv 0 \pmod 8$ and $q \equiv 2 \pmod 4$, then the left hand side of (13) is congruent to 2 (mod 4) but the right hand side of (13) is congruent to 0 (mod 4). This leads to a contradiction. □

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