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Some constructions of supermagic non-regular graphs

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Abstract

A graph is called supermagic if it admits a labeling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex.

In this paper we deal with non-regular supermagic graphs. We introduce some constructions of supermagic non-regular graphs using supermagic labeling of some regular graphs and $(a, 1)$-antimagic labeling of some graphs.

1 Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping $f^\star$ from $V(G)$ into positive integers defined by

$$f^\star(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ to positive integers is called a *magic labeling* of

G for an *index* λ if its index-mapping $f^\star$ satisfies

$$f^\star(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labeling f of G is called a *supermagic labeling* of G if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labeling of G.

In [8] it is proved that if G is an r-regular supermagic graph of order n, then there exists a supermagic labeling $f : E(G) \longrightarrow \{1, 2, \dots, \frac{rn}{2}\}$ of G. The corresponding index is $\lambda = \frac{r}{2}(1 + \frac{rn}{2})$. In this paper we will consider only such supermagic labelings of regular graphs.

The concept of magic graphs was introduced by Sedláček [15]. The regular magic graphs are characterized in [4]. Two different characterizations of all magic graphs are given in [13] and [12]. Supermagic graphs were introduced by Stewart [16]. In [1], [6], [8], [10], [14] and [17] some classes of regular supermagic graphs are described. Hartsfield and Ringel [7] presented a simple construction of non-regular supermagic graphs. They proved

Proposition 1. [7] *Let G be a regular supermagic graph. Then there exists an edge $e \in E(G)$ such that $G - e$ is a supermagic graph.*

In [5] some constructions of some infinite classes of non-regular supermagic graphs are given. These classes, besides some isolated examples presented in [6] and [16], are the only known classes of infinite non-regular supermagic graphs. Thus we paid our attention to the study of non-regular supermagic graphs. In this paper we present some constructions of such graphs.

2 Supermagic regular graphs

In this section we deal with supermagic graphs obtained from a regular graph by contraction of an edge. We prove

Theorem 1. *Let G be a 3-regular triangle-free supermagic graph. Then there exists an edge $e \in E(G)$ such that the graph obtained from G by contraction of the edge e is supermagic.*

Proof. Let G be a 3-regular supermagic graph of order n. In [8] it is proved that $n \equiv 2 \pmod 4$ and there exists a supermagic labeling $f : E(G) \longrightarrow \{1, 2, \dots, \frac{3n}{2}\}$ of G for an index $\frac{3}{2}\left(1 + \frac{3n}{2}\right)$. Let $u_1u_2 \in E(G)$ be the edge of G such that $f(u_1u_2) = \frac{3n}{2}$. By H we denote the graph obtained from G by the contraction of the edge u_1u_2. Let w denote the vertex in $V(H)$ which arose by identification of u_1 and u_2. Consider the bijection $g : E(H) \longrightarrow \{1 + \frac{3(n-2)}{4}, 2 + \frac{3(n-2)}{4}, \dots, \frac{3n}{2} + \frac{3(n-2)}{4}\}$ given by

$$g(e) = f(e) + \frac{3(n-2)}{4} \quad \text{for every } e \in E(H).$$

For its index-mapping we get

$$\begin{aligned} g^\star(w) &= f^\star(u_1) + f^\star(u_2) - 2f(u_1u_2) + 4\tfrac{3(n-2)}{4} \\ &= 2\tfrac{3}{2}\left(1 + \tfrac{3n}{2}\right) - 2\tfrac{3n}{2} + 3(n-2) = \tfrac{3(3n-2)}{2} \end{aligned}$$

and

$$g^{\star}(v) = f^{\star}(v) + 3\tfrac{3(n-2)}{4} = \tfrac{3}{2}\left(1 + \tfrac{3n}{2}\right) + \tfrac{9(n-2)}{4} = \tfrac{3(3n-2)}{2},$$

for every vertex $v \in E(H) - w$.

Thus g is a supermagic labeling of H. □

Let f be a supermagic labeling of an r-regular graph G of order n. The dual labeling g of f is defined by

$$g(e) = 1 + \tfrac{rn}{2} - f(e) \quad \text{for every } e \in E(G).$$

For every $v \in V(G)$ we have

$$g^{\star}(v) = r\left(1 + \tfrac{rn}{2}\right) - f^{\star}(v) = r\left(1 + \tfrac{rn}{2}\right) - \tfrac{r}{2}\left(1 + \tfrac{rn}{2}\right) = \tfrac{r}{2}\left(1 + \tfrac{rn}{2}\right).$$

Thus g is also a supermagic labeling of graph G.

So if G is a 3-regular supermagic triangle-free graph, then there exist at least two edges e_1, e_2 such that by the contraction of the edge e_i, $i = 1, 2$, a supermagic graph is obtained.

Recall that the Möbius ladder M_n, where $6 \leq n \equiv 0 \pmod 2$, is a 3-regular graph consisting of a cycle C_n of length n, in which all pairs of opposite vertices are connected by an edge (chord). Sedláček proved, see [14], that M_n is supermagic if and only if $6 \leq n \equiv 2 \pmod 4$.

In Figure 1 the supermagic labeling of the Möbius ladder M_{10} and the corresponding dual supermagic labeling of M_{10} are illustrated. The edges with the maximal labels are depicted with thick lines.

Figure 2 illustrates the supermagic labelings of the graphs obtained from the graphs in Figure 1 by contraction of the edge with the maximal value.

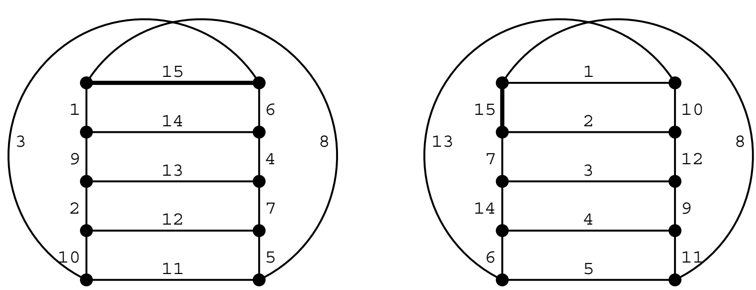

Figure 1.

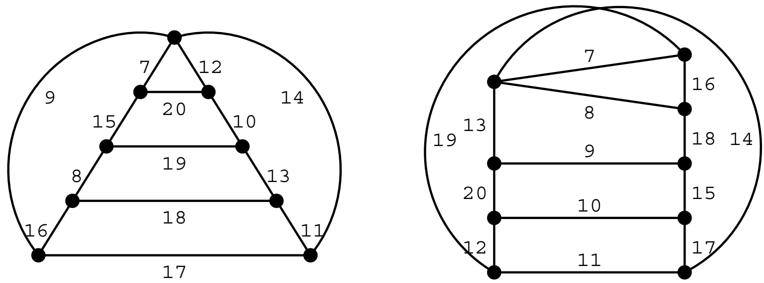

Figure 2.

It is easy to see that the first graph in Figure 2 can also arise from the cartesian product $C_n \times K_2$ of the cycle C_n and the complete graph K_2 by the contraction of an edge joining two edge-disjoint cycles of length n. In [8] it is proved that there does not exist a 3-regular supermagic graph of order $n \equiv 0 \pmod 4$. Thus it is not possible to use Theorem 1 in this case. Despite this fact we found a supermagic labeling of such a graph. First we introduce the following notation. Let A_n be a graph isomorphic to the cartesian product of C_n and K_2, in which one edge joining two edge-disjoint cycles of length n is contracted. So the graph A_n has the vertex set

$$V(A_n) = \{u, v_2^1, \dots, v_n^1, v_2^2, \dots, v_n^2\},$$

(for the sake of clarity let $u = v_1^1 = v_1^2$), and the edge set

$$E(A_n) = \bigcup_{i=1}^{n} \{v_i^1 v_{i+1}^1,\ v_i^2 v_{i+1}^2\} \cup \bigcup_{i=2}^{n} \{v_i^1 v_i^2\},$$

where subscripts are taken modulo n.

Now we are able to prove

Theorem 2. *The graph A_n is supermagic for every positive integer n, $n \geq 3$.*

Proof. If $n \geq 3$ is an odd positive integer, then A_n is isomorphic to a graph which we obtain from the Möbius ladder M_{2n} with one contracted chord. In [14] is constructed a supermagic labeling of M_{2n}, $3 \leq n \equiv 1 \pmod 2$, with the minimal value on the chord. Then we consider the dual labeling to this supermagic labeling and according to Theorem 1, we get that A_n is a supermagic graph.

If $n \geq 4$ is an even positive integer, we consider a mapping $f : E(A_n) \longrightarrow \{\frac{n}{2}, \dots, \frac{7n}{2} - 2\}$ defined by

$$f(v_i^1 v_{i+1}^1) = \begin{cases} 2n + \frac{i-1}{2} & \text{if } i \equiv 1 \pmod 2, \\ \frac{n}{2} + \frac{i-2}{2} & \text{if } i \equiv 0 \pmod 2, \end{cases}$$

$$f(v_i^2 v_{i+1}^2) = \begin{cases} \frac{3n}{2} + \frac{i-1}{2} & \text{if } i \equiv 1 \pmod 2, \\ n + \frac{i-2}{2} & \text{if } i \equiv 0 \pmod 2, \end{cases}$$

$$f(v_i^1 v_i^2) = \tfrac{7n}{2} - i \quad \text{if} \quad i \geq 2.$$

It is easy to check that f is a bijection and

$$f^\star(v) = 6n - 2 \quad \text{for every } v \in V(A_n).$$

Thus f is a supermagic labeling of A_n. □

Another construction of supermagic non-regular graphs provides the following theorem.

Theorem 3. *Let f be a supermagic labeling of a 4-regular graph G such that there exists a vertex $v \in V(G)$ with*

$$f(vu_1) + f(vu_2) = f(vu_3) + f(vu_4),$$

where u_i, $i = 1, \dots, 4$, are the vertices adjacent to v. Let H be a graph with the vertex set $V(H) = (V(G) - v) \cup \{v^1, v^2\}$ and the edge set $E(H) = \big(E(G) - \bigcup_{i=1}^4 \{vu_i\}\big) \cup \{v^1u_1, v^1u_2, v^2u_3, v^2u_4, v^1v^2\}$. Then H is a supermagic graph.

Proof. Let f be a supermagic labeling of 4-regular graph G such that for the vertex $v \in V(G)$,

$$f(vu_1) + f(vu_2) = f(vu_3) + f(vu_4).$$

This expression is equal to $\frac{\lambda}{2}$, where λ is the index of f.

Consider a bijection $g : E(H) \longrightarrow \{1, 2, \dots, |E(G)|, |E(G)| + 1\}$ defined by

$$g(e) = \begin{cases} f(e) & \text{if } \ e \in E(G), \\ f(u_i v) & \text{if } \ e = u_i v^j, \ (i,j) \in \{(1,1),(2,1),(3,2),(4,2)\}, \\ |E(G)| + 1 & \text{if } \ e = v^1 v^2. \end{cases}$$

For the index-mapping of g we have

$$\begin{aligned} g^\star(v^1) &= g(v^1u_1) + g(v^1u_2) + g(v^1v^2) = f(vu_1) + f(vu_2) + 1 + |E(G)| \\ &= 1 + |E(G)| + 1 + |E(G)| = 2(1 + |E(G)|), \\ g^\star(v^2) &= g(v^2u_3) + g(v^2u_4) + g(v^1v^2) = f(vu_3) + f(vu_4) + 1 + |E(G)| \\ &= 1 + |E(G)| + 1 + |E(G)| = 2(1 + |E(G)|), \\ g^\star(u) &= f^\star(u) = 2(1 + |E(G)|) \quad \text{for every } u \in V(H) - \{v^1, v^2\}. \end{aligned}$$

Thus g is a supermagic labeling of H. □

It is obvious that the graph G in Theorem 3 arises from H by the contraction of the edge v^1v^2.

In [9] it is proved that if G is a 4-regular bipartite graph decomposable into two Hamilton cycles then for every vertex $v \in V(G)$ there exists a supermagic labeling f of G such that

$$f(e_1) + f(e_2) = f(e_3) + f(e_4),$$

where the edges e_i, $i = 1, \dots, 4$, are adjacent to v. Thus if G is a 4-regular bipartite graph decomposable into two Hamilton cycles then the graph obtained from G by splitting of an arbitrary vertex and adding a new edge is supermagic. In Figure 3 is illustrated a supermagic labeling of a bipartite graph decomposable into two Hamilton cycles and the supermagic labeling of the corresponding graph obtained from the original by splitting a vertex and adding an edge. In the original graph the vertex which is split and in the corresponding graph the added edge are marked.

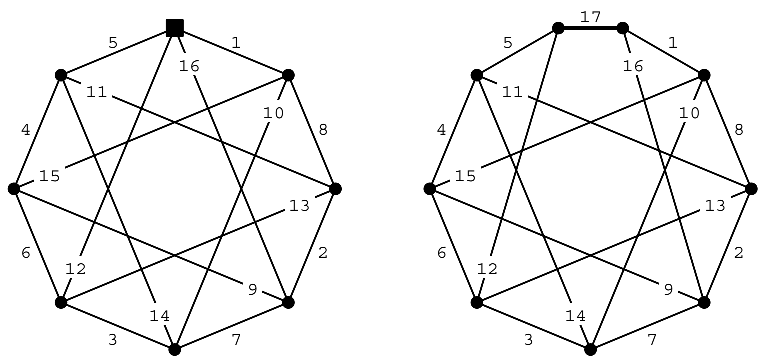

FIGURE 3.

In the next part of this paper we will deal with the disjoint union of two regular graphs. We are able to prove

Theorem 4. *For $i = 1, 2$ let G_i be an r_i-regular supermagic graph of order n_i. If $r_1 > r_2$ and*

$$p := \frac{n_2 r_2^2 - n_1 r_1^2 + 2 r_1 r_2 n_1}{4(r_1 - r_2)} - \frac{1}{2}$$

is a non-negative integer, then the disjoint union of the graphs G_1 and G_2 is a supermagic graph.

Proof. As G_1 is an r_1-regular supermagic graph, there exists a supermagic labeling $f_1 : E(G_1) \longrightarrow \{1, \dots, \frac{r_1 n_1}{2}\}$ of G_1 for an index $\lambda_1 = \frac{r_1}{2}\left(1 + \frac{r_1 n_1}{2}\right)$. Analogously, there exists a supermagic labeling $f_2 : E(G_2) \longrightarrow \{1, \dots, \frac{r_2 n_2}{2}\}$ of G_2 for an index $\lambda_2 = \frac{r_2}{2}\left(1 + \frac{r_2 n_2}{2}\right)$.

If $p = \frac{n_2 r_2^2 - n_1 r_1^2 + 2 r_1 r_2 n_1}{4(r_1 - r_2)} - \frac{1}{2}$ is a non-negative integer then we consider a bijection $g : E(G_1 \cup G_2) \longrightarrow \{1 + p, \dots, \frac{r_1 n_1 + r_2 n_2}{2} + p\}$ defined by

$$g(e) = \begin{cases} f_1(e) + p & \text{for } \ e \in E(G_1), \\ f_2(e) + \frac{r_1 n_1}{2} + p & \text{for } \ e \in E(G_2). \end{cases}$$

For its index-mapping we get

$$g^\star(v) = \begin{cases} \lambda_1 + r_1 p & \text{if } \ v \in V(G_1), \\ \lambda_2 + r_2(\frac{r_1 n_1}{2} + p) & \text{if } \ v \in V(G_2). \end{cases}$$

So $g^{\star}(v) = r_1 r_2(r_1 n_1 + r_2 n_2)/4(r_1 - r_2)$ for every vertex $v \in V(G_1 \cup G_2)$. Thus g is a supermagic labeling of $G_1 \cup G_2$. □

Figure 4 depicts a supermagic labeling of $K_{3,3} \cup K_{4,4}$ obtained by using the construction described in Theorem 4.

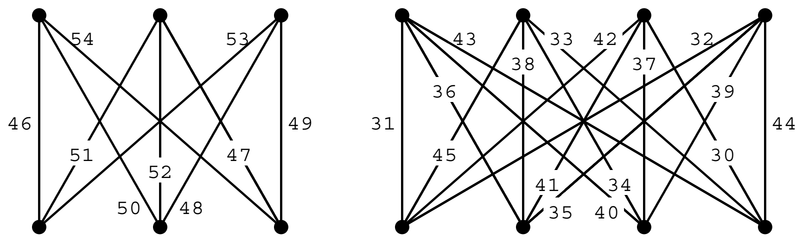

FIGURE 4.

3 $(a, 1)$-antimagic graphs

Let G be a graph. A bijective mapping f from $E(G)$ into the set of integers $\{1, 2, \ldots, |E(G)|\}$ is called an *antimagic labeling* of G if the index-mapping $f^{\star}$ is injective, i.e., if it satisfies

$$f^{\star}(v) \neq f^{\star}(u) \quad \text{for every} \;\; u, v \in V(G),\, u \neq v.$$

The concept of an antimagic labeling was introduced by Hartsfield and Ringel [6]. Bodendiek and Walther [2] introduced the special case of antimagic graphs. For positive integers a, d, a graph G is said to be (a, d)*-antimagic* if it admits an antimagic labeling f such that

$$\{f^{\star}(v) : v \in V(G)\} = \{a, a + d, \ldots, a + (|V(G)| - 1)d\}.$$

Obviously, $a = \dfrac{|E(G)|(|E(G)| + 1)}{|V(G)|} - \dfrac{(|V(G)| - 1)d}{2}$ in this case. Since a is an integer, for an arbitrary r-regular $(a, 1)$-antimagic graph of order n: if $r \equiv 0 \pmod 2$, then $n \equiv 1 \pmod 2$ and if $r \equiv 1 \pmod 2$, then $n \equiv 0 \pmod 4$.

In this part we will describe two constructions of non-regular supermagic graphs using $(a, 1)$-antimagic graphs.

Theorem 5. *Let G_1, G_2, G_3 be 2-regular $(a, 1)$-antimagic graphs each of order n. Then there exists a supermagic graph G, which is decomposable into two edge-disjoint spanning subgraphs F_1 and F_2, where F_1 is isomorphic to the disjoint union of G_1, G_2, G_3 (i.e., $F_1 \cong G_1 \cup G_2 \cup G_3$) and the subgraph F_2 is isomorphic to n copies of a path on 3 vertices (i.e., $F_2 \cong nP_3$).*

Proof. Since G_1, G_2, G_3 are 2-regular $(a,1)$-antimagic graphs of the same order n, it follows that $n \equiv 1 \pmod 2$. Moreover, for $i = 1,2,3$ there exists an $(a,1)$-antimagic labeling $f_i : E(G_i) \longrightarrow \{1,2,\dots,n\}$ of G_i, such that its index-mapping $f_i^\star$ satisfies

$$\{f_i^\star(v) : v \in V(G_i)\} = \left\{\tfrac{n+3}{2}, \tfrac{n+5}{2}, \dots, \tfrac{3n+1}{2}\right\}.$$

We denote the vertices of graph G_i, $i = 1,2,3$, by $v_1^i, v_2^i \dots, v_n^i$ in such a way that

$$f_1^\star(v_i^1) = \frac{n+1}{2} + i \quad \text{for} \quad i = 1,\dots,n,$$

$$f_2^\star(v_i^2) = \begin{cases} \frac{n-1}{2} + 2i & \text{for} \quad i = 1,\dots,\frac{n+1}{2}, \\ 2i - \frac{n+1}{2} & \text{for} \quad i = \frac{n+3}{2},\dots,n, \end{cases}$$

$$f_3^\star(v_i^3) = \begin{cases} n + i & \text{for} \quad i = 1,\dots,\frac{n+1}{2}, \\ i & \text{for} \quad i = \frac{n+3}{2},\dots,n. \end{cases}$$

Evidently the following holds:

$$f_1^\star(v_1^1) < f_1^\star(v_2^1) < \dots < f_1^\star(v_n^1),$$
$$f_2^\star(v_1^2) < f_2^\star(v_{\frac{n+3}{2}}^2) < f_2^\star(v_2^2) < f_2^\star(v_{\frac{n+5}{2}}^2) < \dots < f_1^\star(v_{\frac{n+1}{2}}^2),$$
$$f_3^\star(v_{\frac{n+3}{2}}^3) < f_3^\star(v_{\frac{n+5}{2}}^3) < \dots < f_3^\star(v_n^3) < f_3^\star(v_1^3) < f_3^\star(v_2^3) < \dots < f_3^\star(v_{\frac{n+1}{2}}^2).$$

Let G be a graph obtained from the disjoint union of G_1, G_2, G_3 with added edges $v_i^1 v_i^2$, $v_i^2 v_i^3$, $i = 1,\dots,n$. Now we consider a mapping $g : E(G) \longrightarrow \left\{\frac{7n+1}{2}, \frac{7n+3}{2}, \dots, \frac{17n-1}{2}\right\}$ defined by

$$g(e) = \begin{cases} f_1(e) + \frac{15n-1}{2} & \text{for} \quad e \in E(G_1), \\ f_2(e) + \frac{9n-1}{2} & \text{for} \quad e \in E(G_2), \\ f_3(e) + \frac{13n-1}{2} & \text{for} \quad e \in E(G_3), \\ \frac{9n+1}{2} - i & \text{for} \quad e = v_i^1 v_i^2 \quad i = 1,\dots,n, \\ 6n + 1 - i & \text{for} \quad e = v_i^2 v_i^3 \quad i = 1,\dots,\frac{n+1}{2}, \\ 7n + 1 - i & \text{for} \quad e = v_i^2 v_i^3 \quad i = \frac{n+3}{2},\dots,n. \end{cases}$$

It is easy to check that g is a bijection and for its index-mapping we get

$$g^\star(v_i^1) = f_1^\star(v_i^1) + 2\left(\tfrac{15n-1}{2}\right) + g(v_i^1 v_i^2) = \tfrac{n+1}{2} + i + 15n - 1 + \tfrac{9n+1}{2} - i = 20n$$

for every $i = 1,\dots,n$.

Analogously for $j = 2,3$ and every $i = 1,\dots,n$ we have

$$g^\star(v_i^j) = 20n.$$

Thus g is a supermagic labeling of G. □

Let $n \geq 3$ be an odd positive integer. Let H_n be a graph with the vertex set

$$V(H_n) = \{v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2, v_1^3, \ldots, v_n^3\}$$

and the edge set

$$E(H_n) = \bigcup_{i=1}^{n} \{v_i^1 v_{i+1}^1,\ v_i^2 v_{i+\frac{n+1}{2}}^2,\ v_i^3 v_{i+1}^3,\ v_i^1 v_i^2,\ v_i^2 v_i^3\},$$

where subscripts are taken modulo n.

It is easy to see that H_n is decomposable into two edge-disjoint spanning subgraphs F_1 and F_2, where F_1 is isomorphic to three copies of the odd cycle C_n (i.e., $F_1 \cong 3C_n$) and the spanning subgraph F_2 is isomorphic to n copies of the path on 3 vertices (i.e., $F_2 \cong nP_3$). It is known, see [3], that there exists an $(a, 1)$-antimagic labeling of the odd cycle $C_n = v_1 v_2 \ldots v_n v_1$ such that $f^\star(v_i) = \frac{n+1}{2} + i$ for $i = 1, \ldots, n$. According to the proof of the previous theorem we get

Corollary 1. *For every odd positive integer $n \geq 3$ the graph H_n is supermagic.*

Figure 5 depicts a supermagic labeling of H_5.

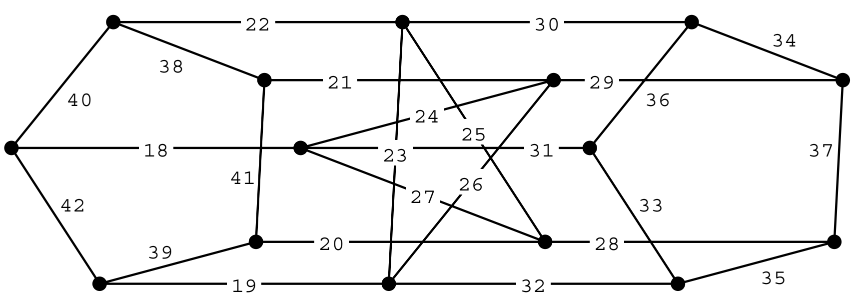

FIGURE 5.

In [11] the following is proved:

Proposition 2. *Let k, n and m be positive integers. For odd k each of the following graphs is $(a, 1)$-antimagic:*

(i) kC_n *if* $3 \leq n \equiv 1 \pmod 2$,

(ii) $k(C_3 \cup C_n)$ *if* $6 \leq n \equiv 0 \pmod 2$,

(iii) $k(C_4 \cup C_n)$ *if* $5 \leq n \equiv 1 \pmod 2$,

(iv) $k(C_5 \cup C_n)$ *if* $4 \leq n \equiv 0 \pmod 2$,

(v) $k(C_m \cup C_n)$ *if* $6 \leq m \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$, $n \geq \frac{m}{2} + 2$.

Combining Theorem 5 and Proposition 2 we can find many other supermagic non-regular graphs. In Figure 6 is depicted a supermagic labeling of one such graph.

Before we present another construction of supermagic graphs using $(a, 1)$-antimagic graphs, we remind the reader that if G and H are regular $(a, 1)$-antimagic graphs of the same order, then the degrees of the graphs must have the same parity.

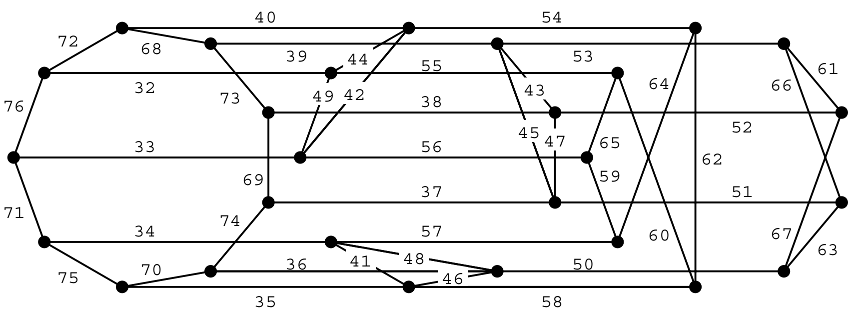

Figure 6.

Theorem 6. *Let k be a positive integer. Let G_1 (G_2) be an r-regular ($(r+2k)$-regular) $(a, 1)$-antimagic graph of order n. Suppose that*

$$p_1 = \frac{nr^2}{4k} - \frac{nk+1}{2} \quad or \quad p_2 = \frac{nr(r+2)}{4k} - \frac{nk+1}{2}$$

is a non-negative integer. Then there exists a supermagic graph G, which is decomposable into two edge-disjoint spanning subgraphs F_1 and F_2, where F_1 is isomorphic to the disjoint union of G_1 and G_2 (i.e., $F_1 \cong G_1 \cup G_2$) and the subgraph F_2 is isomorphic to n copies of K_2 (i.e., $F_2 \cong nK_2$).

Proof. Since G_1 is an r-regular $(a_1, 1)$-antimagic graph of order n, there exists an $(a_1, 1)$-antimagic labeling $f_1 : E(G_1) \longrightarrow \{1, 2, \dots, \frac{nr}{2}\}$ of G_1, such that for its index-mapping $f_1^\star$ we have

$$\{f_1^\star(v) : v \in V(G_i)\} = \{a_1, a_1 + 1, \dots, a_1 + n - 1\}.$$

Clearly $a_1 = (nr^2 + 2r - 2n + 2)/4$ in this case.

Analogously there exists an $(a_2, 1)$-antimagic labeling $f_2 : E(G_2) \longrightarrow \{1, 2, \dots, \frac{n(r+2k)}{2}\}$ of G_2. For its index-mapping we have

$$\{f_2^\star(v) : v \in V(G_i)\} = \{a_2, a_2 + 1, \dots, a_2 + n - 1\},$$

where $a_2 = (n(r+2k)^2 + 2(r+2k) - 2n + 2)/4$.

We denote the vertices of G_i, $i = 1, 2$, by $v_1^i, v_2^i \dots, v_n^i$ in such a way that

$$f_1^\star(v_i^1) = a_1 - 1 + i \quad \text{for} \quad i = 1, \dots, n,$$

$$f_2^\star(v_i^1) = a_2 - 1 + i \quad \text{for} \quad i = 1, \dots, n.$$

Let G be a graph obtained from the disjoint union of G_1 and G_2 with added edges $v_i^1 v_i^2$, $i = 1, \dots, n$.

If $p_1 = \frac{nr^2}{4k} - \frac{nk+1}{2}$ is a non-negative integer, then we consider a bijection $g_1 : E(G) \longrightarrow \{1 + p_1, \dots, n(r + k + 1) + p_1\}$ defined by

$$g_1(e) = \begin{cases} f_1(e) + \frac{n(r+2k)}{2} + p_1 & \text{for} \quad e \in E(G_1), \\ f_2(e) + p_1 & \text{for} \quad e \in E(G_2), \\ n(r + k + 1) + 1 - i + p_1 & \text{for} \quad e = v_i^1 v_i^2, \ \ i = 1, \dots, n. \end{cases}$$

For its index-mapping we get

$$\begin{aligned} g_1^\star(v_i^1) &= f_1^\star(v_i^1) + r(\tfrac{n(r+2k)}{2} + p_1) + g(v_i^1 v_i^2) \\ &= a_1 - 1 + i + r(\tfrac{n(r+2k)}{2} + p_1) + n(r + k + 1) + 1 - i + p_1 \\ &= \tfrac{n(3r^2+4r+2)}{4} + \tfrac{kn(r+1)}{2} + \tfrac{nr^2(r+1)}{4k}, \end{aligned}$$

and

$$\begin{aligned} g_1^\star(v_i^2) &= f_2^\star(v_i^1) + (r + 2k)p_1 + g(v_i^1 v_i^2) \\ &= a_2 - 1 + i + (r + 2k)p_1 + n(r + k + 1) + 1 - i + p_1 \\ &= \tfrac{n(3r^2+4r+2)}{4} + \tfrac{kn(r+1)}{2} + \tfrac{nr^2(r+1)}{4k} \end{aligned}$$

for $i = 1, \dots, n$.

Thus g_1 is a supermagic labeling of G.

If $p_2 = \frac{nr(r+2)}{4k} - \frac{nk+1}{2}$ is a non-negative integer, then we consider a bijection $g_2 : E(G) \longrightarrow \{1 + p_2, \dots, n(r + k + 1) + p_2\}$ defined by

$$g_2(e) = \begin{cases} f_1(e) + \frac{n(r+2k+2)}{2} + p_2 & \text{for} \quad e \in E(G_1), \\ f_2(e) + p_2 & \text{for} \quad e \in E(G_2), \\ \frac{n(r+2k+2)}{2} + 1 - i + p_2 & \text{for} \quad e = v_i^1 v_i^2, \ \ i = 1, \dots, n. \end{cases}$$

For its index-mapping we get

$$g_2^\star(v) = \frac{n(3r^2 + 6r + 2)}{4} + \frac{kn(r + 1)}{2} + \frac{nr(r + 1)(r + 2)}{4k} \qquad \text{for } v \in V(G).$$

So g_2 is a supermagic labeling of G in this case. □

For r even and $k = 1$ in the previous theorem we immediately obtain:

Corollary 2. *Let G_1 be a $2r$-regular $(a, 1)$-antimagic graph of order n and let G_2 be a $2(r + 1)$-regular $(a, 1)$-antimagic graph of order n. Then there exists a supermagic graph G which is decomposable into two edge-disjoint spanning subgraphs F_1 and F_2, where F_1 is isomorphic to the disjoint union of G_1 and G_2 (i.e., $F_1 \cong G_1 \cup G_2$) and the subgraph F_2 is isomorphic to n copies of K_2 (i.e., $F_2 \cong nK_2$).*

Proof. Since the graphs G_1 and G_2 are $(a, 1)$-antimagic graphs of even degree, then $n \equiv 1 \pmod 2$. Thus $p_1 = \frac{n(2r)^2}{4\cdot 1} - \frac{n\cdot 1+1}{2} = nr^2 - \frac{n+1}{2}$ is a positive integer and according to Theorem 6, there exists a desired supermagic graph G. □

The construction described in Theorem 6 is illustrated in Figure 7.

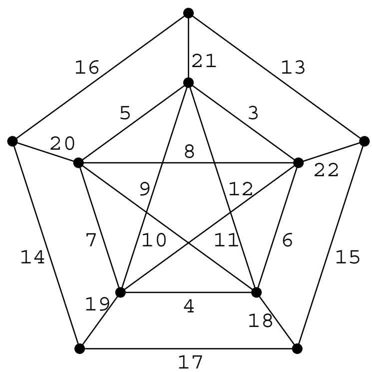

FIGURE 7.

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